Stability of singular perturbed switched systems and their corresponding switched DAEs

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Switched systems

$$\dot{x} = A_\sigma x$$

with

- $x : [t_0, \infty) \to \mathbb{R}^n$ the state
- $\sigma : [t_0, \infty) \to \{1, 2, \ldots, \mathrm{p}\}$ the switching signal
- $A_1, A_2, \ldots, A_\mathrm{p} \in \mathbb{R}^{n \times n}$ the coefficient matrices for each mode

Nontrivial stability behavior

Each mode $\dot{x} = A_p x$ stable $\not\Rightarrow$ $\dot{x} = A_\sigma x$ stable

Illustration ...

Linearity and solution sets

$$\Sigma_\sigma : \quad \dot{x} = A_\sigma x, \quad x(t_0) = x_0$$

Solution formula for specific σ

x solve Σ_σ $\iff$ $x(t) = e^{A_{\sigma_k}(t-t_k)} e^{A_{\sigma_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{\sigma_0}(t_1 - t_0)} x_0 =: \Phi_\sigma(t) x_0$

$\rightsquigarrow$ time-varying linear system

$\rightsquigarrow$ solution set is finite-dimensional linear subspace of $\mathcal{C}^1_{\text{pw}}([t_0, \infty) \to \mathbb{R}^n)$

Nonlinear solution set

Solution set for arbitrary σ is non-linear!

x_1 solves $\dot{x}_1 = A_{\sigma_1} x_1$ and x_2 solves $\dot{x}_2 = A_{\sigma_2} x_2$ $\quad \not\Rightarrow \quad$ $x = x_1 + x_2$ solves any $\dot{x} = A_\sigma x$

BUT: Set of solution operators has semigroup property ...

Semigroup property

Definition

A family of sets $(\mathcal{S}_t)_{t\geq 0}$ has the semigroup property $:\iff \forall\, s,t \geq 0 : \mathcal{S}_s \cdot \mathcal{S}_t = \mathcal{S}_{s+t}$

Here $\mathcal{S}_s \cdot \mathcal{S}_t := \{\Phi_s \cdot \Phi_t \mid \Phi_s \in \mathcal{S}_s, \Phi_t \in \mathcal{S}_t\} \subseteq \mathbb{R}^{n\times n}$

Semigroups for non-switched and switched systems

- Non-switched case: $\mathcal{S}_t = \mathcal{S}_t(A) := \{e^{At}\}$ has semigroup property
- Switched case:

$$\mathcal{S}_t = \mathcal{S}_t(A_1, A_2, \ldots, A_\mathrm{p}) := \left\{ e^{A_{\sigma_k}\tau_k} e^{A_{\sigma_{k-1}}\tau_{k-1}} \cdots e^{A_{\sigma_0}\tau_0} \;\middle|\; \begin{array}{l} k \geq 0,\ \tau_i \geq 0, \\ \sigma_i \in \{1,2,\ldots,\mathrm{p}\}, \\ t = \sum_{i=0}^k \tau_i \end{array} \right\}$$

Key property

x solve switched system $\iff x(t) = \Phi_{t-t_0} \cdot x_0$ with $\Phi_{t-t_0} \in \mathcal{S}_{t-t_0}$

Illustration ...

Lyapunov exponent

In the following let $\mathcal{S} := \bigcup_{t\geq 0} \mathcal{S}_t$.

Definition (Lyapunov exponent of semigroup)

Exponential growth bound: $\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t\in\mathcal{S}_t} \frac{\ln\|\Phi_t\|}{t}$

Lyapunov exponent: $\lambda(\mathcal{S}) := \limsup_{t\to\infty} \lambda_t(\mathcal{S}_t)$

Some observations (for $t_0 = 0$):

