On the equivalence of Lyapunov exponents for switched DAEs and corresponding singular perturbed system

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***Abstract*— We are interested in the relationship between the stability properties of a singularly perturbed switched system and those of the corresponding limiting switched system. In some recent work, it has been shown that for a large class of singularly perturbed systems the worst case exponential growth rate for sufficiently small perturbation parameters is lower bounded by the growth rate of the limiting switched system. However, examples show that there could be a positive gap between these bounds. Here we want to investigate this gap further by first observing that the limiting switching system is in fact a switched differential-algebraic equation (switched DAE) for which numerous stability results are already available. Based on the underlying geometric structure of the switched DAE we introduce the concept of structurally aligned singular perturbations and show for the commuting case that indeed there is no gap. We also provide an example which has a commuting limiting switched DAE, but for which the singular perturbations are not structurally aligned and there is indeed a gap between the growth bounds.**

I. Introduction

We consider singular perturbed switched systems of the form

$$\Sigma^\varepsilon : \quad E^\varepsilon_\sigma \dot{x} = A_\sigma x \tag{1}$$

where $\varepsilon > 0$ is a small parameter corresponding to the presence of two time-scales (fast and slow dynamics), $\sigma : [0,\infty) \to \mathcal{M}$ is a piecewise-constant switching signal with values in the finite mode-set $\mathcal{M}$ and having only finitely many discontinuities in each finite time-interval. We assume that for each $p \in \mathcal{M}$ and $\varepsilon > 0$ the matrix $E^\varepsilon_p \in \mathbb{R}^{n\times n}$ is invertible, consequently for each $x_0 \in \mathbb{R}^n$, $\varepsilon > 0$, and switching signal σ there exists a unique (Carathéodory) solution x^ε_σ of (1) with $x^\varepsilon_\sigma(0) = x_0$. Stability issues of singularly perturbed hybrid or switched systems have been extensively studied in the literature, especially in recent years, see e.g. [1], [8], [11], [12], [13], [14], [15], [16], [19], [20]. We are here interested in the question on how stability properties of (1) are related to those of the switched differential-algebraic equation (DAE)

$$\Sigma^0 : \quad E_\sigma \dot{x} = A_\sigma x \tag{2}$$

which is obtained from (1) by letting $\varepsilon \to 0$, i.e. for each $p \in \mathcal{M}$ we have

$$E_p := \lim_{\varepsilon\to 0} E^\varepsilon_p,$$

assuming, in particular, that such limit exists. Here we consider stability in the sense of the so-called Lyapunov exponent which is the worst case exponential growth rate under all possible switching signals. In the recent contribution [3] a sub-class of (1) was considered where $A_p = A$ for some mode-independent $A \in \mathbb{R}^{n\times n}$ and E^ε_p are diagonal matrices with diagonal entries either one or ε. Under some further standard assumptions it was shown there that the limit of the Lyapunov exponent of (1) for $\varepsilon \to 0$ is always lower bounded by the Lyapunov exponent of the corresponding switched DAE (2). In the recently submitted manuscript [6], the system class of switched singular perturbed systems was significantly extended, allowing for state jumps and dropping the assumption that A_p is mode-independent.

In this note we want to understand better when there is no gap between the two Lyapunov exponents, i.e. when the worst case growth rate of the singular perturbed system (as $\varepsilon \to 0$) is equal to the worst case growth rate of the limiting switched DAE. In general there is a positive gap, which is illustrated by the example in [4, Section 6], which has a uniformly exponentially stable limit switched DAE (i.e. the Lyapunov exponent is negative), whereas for each $\varepsilon > 0$ the singular perturbed switched system has unbounded solutions (i.e. the Lyapunov exponent is non-negative).

In [10] a similar setup was considered, with the goal of understanding when stabilizability of the switched DAE (2) can be concluded from the stabilizability of a specific relaxed singular pertubation based on the quasi-Weierstrass form. Since this relaxation preserves the already existing fast (i.e. jumping) and slow dynamics of the DAE, we conjecture that in general this choice of singular perturbation will result in a zero gap between the corresponding Lyapunov exponents. While at the present stage, we have not been able to prove this conjecture in general, we will highlight a special case where equality is guaranteed.

