Stability of singular perturbed switched systems and their corresponding switched DAEs

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Switched systems

$$\dot{x} = A_\sigma x$$

with

- $x : [t_0, \infty) \to \mathbb{R}^n$ the state
- $\sigma : [t_0, \infty) \to \{1, 2, \ldots, \mathrm{p}\}$ the switching signal
- $A_1, A_2, \ldots, A_\mathrm{p} \in \mathbb{R}^{n \times n}$ the coefficient matrices for each mode

Nontrivial stability behavior

Each mode $\dot{x} = A_p x$ stable $\quad \not\Rightarrow \quad$ $\dot{x} = A_\sigma x$ stable

Semigroup property

Definition

A family of sets $(\mathcal{S}_t)_{t\geq 0}$ has the semigroup property $:\iff \; \forall\, s,t \geq 0 : \mathcal{S}_s \cdot \mathcal{S}_t = \mathcal{S}_{s+t}$

Here $\mathcal{S}_s \cdot \mathcal{S}_t := \{\Phi_s \cdot \Phi_t \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t\} \subseteq \mathbb{R}^{n\times n}$

Semigroups for non-switched and switched systems

- Non-switched case: $\mathcal{S}_t = \mathcal{S}_t(A) := \{e^{At}\}$ has semigroup property
- Switched case:
$$\mathcal{S}_t = \mathcal{S}_t(A_1, A_2, \ldots, A_{\mathtt{p}}) := \left\{ e^{A_{\sigma_k}\tau_k} e^{A_{\sigma_{k-1}}\tau_{k-1}} \cdots e^{A_{\sigma_0}\tau_0} \;\middle|\; \begin{array}{l} k \geq 0,\ \tau_i \geq 0, \\ \sigma_i \in \{1,2,\ldots,\mathtt{p}\}, \\ t = \sum_{i=0}^{k} \tau_i \end{array} \right\}$$

Key property

x solve switched system $\iff x(t) = \Phi_{t-t_0} \cdot x_0$ with $\Phi_{t-t_0} \in \mathcal{S}_{t-t_0}$

Lyapunov exponent

In the following let $\mathcal{S} := \bigcup_{t\geq 0} \mathcal{S}_t$.

Definition (Lyapunov exponent of semigroup)

Exponential growth bound: $\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t}$

Lyapunov exponent: $\lambda(\mathcal{S}) := \limsup_{t\to\infty} \lambda_t(\mathcal{S}_t)$

Stability characterization

$\lambda(\mathcal{S}) < 0 \quad \Longleftrightarrow \quad$ all solutions of $\dot{x} = A_\sigma x$ with arbitrary σ exponentially converge to zero

$\lambda(\mathcal{S}) > 0 \quad \Longleftrightarrow \quad$ $\exists x_0\ \exists \sigma$ (periodic) : $x(t) \to \infty$ exponentially

Content overview

Introduction: Switched Systems and Lyapunov exponent

Singular perturbed systems

Switched DAEs and structurally aligned singular perturbations

Classical singular perturbed systems

$$\begin{aligned} \dot{x}^s &= A^s x^s + A^{sf} x^f \\ \varepsilon \dot{x}^f &= A^{fs} x^s + A^f x^f \end{aligned} \quad \overset{\varepsilon \to 0}{\rightsquigarrow} \quad \dot{x}^s \approx (A^s - A^{sf}(A^f)^{-1}A^{fs})x_s =: A^b x^s$$

Theorem (cf. Tikhonov's Theorem)

Singular perturbed system is stable for sufficiently small $\varepsilon \iff A^f$ *and* A^b *are Hurwitz*

$\rightsquigarrow$ Divide and Conquer / dimension reduction

Boundary layer system as DAE

Instead of $\dot{x}_s = A^b x_s$ consider the boundary layer DAE:
$$\begin{aligned} \dot{x}_s &= A^s x^s + A^{sf} x^f \\ 0 &= A^{fs} x^s + A^f x^f \end{aligned}$$

DAE = differential-algebraic equation

Singular perturbed switched systems

simple extension

$$\begin{aligned} \dot{x}^s &= A_\sigma^s x^s + A_\sigma^{sf} x^f \\ \varepsilon \dot{x}^f &= A_\sigma^{fs} x^s + A_\sigma^f x^f \end{aligned}$$

[Chitour, Haidar, Mason, Sigalotti; Automatica 2023]

more general extension

$$D_\sigma^\varepsilon \dot{x} = A_\sigma x$$

[Chitour, Daafouz, Haidar, Mason, Sigalotti; CDC 2024]

most general extension

$$E_\sigma^\varepsilon \dot{x} = A_\sigma x$$

[Haidar, Chitour, Daafouz, Mason, Sigalotti; arXiv 2026]

In all three cases (under some reasonable assumptions): $\lambda(\mathcal{S}^0) \leq \liminf_{\varepsilon \to 0} \lambda(\mathcal{S}^\varepsilon)$

Here $\mathcal{S}^\varepsilon$ is the semigroup of sing. pert. switched ODE, whereas $\mathcal{S}^0$ is semigroup of cor. switched DAE ($\varepsilon = 0$)

Inequality can be strict

Already for the simple extension the inequality can be strict, in particular, the boundary layer dynamics may be stable but the singular perturbed system is unstable for arbitrarily small ε!