- $\|x(t)\| = \|\Phi_t x_0\| \leq \|\Phi_t\|\,\|x_0\| = e^{\ln\|\Phi_t\|}\|x_0\| \leq e^{\lambda_t(\mathcal{S}_t)\cdot t}\|x_0\|$
- For $\dot{x} = A_\sigma x$ with finitely many modes, both $\lambda_t(\mathcal{S}_t)$ and $\lambda(\mathcal{S})$ are finite
- $\lambda_t(\mathcal{S}_t)$ depends on chosen norm, $\lambda(\mathcal{S})$ does not
- In general $\|x(t)\| \not\leq e^{\lambda(\mathcal{S})t}\|x_0\|$, but $\forall\varepsilon>0\,\exists M_\varepsilon\geq 1 : \|x(t)\| \leq M_\varepsilon e^{(\lambda(\mathcal{S})+\varepsilon)t}\|x_0\|$
- For $\mathcal{S}_t = \{e^{At}\}$, $\lambda(A) := \lambda(\mathcal{S}) = \max\Re(\sigma(A))$ (maximal real part of all eigenvalues of A)
- For $\dot{x} = A_\sigma x$, $\lambda(\mathcal{S}) \geq \max_i \lambda(A_i)$ with strict inequality in general
- For $\dot{x} = A_\sigma x$ with commuting A_i, $\lambda(\mathcal{S}) = \max_i \lambda(A_i)$

Lyapunov exponent and stability

Corollary

$\lambda(\mathcal{S}) < 0 \quad \Longleftrightarrow \quad$ all solutions of $\dot{x} = A_\sigma x$ with arbitrary σ exponentially converge to zero

$\lambda(\mathcal{S}) > 0 \quad \Longleftrightarrow \quad \exists x_0\ \exists \sigma$ (periodic) : $x(t) \to \infty$ exponentially

Shifted systems

Shifted system, $\delta \in \mathbb{R}$:

$$\dot{x} = (A_\sigma - \delta I)x$$

with corresponding semigroup $\mathcal{S}^\delta$

Lemma

$\lambda(\mathcal{S}^\delta) = \lambda(\mathcal{S}) - \delta$

Important consequence when comparing Lyapunov exponents of two (shifted) systems

$\dot{x} = A_\sigma x$	with semigroup $\mathcal{S}$	$\dot{x} = \overline{A}_\sigma x$	with semigroup $\overline{\mathcal{S}}$
$\dot{x} = (A_\sigma - \delta I)x$	with $\mathcal{S}^\delta$	$\dot{x} = (\overline{A}_\sigma - \delta I)x$	with $\overline{\mathcal{S}}^\delta$

Corollary

$\lambda(\mathcal{S}) > \lambda(\overline{\mathcal{S}}) \iff \exists \delta \in \mathbb{R} : \lambda(\mathcal{S}^\delta) > 0 > \lambda(\overline{\mathcal{S}}^\delta)$

In other words: two systems have different Lyapunov exponents
$\iff$ $\exists$ shift δ s.t. one system is exponentially stable and the other is unstable

Content overview

Introduction: Switched Systems and Lyapunov exponent

Singular perturbed systems

Switched DAEs and structurally aligned singular perturbations

Classical singular perturbed systems

$$\begin{aligned} \dot{x}^s &= A^s x^s + A^{sf} x^f \\ \varepsilon \dot{x}^f &= A^{fs} x^s + A^f x^f \end{aligned} \quad \overset{\varepsilon\to 0}{\leadsto} \quad \dot{x}^s \approx (A^s - A^{sf}(A^f)^{-1}A^{fs})x_s =: A^b x^s$$

Theorem (cf. Tikhonov's Theorem)

Singular perturbed system is stable for sufficiently small ε $\iff$ A^f and A^b are Hurwitz

$\leadsto$ Divide and Conquer / dimension reduction

Boundary layer system as DAE

Instead of $\dot{x}_s = A^b x_s$ consider the boundary layer DAE:

$$\begin{aligned} \dot{x}_s &= A^s x^s + A^{sf} x^f \\ 0 &= A^{fs} x^s + A^f x^f \end{aligned}$$