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II. Preliminaries

A. Regularity and impulse-freeness of matrix pairs

Definition 1: A matrix pair $(E, A) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n}$ is called *regular* if the polynomial $\det(sE - A) \in \mathbb{R}[s]$ is not identically zero.

It is well known (see e.g. [2]) that (E, A) is regular if, and only if, there exists coordinate transformations $S, T \in \mathbb{R}^{n\times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{3}$$

where $J \in \mathbb{R}^{\ell\times\ell}$, $0 \le \ell \le n$ is some matrix and $N \in \mathbb{R}^{(n-\ell)\times(n-\ell)}$ is a nilpotent matrix (i.e. there is $1 \le \nu \le n - \ell$ such that $N^\nu = 0$). Following [2] we call (3) a quasi-Weierstrass form (QWF) of (E, A). Furthermore, the *index* of a regular matrix pair (E, A) with QWF (3) is defined as the minimal number ν such that $N^\nu = 0$ (if $\ell = 0$, then the index is defined to be zero). The index-one case is of special importance and from the QWF it can easily be seen that index-one can be characterized by the condition

$$\deg\det(sE - A) = \operatorname{rank} E.$$

The index-one property is also called *impulse-freeness* in the literature.

B. Solution theory for switched DAEs

It can be shown that the switched DAE (2) has unique solutions for all switching signals and all initial values, if each matrix pair (E_p, A_p), $p \in \mathcal{M}$, is regular [17]. In general, the solution space has to be extended to allow for jumps as well as Dirac impulses in the solution. However, if each matrix pair (E_p, A_p) is index-one, then solutions exists in the space of piecewise-smooth functions (with possible jumps at the switching instants). In that case, for any switching signal σ with switching times $t_1, t_2, \ldots$, with corresponding mode sequence $\sigma_i := \sigma(t_i^+)$, $i = 0, 1, 2, \ldots$, the unique solution of (2) with initial condition $x(t_0^-) = x_0$ is given by, for $t \in [t_k, t_{k+1})$,

$$x(t) = \mathrm{e}^{A^{\text{diff}}_{\sigma_k}(t-t_k)}\Pi_{\sigma_k} \cdot \mathrm{e}^{A^{\text{diff}}_{\sigma_{k-1}}(t_k-t_{k-1})}\Pi_{\sigma_{k-1}} \cdots \cdots \mathrm{e}^{A^{\text{diff}}_{\sigma_0}(t_1-t_0)}\Pi_{\sigma_0} x_0,$$

where A^{diff}_p and Π_p are defined in terms of the corresponding QWF for mode $p \in \mathcal{M}$ as

$$A^{\text{diff}}_p = T_p \begin{bmatrix} J_p & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}, \quad \Pi_p = T_p \begin{bmatrix} I_{\ell_p} & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1}.$$

The matrices A^{diff}_p are called *flow matrices* and Π_p are called *consistency projectors*. For later use let

$$\mathcal{P} := \left\{ (A^{\text{diff}}_p, \Pi_p) \;\middle|\; p \in \mathcal{M} \right\}.$$

Note that for all non-switching time $t \ge t_0$ it holds that $x(t^+) = x(t^-) = x(t) = \Pi_{\sigma(t)} x(t)$ and for switching times $t = t_k$ it holds that $x(t_k^+) = x(t_k) = \Pi_{\sigma_k} x(t_k^-)$. The above solution formula is given for one specific switching signal, however, it is possible to characterize the solution set of the switched DAE (2) by the following family of sets $\{\mathcal{S}_t\}_{t\ge0}$, cf. [18],

$$\mathcal{S}^0_t := \left\{ \prod_{j=1}^{k} \mathrm{e}^{A^{\text{diff}}_j \tau_j} \Pi_j \;\middle|\; \begin{array}{l} k \in \mathbb{N}\setminus\{0\},\ (A^{\text{diff}}_j, \Pi_j) \in \mathcal{P}, \\ \tau_j > 0,\ \sum_{j=1}^k \tau_j = t \end{array} \right\}$$

and $\mathcal{S}^0_0 := \{I\}$. Note that we have to exclude zero duration mode-lengths (i.e. $\tau_j = 0$ for some j), because that would allow double jumps $\Pi_{j+1}\Pi_j$, which cannot be produced by the switched DAE (2). Taking this limitation into account, it is now easy to see that x is a solution of the (regular and index-1) switched DAE (2) for some σ if, and only if, for all $t \ge t_0$ there is $\Phi_{t-t_0} \in \mathcal{S}^0_{t-t_0}$ such that