[Malloci, Daafouz, Iung; CDC 2009]

Research question

$$\lambda(\mathcal{S}^0) \stackrel{?}{=} \liminf_{\varepsilon \to 0} \lambda(\mathcal{S}^\varepsilon)$$

Question

Which additional assumptions prevent a gap, i.e. when are the stability behaviors asymptotically equal?

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Geometric analysis of DAEs

$$E\dot{x} = Ax + Bu \qquad \text{(DAE)}$$

Theorem (Characteristic invariant subspace decomposition)

$\det(sE - A) \not\equiv 0$ then $\exists$ unique decomposition $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$ such that:
x solves (DAE) $\iff$ $x = v \oplus w \in \mathcal{V} \oplus \mathcal{W}$ and

$$\begin{aligned} \text{in } \mathcal{V}: &\quad \dot{v} = A^{\text{diff}} v + B^{\text{diff}} u, &\quad v(0^-) = \Pi x(0^-) \\ \text{in } \mathcal{W}: &\quad E^{\text{imp}} \dot{w} = w + B^{\text{imp}} u, &\quad w(0^-) = (I - \Pi) x(0^-) \end{aligned}$$

for some suitably defined $A^{\text{diff}}, E^{\text{imp}}, B^{\text{diff}}, B^{\text{imp}}, \Pi$ with E^{imp} nilpotent and Π is projector onto $\mathcal{V}$ along $\mathcal{W}$ (consistency projector)

Corollary

$w = -\sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}$ and hence, for $u \equiv 0$, $w(0^+) = 0$

$\rightsquigarrow x(0^+) - x(0^-) = v(0^+) + w(0^+) - v(0^-) - w(0^-) = -w(0^-) \in \operatorname{im}(I - \Pi) = \mathcal{W}$

Solution semigroup of switched DAEs

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \qquad \text{(swDAE)}$$

with corresponding A_i^{diff}, E_i^{imp}, Π_i, $i \in \{1, \ldots, \mathtt{p}\}$
regular & index-1 assumption: $\det(sE_i - A_i) \not\equiv 0$ and $E_i^{\text{imp}} = 0$

Theorem (Trenn 2009)

x solves regular & index-1 (swDAE) $\iff$
$$x(t^-) = e^{A_{\sigma_k}^{\text{diff}}(t-t_k)} \Pi_{\sigma_k} e^{A_{\sigma_{k-1}}^{\text{diff}}(t_k - t_{k-1})} \Pi_{\sigma_{k-1}} \cdots e^{A_{\sigma_0}^{\text{diff}}(t_1 - t_0)} \Pi_{\sigma_0} x_0$$

Corresponding semigroup $\mathcal{S}_t^0 := \left\{ \prod_{i=0}^k e^{A_{\sigma_i}^{\text{diff}} \tau_i} \Pi_{\sigma_i} \;\middle|\; \begin{array}{l} k > 0,\ \tau_i > 0, \\ \sigma_i \in \{1, 2, \ldots, \mathtt{p}\}, \\ t = \sum_{i=0}^k \tau_i \end{array} \right\}$, $\quad \mathcal{S}_0^0 := \{I\}$

$\rightsquigarrow x(t^-) = \Phi_{t-t_0} x_0$ for some $\Phi_{t-t_0} \in \mathcal{S}_{t-t_0}^0$

Lyapunov exponents for switched DAEs

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \qquad \text{(swDAE)}$$

with corresponding semigroup $\mathcal{S}_t^0 := \left\{ \prod_{i=0}^k e^{A_{\sigma_i}^{\text{diff}} \tau_i} \Pi_{\sigma_i} \;\middle|\; \begin{array}{l} k > 0,\ \tau_i > 0, \\ \sigma_i \in \{1, 2, \ldots, \mathrm{p}\}, \\ t = \sum_{i=0}^k \tau_i \end{array} \right\}$

Infinite exponential growth bound

$\lambda_t(\mathcal{S}_t^0) = \pm\infty$ is possible!