DAE = differential-algebraic equation

Singular perturbed switched systems

simple extension

$$\begin{aligned} \dot{x}^s &= A^s_\sigma x^s + A^{sf}_\sigma x^f \\ \varepsilon \dot{x}^f &= A^{fs}_\sigma x^s + A^f_\sigma x^f \end{aligned}$$

[Chitour, Haidar, Mason, Sigalotti; Automatica 2023]

more general extension

$$D^\varepsilon_\sigma \dot{x} = A_\sigma x$$

[Chitour, Daafouz, Haidar, Mason, Sigalotti; CDC 2024]

most general extension

$$E^\varepsilon_\sigma \dot{x} = A_\sigma x$$

[Haidar, Chitour, Daafouz, Mason, Sigalotti; arXiv 2026]

In all three cases (under some reasonable assumptions): $\lambda(\mathcal{S}^0) \leq \liminf_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon)$

Inequality can be strict

Already for the simple extension the inequality can be strict, in particular, the boundary layer dynamics may be stable but the singular perturbed system is unstable for arbitrarily small ε!

[Malloci, Daafouz, Iung; CDC 2009]

Destabilizing switching signal depends on ε $\rightsquigarrow$ gap disappears with dwell-time condition

Research question

$$\lambda(\mathcal{S}^0) \stackrel{?}{=} \liminf_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon)$$

Question

Which additional assumptions prevent a gap, i.e. when are the stability behaviors asymptotically equal?

Content overview

Geometric analysis of DAEs

$$E\dot{x} = Ax + Bu \qquad \text{(DAE)}$$

Theorem (Characteristic invariant subspace decomposition)

$\det(sE - A) \not\equiv 0$ *then* $\exists$ *unique decomposition* $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$ *such that:*
x solves (DAE) $\iff$ $x = v \oplus w \in \mathcal{V} \oplus \mathcal{W}$ *and*

$$\begin{aligned}
\text{in } \mathcal{V}: &\quad \dot{v} = A^{\text{diff}} v + B^{\text{diff}} u, &\quad v(0^-) = \Pi x(0^-)\\
\text{in } \mathcal{W}: &\quad E^{\text{imp}} \dot{w} = w + B^{\text{imp}} u, &\quad w(0^-) = (I - \Pi) x(0^-)
\end{aligned}$$

for some suitably defined $A^{\text{diff}}, E^{\text{imp}}, B^{\text{diff}}, B^{\text{imp}}, \Pi$ *with* E^{imp} *nilpotent and*
Π *is projector onto* $\mathcal{V}$ *along* $\mathcal{W}$ *(consistency projector)*

Corollary

$w = -\sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}$ and hence, for $u \equiv 0$, $w(0^+) = 0$

$\leadsto x(0^+) - x(0^-) = v(0^+) + w(0^+) - v(0^-) - w(0^-) = -w(0^-) \in \operatorname{im}(I - \Pi) = \mathcal{W}$

Solution semigroup of switched DAEs

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \qquad \text{(swDAE)}$$

with corresponding A_i^{diff}, E_i^{imp}, Π_i, $i \in \{1,\ldots,\mathtt{p}\}$
regular & index-1 assumption: $\det(sE_i - A_i) \not\equiv 0$ and $E_i^{\text{imp}} = 0$

Theorem (Trenn 2009)

x solves regular & index-1 (swDAE) $\iff$
$$x(t^-) = e^{A^{\text{diff}}_{\sigma_k}(t-t_k)}\Pi_{\sigma_k} e^{A^{\text{diff}}_{\sigma_{k-1}}(t_k - t_{k-1})}\Pi_{\sigma_{k-1}} \cdots e^{A^{\text{diff}}_{\sigma_0}(t_1-t_0)}\Pi_{\sigma_0} x_0$$