$$x(t^-) = \Phi_{t-t_0} x_0.$$

Furthermore, $\{\mathcal{S}^0_t\}_{t\ge0}$ satisfies the semi-group property $\mathcal{S}^0_{t_1+t_2} = \mathcal{S}^0_{t_2}\mathcal{S}^0_{t_1} := \left\{ \Phi_{t_2}\Phi_{t_1} \;\middle|\; \Phi_{t_1} \in \mathcal{S}^0_{t_1}, \Phi_{t_2} \in \mathcal{S}^0_{t_2} \right\}$ for all $t_1, t_2 \ge 0$.

C. Lyapunov exponent of a semi-group

For the switched system (1) we can also define a semi-group characterizing the solution set of the switched system as follows:

$$\mathcal{S}^\varepsilon_t := \left\{ \prod_{j=1}^{k} \mathrm{e}^{A^\varepsilon_{p_j} \tau_j} \;\middle|\; \begin{array}{l} k\in\mathbb{N},\ A^\varepsilon_{p_j} \in \mathcal{A}^\varepsilon,\ j=1,\ldots,k, \\ \tau_j \ge 0,\ \sum_{j=1}^k \tau_j = t \end{array} \right\},$$

where $A^\varepsilon_p := (E^\varepsilon_p)^{-1} A_p$ and $\mathcal{A}^\varepsilon := \left\{ A^\varepsilon_p \;\middle|\; p \in \mathcal{M} \right\}$. To characterize the stability properties of both the singular perturbed switched system and the corresponding switched DAE we introduce the exponential growth bound and the Lyapunov exponent of a general semi-group as follows.

Definition 2: For $\mathcal{S} := \bigcup_{t\ge0} \mathcal{S}_t$, with either $\mathcal{S}_t = \mathcal{S}^\varepsilon_t$ or $\mathcal{S}_t = \mathcal{S}^0_t$, let for every $t \ge 0$ the *exponential growth bound* be defined as

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t\in\mathcal{S}_t} \frac{\ln\|\Phi_t\|}{t}.$$

The *Lyapunov exponent* of $\mathcal{S}$ is then defined as

$$\lambda(\mathcal{S}) = \limsup_{t\to\infty} \lambda_t(\mathcal{S}_t).$$

At this point it is not excluded that $\lambda_t(\mathcal{S}_t) = \pm\infty$, but in any case, it holds by definition that for all $\Phi_t \in \mathcal{S}_t$

$$\|\Phi_t x_0\| \le \|\Phi_t\|\|x_0\| \le \mathrm{e}^{\ln\|\Phi_t\|}\|x_0\| \le \mathrm{e}^{\lambda_t(\mathcal{S}_t)t}\|x_0\|,$$

justifying the name “exponential growth bound”.

Note that the exponential growth bound depends on the specifically chosen norm on $\mathbb{R}^n$, whereas, the Lyapunov exponent does *not* depend on the norm.

For a finite mode set $\mathcal{M}$ and a fixed $\varepsilon > 0$ it is well known (see e.g. [5] or [7]) that $\lambda_t(\mathcal{S}^\varepsilon_t)$ and the corresponding Lyapunov exponent $\lambda(\mathcal{S}^\varepsilon)$ are finite. However, for switched DAEs this is not always the case: $\lambda_t(\mathcal{S}^0_t) = -\infty$, if, and only if, all consistency projectors are zero (which is the case if all E matrices are zero), because then clearly the only solution of the switched DAE is the zero solution for any (non-zero) initial value and every solution can be bounded by $\mathsf{e}^{-\infty t}\|x_0\| = 0$. On the other end of the spectrum $\lambda_t(\mathcal{S}_t^0) = \infty$ if (and only if) the set of consistency projectors is not *product bounded*, i.e. there is a sequence of consistency projectors, whose product grows unbounded, cf. [18].