In fact, $\lambda_t(\mathcal{S}_t^0) = -\infty \iff \forall i: \ E_i = 0$ (then $\mathcal{S}_t^0 = \{0\}$)

and $\lambda_t(\mathcal{S}_t^0) < \infty \iff$ the set $\{\Pi_1, \Pi_2, \ldots, \Pi_\mathrm{p}\}$ is product bounded

Shifted systems

$$E_\sigma^\varepsilon \dot{x} = A_\sigma^\varepsilon x$$

shifted version for $\delta \in \mathbb{R}$:

$$E_\sigma^\varepsilon \dot{x} = (A_\sigma - \delta E_\sigma^\varepsilon)x$$

Lemma

$\lambda(\mathcal{S}^{\varepsilon,\delta}) = \lambda(\mathcal{S}^\varepsilon) - \delta$ *for* $\varepsilon > 0$ *and* $\varepsilon = 0$

Consequence

Understanding the gap between $\lambda(\mathcal{S}^0)$ and $\liminf_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon)$ reduces to understanding when stability of switched DAE does not imply stability of singular perturbed system

Structurally aligned singular perturbations (SASP)

Conjecture

A misaligned jump-approximation can lead to Lyapunov exponent gap.

- Singular perturbation analysis: Fast dynamics approximate jump $x(0^-) \to x(0^+)$ which is then followed by ODE-solution in boundary layer.
- Key geometric insight: jump direction $x(0^+) - x(0^-)$ lies in invariant subspace $\mathcal{W} \rightsquigarrow$ dynamics approximating jump should also evolve within $\mathcal{W}$

Definition (cf. Mironchenko, Wirth, Wulff; CDC 2013 & TAC 2015)

A singular perturbed system $E^\varepsilon \dot{x} = Ax$ is structurally aligned with underlying regular index-1 DAE $E^0 \dot{x} = Ax :\Leftrightarrow$

$$A^\varepsilon := (E^\varepsilon)^{-1} A = A^{\text{diff}} + \frac{1}{\varepsilon}(\Pi - I)$$

Key decoupling features

Let $E^\varepsilon \dot{x} = Ax$ with $A^\varepsilon := (E^\varepsilon)^{-1}A$ have structurally aligned singular perturbations, i.e.

$$A^\varepsilon = A^{\text{diff}} + \frac{1}{\varepsilon}(\Pi - I)$$

then

- In decoupling coordinates (according to $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$): $A^\varepsilon = \begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{bmatrix}$
- $\mathcal{V}$ and $\mathcal{W}$ are A^ε-invariant and $A^\varepsilon|_{\mathcal{V}} = A^{\text{diff}}|_{\mathcal{V}}$ and $A^\varepsilon|_{\mathcal{W}} = \frac{1}{\varepsilon}(\Pi - I)|_{\mathcal{W}}$
- $e^{A^\varepsilon t} = \underbrace{e^{A^{\text{diff}} t}}_{\text{slow}} \cdot \underbrace{e^{\frac{1}{\varepsilon}(\Pi - I)t}}_{\text{fast}} = e^{A^{\text{diff}} t}\Pi + e^{\frac{1}{\varepsilon}(\Pi - I)t}(I - \Pi)$
- For $x_0 = v_0 \oplus w_0 \in \mathcal{V} \oplus \mathcal{W}$: $x(t) = \underbrace{e^{A^{\text{diff}} t} v_0}_{\in \mathcal{V}} + \underbrace{e^{\frac{1}{\varepsilon}(\Pi - I)t} w_0}_{\in \mathcal{W}}$

Commutativity and SASP

Definition (cf. Liberzon, Trenn, Wirth; CDC 2011)

A switched DAE $E_\sigma \dot{x} = A_\sigma x$ is called commutative $:\Leftrightarrow \forall \Phi_t \in \mathcal{S}_t^0, \Phi_s \in \mathcal{S}_s^0: \quad \Phi_t \cdot \Phi_s = \Phi_s \cdot \Phi_t$

It can be shown that for invertible A_i, commutativity is equivalent to $A_i^{\text{diff}} A_j^{\text{diff}} = A_j^{\text{diff}} A_i^{\text{diff}}$.

Theorem

Consider singular perturbed switched system $E_\sigma^\varepsilon \dot{x} = A_\sigma x$ and assume

- *each $(E_i, A_i) := (E_i^0, A_i)$ is regular and index-1,*
- *$E_\sigma \dot{x} = A_\sigma x$ is commuting,*
- *the singular perturbation is structurally aligned,*

then $\lambda(\mathcal{S}^0) = \lim_{\varepsilon \to \infty} \lambda(\mathcal{S}^\varepsilon)$

Attention: Commutativity (and regular, index-1) not sufficient (counter example in paper)

Open problem: Commutativity is most likely too strong, what is the necessary property?

Summary

- For non-switched systems: Stability properties of singular perturbed system $\iff$ stability property of boundary system (or DAE)
- For switched systems: In general gap between stability properties of

$$E_\sigma^\varepsilon \dot{x} = A_\sigma x \qquad \text{and} \qquad E_\sigma \dot{x} = A_\sigma x$$

- Regular DAEs have intrinsic subspace decoupling $\mathbb{R} = \mathcal{V} \oplus \mathcal{W}$
- Singular perturbations should be structurally aligned to this decoupling
- For commuting switched DAEs, structural alignment indeed closes the stability gap
- Open problem: Is structural alignment alone already sufficient for zero gap?