Corresponding semigroup $\mathcal{S}_t^0 := \left\{ \prod_{i=0}^k e^{A^{\text{diff}}_{\sigma_i}\tau_i}\Pi_{\sigma_i} \;\middle|\; \begin{array}{l} k>0,\ \tau_i>0, \\ \sigma_i \in \{1,2,\ldots,\mathtt{p}\}, \\ t = \sum_{i=0}^k \tau_i \end{array} \right\}$, $\quad \mathcal{S}_0^0 := \{I\}$

$\rightsquigarrow x(t^-) = \Phi_{t-t_0} x_0$ for some $\Phi_{t-t_0} \in \mathcal{S}^0_{t-t_0}$

Lyapunov exponents for switched DAEs

$$E_\sigma \dot{x} = A_\sigma x, \quad x(0^-) = x_0 \qquad \text{(swDAE)}$$

with corresponding semigroup $\mathcal{S}_t^0 := \left\{ \prod_{i=0}^k e^{A_{\sigma_i}^{\text{diff}} \tau_i} \Pi_{\sigma_i} \;\middle|\; \begin{array}{l} k > 0,\ \tau_i > 0, \\ \sigma_i \in \{1,2,\ldots,\mathrm{p}\}, \\ t = \sum_{i=0}^k \tau_i \end{array} \right\}$

Infinite exponential growth bound

$\lambda_t(\mathcal{S}_t^0) = \pm\infty$ is possible!

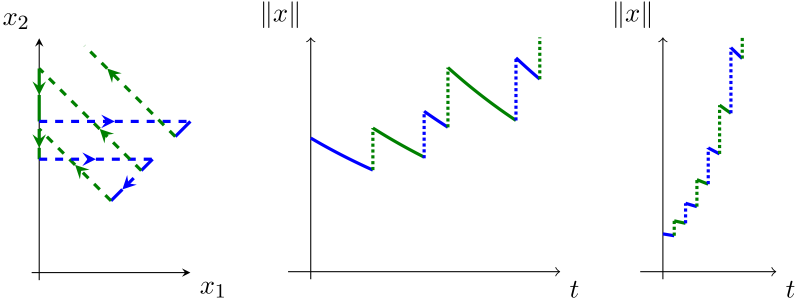
In fact, $\lambda_t(\mathcal{S}_t^0) = -\infty \iff \forall i: \ E_i = 0$ (then $\mathcal{S}_t^0 = \{0\}$)

and $\lambda_t(\mathcal{S}_t^0) < \infty \iff$ the set $\{\Pi_1, \Pi_2, \ldots, \Pi_\mathrm{p}\}$ is product bounded

Infinite exponential growth bound

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}\right) \qquad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$$\Pi_1 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad \Pi_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

What about shifts?

$$E_\sigma^\varepsilon \dot{x} = A_\sigma^\varepsilon x$$

shifted version for $\delta \in \mathbb{R}$:

$$E_\sigma^\varepsilon \dot{x} = (A_\sigma - \delta E_\sigma^\varepsilon) x$$

Lemma (Trenn, Haidar, Mason, Sigalotti; ECC 2026)

$\lambda(\mathcal{S}^{\varepsilon,\delta}) = \lambda(\mathcal{S}^\varepsilon) - \delta$ *for* $\varepsilon > 0$ *and* $\varepsilon = 0$

Consequence

Understanding the gap between $\lambda(\mathcal{S}^0)$ and $\liminf_{\varepsilon \to 0} \lambda(\mathcal{S}^\varepsilon)$ reduces to understanding when stability of switched DAE does not imply stability of singular perturbed system

Structurally aligned singular perturbations (SASP)

Conjecture

A misaligned jump-approximation can lead to Lyapunov exponent gap.