The following result shows that we can "shift" a (finite) exponential growth bound (and hence also the Lyapunov exponent) by an arbitrary amount. To formulate this result we introduce the shifted versions of (1) and (2) as follows: given $\delta \in \mathbb{R}$, let

$$\begin{aligned} \Sigma_\delta^\varepsilon : \quad & E_\sigma^\varepsilon \dot{x} = (A_\sigma - \delta E_\sigma^\varepsilon)x, \\ \Sigma_\delta^0 : \quad & E_\sigma \dot{x} = (A_\sigma - \delta E_\sigma)x. \end{aligned}$$

It is easily seen that the corresponding shifted matrices in $\mathcal{A}^\varepsilon$ and $\mathcal{P}$ are given by $A_{p,\delta}^\varepsilon = (E_p^\varepsilon)^{-1}(A_p - \delta E_p^\varepsilon) = A_p^\varepsilon - \delta I$ and $A_{p,\delta}^{\text{diff}} = T_p \left[\begin{smallmatrix} J_p - \delta I & 0 \\ 0 & 0 \end{smallmatrix}\right] T_p^{-1}$, whereas the consistency projectors remain unchanged.

Lemma 3: Consider the semi-groups $\mathcal{S}_t^\varepsilon$ and $\mathcal{S}_t^0$ associated with the switched systems Σ^ε and Σ^0 together with the shifted semi-groups $\mathcal{S}_{t,\delta}^\varepsilon$ and $\mathcal{S}_{t,\delta}^0$ associated with $\Sigma_\delta^\varepsilon$ and Σ_δ^0. Assume that all exponential growth bounds $\lambda_t(\mathcal{S}_t^\varepsilon)$ and $\lambda_t(\mathcal{S}_t^0)$ are bounded. Then

$$\lambda_t(\mathcal{S}_{t,\delta}^\varepsilon) = \lambda_t(\mathcal{S}_t^\varepsilon) - \delta \quad \text{and} \quad \lambda_t(\mathcal{S}_{t,\delta}^0) = \lambda_t(\mathcal{S}_t^0) - \delta.$$

Consequently, the Lyapunov exponents $\lambda(\mathcal{S}^\varepsilon)$ and $\lambda(\mathcal{S}^0)$ then also satisfy

$$\lambda(\mathcal{S}_\delta^\varepsilon) = \lambda(\mathcal{S}^\varepsilon) - \delta \quad \text{and} \quad \lambda(\mathcal{S}_\delta^0) = \lambda(\mathcal{S}^0) - \delta.$$

Proof: Since $A_{p,\delta}^\varepsilon = A_p^\varepsilon - \delta I$ it is clear that for any switching signal σ we have that x is a solution of Σ^ε if, and only if, x^δ given by $x^\delta(t) = \mathsf{e}^{-\delta t}x(t)$ is a solution of $\Sigma_\delta^\varepsilon$. Consequently, $\Phi_t^\varepsilon \in \mathcal{S}_t^\varepsilon$ if, and only if, $\mathsf{e}^{-\delta t}\Phi_t^\varepsilon \in \mathcal{S}_{t,\delta}^\varepsilon$. From this it immediately follows that $\lambda_t(\mathcal{S}_{t,\delta}^\varepsilon) = \lambda_t(\mathcal{S}_t^\varepsilon) - \delta$. For the switched DAE case, we observe that

$$\mathsf{e}^{A_{p,\delta}^{\text{diff}} t}\Pi_p = \mathsf{e}^{(A_p^{\text{diff}} - \delta I)t}\Pi_p = \mathsf{e}^{-\delta t}\mathsf{e}^{A_p^{\text{diff}} t}\Pi_p.$$

Consequently, $\Phi_t^0 \in \mathcal{S}_t^0$ if, and only if, $\mathsf{e}^{-\delta t}\Phi_t^0 \in \mathcal{S}_{t,\delta}^0$ and hence $\lambda_t(\mathcal{S}_{t,\delta}^0) = \lambda_t(\mathcal{S}_t^0) - \delta$. ∎

Let us call the switched systems Σ^ε and Σ^0 *uniformly exponentially stable* if the corresponding Lyapunov exponents $\lambda(\mathcal{S}^\varepsilon)$ and $\lambda(\mathcal{S}^0)$ are finite and negative. We next show that asymptotic equality between the Lyapunov exponents of Σ^ε (for $\varepsilon \to 0$) and the Lyapunov exponent of the limit switched DAE Σ^0 can be reduced to a suitable stability implication.