- Singular perturbation analysis: Fast dynamics approximate jump $x(0^-) \to x(0^+)$ which is then followed by ODE-solution in boundary layer.
- Key geometric insight: jump direction $x(0^+) - x(0^-)$ lies in invariant subspace $\mathcal{W} \rightsquigarrow$ dynamics approximating jump should also evolve within $\mathcal{W}$

Definition (cf. Mironchenko, Wirth, Wulff; CDC 2013 & TAC 2015)

A singular perturbed system $E^\varepsilon \dot{x} = Ax$ is structurally aligned with underlying regular index-1 DAE $E^0 \dot{x} = Ax$:⇔

$$A^\varepsilon := (E^\varepsilon)^{-1} A = A^{\text{diff}} + \frac{1}{\varepsilon}(\Pi - I)$$

Key decoupling features

Let $E^\varepsilon \dot{x} = Ax$ with $A^\varepsilon := (E^\varepsilon)^{-1}A$ have structurally aligned singular perturbations, i.e.

$$A^\varepsilon = A^{\mathsf{diff}} + \frac{1}{\varepsilon}(\Pi - I)$$

then

- In decoupling coordinates (according to $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W}$): $A^\varepsilon = \begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon}I \end{bmatrix}$
- $\mathcal{V}$ and $\mathcal{W}$ are A^ε-invariant and $A^\varepsilon\big|_{\mathcal{V}} = A^{\mathsf{diff}}\big|_{\mathcal{V}}$ and $A^\varepsilon\big|_{\mathcal{W}} = \frac{1}{\varepsilon}(\Pi - I)\big|_{\mathcal{W}}$
- $e^{A^\varepsilon t} = \underbrace{e^{A^{\mathsf{diff}}t}}_{\text{slow}} \cdot \underbrace{e^{\frac{1}{\varepsilon}(\Pi - I)t}}_{\text{fast}} = e^{A^{\mathsf{diff}}t}\Pi + e^{\frac{1}{\varepsilon}(\Pi-I)t}(I - \Pi)$
- For $x_0 = v_0 \oplus w_0 \in \mathcal{V} \oplus \mathcal{W}$: $x(t) = \underbrace{e^{A^{\mathsf{diff}}t}v_0}_{\in\mathcal{V}} + \underbrace{e^{\frac{1}{\varepsilon}(\Pi-I)t}w_0}_{\in\mathcal{W}}$

Commutativity and SASP

Definition (cf. Liberzon, Trenn, Wirth; CDC 2011)

A switched DAE $E_\sigma \dot{x} = A_\sigma x$ is called commutative $:\Leftrightarrow \forall \Phi_t \in \mathcal{S}_t^0, \Phi_s \in \mathcal{S}_s^0 : \quad \Phi_t \cdot \Phi_s = \Phi_s \cdot \Phi_t$

It can be shown that for invertible A_i, commutativity is equivalent to $A_i^{\text{diff}} A_j^{\text{diff}} = A_j^{\text{diff}} A_i^{\text{diff}}$.

Theorem (Trenn, Haidar, Mason, Sigalotti; ECC 2026)

Consider singular perturbed switched system $E_\sigma^\varepsilon \dot{x} = A_\sigma x$ and assume

- *each $(E_i, A_i) := (E_i^0, A_i)$ is regular and index-1,*
- $E_\sigma \dot{x} = A_\sigma x$ *is commuting,*
- *the singular perturbation is structurally aligned,*

then $\lambda(\mathcal{S}^0) = \lim_{\varepsilon \to \infty} \lambda(\mathcal{S}^\varepsilon)$

Attention: Commutativity (and regular, index-1) not sufficient (counter example exists)

Open problem: Commutativity is most likely too strong, what is the necessary property?

Summary

- For non-switched systems: Stability properties of singular perturbed system $\Longleftrightarrow$ stability property of boundary system (or DAE)
- For switched systems: In general gap between stability properties of

$$E_\sigma^\varepsilon \dot{x} = A_\sigma x \qquad \text{and} \qquad E_\sigma \dot{x} = A_\sigma x$$

- Regular DAEs have intrinsic subspace decoupling $\mathbb{R} = \mathcal{V} \oplus \mathcal{W}$
- Singular perturbations should be structurally aligned to this decoupling
- For commuting switched DAEs, structural alignment indeed closes the stability gap
- Open problem: Is structural alignment alone already sufficient for zero gap?