Towards this goal, we first recall the following recent result from [6]:

Lemma 4 ([6, Case 2 of Thm. 5]): For Σ^ε assume that there exist invertible S_p, T_p such that $S_p E_p^\varepsilon T_p = \left[\begin{smallmatrix} I_{\ell_p} & 0 \\ 0 & -\varepsilon I_{n-\ell_p} \end{smallmatrix}\right]$ and $S_p A_p T_p = \left[\begin{smallmatrix} * & * \\ C_p & D_p \end{smallmatrix}\right]$ with $-D_p$ being Hurwitz; furthermore, for Σ^0 assume that the set of consistency projectors is product bounded[1]. Then $\liminf_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) \geq \lambda(\mathcal{S}^0)$.

Furthermore, we can utilize Lemma 3 to establish a characterization of the opposite inequality:

Lemma 5: The inequality $\limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) \leq \lambda(\mathcal{S}^0)$ holds if, and only if, for all $\delta \in \mathbb{R}$ the following implication holds

$$\lambda(\mathcal{S}_\delta^0) < 0 \implies \limsup_{\varepsilon\to 0} \lambda(\mathcal{S}_\delta^\varepsilon) < 0, \tag{4}$$

or, in other words, uniform exponential stability of the (shifted) switched DAE implies uniform exponential stability of the (shifted) singular perturbed system for sufficiently small $\varepsilon > 0$.

Proof: In view of Lemma 3 we have that $\lambda(\mathcal{S}_\delta^0) < 0$ if, and only if, $\delta > \lambda(\mathcal{S}^0)$. Hence if $\lambda(\mathcal{S}^0) \geq \limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon)$ and $\delta > \lambda(\mathcal{S}^0)$, then

$$\limsup_{\varepsilon\to 0} \lambda(\mathcal{S}_\delta^\varepsilon) \overset{\text{Lem. 3}}{=} \limsup_{\varepsilon\to 0} (\lambda(\mathcal{S}^\varepsilon) - \delta) \leq \lambda(\mathcal{S}^0) - \delta < 0,$$

which results in the desired implication (4). Conversely, assume that implication (4) is true for all $\delta \in \mathbb{R}$. Consider a monotonically decreasing sequence $(\delta_n)_{n\in\mathbb{N}}$ with $\delta_n > \lambda(\mathcal{S}^0)$ and $\lim_{n\to\infty} \delta_n = \lambda(\mathcal{S}^0)$. Then, due to Lemma 3, $\lambda(\mathcal{S}_{\delta_n}^0) = \lambda(\mathcal{S}^0) - \delta_n < 0$ and hence (4) yields that $0 > \limsup_{\varepsilon\to 0}(\mathcal{S}_{\delta_n}^\varepsilon) = \limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) - \delta_n$ for all $n \in \mathbb{N}$. Hence

$$0 \geq \limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) - \lim_{n\to\infty} \delta_n = \limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) - \lambda(\mathcal{S}^0).$$

This is the required inequality. ∎

Altogether we arrive at the following corollary:

Corollary 6: Consider Σ^ε and Σ^0 as in Lemma 4. Then $\limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) = \lambda(\mathcal{S}^0)$ if, and only if, the implication (4) holds. In that case $\limsup_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) = \liminf_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) = \lim_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon)$.

The interest of Corollary 6 is that it allows to simplify the problem of comparing Lyapunov exponents (which are in general quite hard to calculate precisely) to a standard stability test, which e.g. can be done with suitable Lyapunov functions. In particular, the existence of a gap between the asymptotic Lyapunov exponent of the singular perturbed system and the Lyapunov exponent of the corresponding singular system can be shown with a simple counterexample which for $\varepsilon > 0$ is unstable, but the singular system is exponentially stable. Conversely, if for a certain class of systems the implication (4) can be shown (e.g. with a suitable Lyapunov function) then it can be concluded that for this system class there is no structural gap between the two Lyapunov exponents.

[1] Note that the matrices S_p and T_p in Lemma 4 in general do not lead to a QWF (3), but it is easily seen that the consistency projectors are given by $T_p \left[\begin{smallmatrix} I & 0 \\ -D_p^{-1}C_p & 0 \end{smallmatrix}\right] T_p^{-1}$ in agreement with the jump maps in $\overline{\mathcal{R}}$ in [6].

III. Structurally aligned singular perturbation

While for non-switched singular perturbed systems the direction of convergence towards the singular manifold usually does not play a big role (as long as it does not diverge), for switched systems the subspace within which the fast dynamics evolve plays a significant role for the overall convergence. At the same time, for switched DAEs there is a unique jump space in the form of the second Wong limit (which for the index-one case is simply the null-space of the corresponding E-matrix), cf. [17].

Inspired by the singular approximation of the jumping behavior in switched DAEs in [10] we consider the following class of singular perturbed systems.

Definition 7 (Structurally aligned singular perturbation): Consider a singular perturbed (non-switched) system $E^\varepsilon \dot{x} = Ax$ and assume that the corresponding DAE (E^0, A) is regular and index-one with corresponding consistency projector Π and flow matrix A^{diff}. The singular perturbation is called *structurally aligned* if there exists a scalar function $\kappa : (0, +\infty) \to (0, +\infty)$ with $\lim_{\varepsilon\to 0} \kappa(\varepsilon) = \infty$ such that

$$A^\varepsilon := (E^\varepsilon)^{-1}A = A^{\text{diff}} + \kappa(\varepsilon)(\Pi - I).$$

Note that based on the QWF (3) the matrix A^ε in [10] was chosen as $A^\varepsilon = T \begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon}I \end{bmatrix} T^{-1} = A^{\text{diff}} + \frac{1}{\varepsilon}(\Pi - I)$, i.e. it is structurally aligned in the sense of Definition 7 with $\kappa(\varepsilon) = 1/\varepsilon$. Furthermore, $A^\varepsilon = (E^\varepsilon)^{-1}A$ with $E^\varepsilon = S^{-1} \begin{bmatrix} I & 0 \\ 0 & -\varepsilon I \end{bmatrix} T^{-1}$.

Remark 8 (Decoupling of fast and slow dynamics): A key feature of structurally aligned perturbation is the fact that the fast and slow dynamics are completely decoupled:

$$\mathsf{e}^{A^\varepsilon t} = \mathsf{e}^{A^{\text{diff}}t}\mathsf{e}^{\kappa(\varepsilon)(\Pi - I)t} = \mathsf{e}^{\kappa(\varepsilon)(\Pi - I)t}\mathsf{e}^{A^{\text{diff}}t},$$

which is a consequence of the commutativity property $A^{\text{diff}}\Pi = \Pi A^{\text{diff}} (= A^{\text{diff}})$. Furthermore, the dynamics of $\dot{x} = A^\varepsilon x$ are aligned with respect to the decomposition $\mathbb{R}^n = \mathcal{V} \oplus \mathcal{W} := \operatorname{im}\Pi \oplus \ker \Pi$ in the sense that

$$A^\varepsilon\big|_{\mathcal{V}} = A^{\text{diff}}\big|_{\mathcal{V}} \quad \text{and } A^\varepsilon\big|_{\mathcal{W}} = \kappa(\varepsilon)(\Pi - I)\big|_{\mathcal{W}}$$

and for $x = \Pi x + (I - \Pi)x =: v + w$ with $v \in \mathcal{V}$ and $w \in \mathcal{W}$ we have

$$A^\varepsilon x = A^{\text{diff}}v - \kappa(\varepsilon)w.$$

The lack of decoupling of the slow and fast dynamics in a singular perturbed *switched* system seems to be one key reason for the gap between the asymptotic Lyapunov exponent of the singular perturbed switched system Σ^ε and the Lyapunov exponent of the corresponding switched DAE Σ^0; this viewpoint is supported by [4, Section 6], which provides an example with a non-zero gap in which the singular perturbed modes are *not* structurally aligned.

We believe that having structurally aligned singular perturbations is strongly related to a sufficient condition for asymptotic equality of the Lyapunov exponents of Σ^ε and Σ^0. In fact, we will show in the following that for a certain commuting case, this belief is indeed true.

IV. The commuting case

Definition 9 (cf. [9]): We call the switched DAE Σ^0 or the switched singular perturbed system Σ^ε *commuting*, if the elements of the corresponding semigroup $\mathcal{S}^0$ or $\mathcal{S}^\varepsilon$ are commuting.

While for the switched system Σ^ε commutativity is equivalent to commutativity of the matrices $A_p^\varepsilon = (E_p^\varepsilon)^{-1}A_p$, the commutativity property of Σ^0 cannot be directly read off from the commutativity properties of the system matrices E_p and A_p. Instead, the flow matrices A_p^{diff} have to be considered:

Lemma 10 ([9, Thm. 7]): Consider the regular, index-one switched DAE Σ^0 with A_p invertible and corresponding flow matrices A_p^{diff}. Then Σ^0 is commuting if, and only if, the flow matrices A_p^{diff} commute with each other.

Note that the invertibility assumption for A_p is not restrictive: Lemma 3 allows us to shift the system without altering the qualitative relationship between the Lyapunov exponents. Furthermore, we will focus on utilizing Corollary 6 and assuming that the switched DAE Σ^0 is exponentially stable implies invertibility of each A_p matrix (the matrices J_p in the QWF (3) have to be Hurwitz, which then implies that $A_p = S_p^{-1} \begin{bmatrix} J_p & 0 \\ 0 & I \end{bmatrix} T_p^{-1}$ are also invertible).

Furthermore, note that from [9, Lem. 9] it also follows that for a commuting switched DAE Σ^0 not only the A^{diff}-matrices are commuting, but additionally the consistency projectors are commuting with each other and also with the other A^{diff}-matrices. Consequently, we arrive at the following result:

Corollary 11: Consider the switched system Σ^ε and assume that the singular perturbations are structurally aligned for each mode. Furthermore, assume that the underlying switched DAE Σ^0 is commutative. Then the singular perturbed switched system Σ^ε is also commutative.

We now provide the main result of this paper:

Theorem 12: Assume that Σ^0 is commutative with a finite mode set $\mathcal{M}$ and that the singular perturbations leading to Σ^ε are structurally aligned with[2] $\kappa(\varepsilon) = \frac{1}{\varepsilon}$. Then the Lyapunov exponents are asymptotically equal, i.e. for the corresponding semi-groups $\mathcal{S}^\varepsilon$ and $\mathcal{S}^0$ we have $\lim_{\varepsilon\to 0} \lambda(\mathcal{S}^\varepsilon) = \lambda(\mathcal{S}^0)$.

Proof: In view of Lemma 3, Corollary 6 and the fact that shifting does not effect commutativity, it suffices to show that exponential stability of Σ^0 implies exponential stability of Σ^ε. Denote by $\Phi_\sigma^\varepsilon(t, t_0) \in \mathcal{S}_{t-t_0}^\varepsilon$ the solution operator for Σ^ε with switching signal σ, i.e. all solutions of (1) with initial value $x_0 \in \mathbb{R}$ and given σ take the form

$$x(t) = \Phi_\sigma^\varepsilon(t, t_0)x_0.$$

[2]The assumption that $\kappa(\varepsilon) = \frac{1}{\varepsilon}$ is only necessary to apply Lemma 4. Although we suspect this restriction to be not essential, verifying this remains an open question for future work.

In view of Corollary 11, we can rearrange the matrix exponential in the solution operator in such a way that all modes are grouped, i.e.

$$\Phi_\sigma^\varepsilon(t,t_0) = \prod_{p\in\mathcal{M}} \mathsf{e}^{A_p^\varepsilon \tau_p^\sigma(t)},$$

where $\tau_p^\sigma(t)$ is the total time duration of mode p in the interval $[t_0,t]$ for the switching signal σ. Furthermore, based on the QWF (3) we have

$$\begin{aligned}\mathsf{e}^{A_p^{\text{diff}}\tau}\Pi &= T_p \begin{bmatrix} \mathsf{e}^{J_p\tau} & 0 \\ 0 & 0 \end{bmatrix} T_p^{-1} \quad \text{and} \\ \mathsf{e}^{A_p^\varepsilon\tau} &= T_p \begin{bmatrix} \mathsf{e}^{J_p\tau} & 0 \\ 0 & \mathsf{e}^{-\tau/\varepsilon} I \end{bmatrix} T_p^{-1} \\ &= \mathsf{e}^{A_p^{\text{diff}}\tau}\Pi_p - \mathsf{e}^{-\tau/\varepsilon}(\Pi_p - I).\end{aligned}$$

By assumption, Σ^0 is exponentially stable, hence there exists $C_p > 0$ and $\lambda_p > 0$ such that

$$\|\mathsf{e}^{A_p^{\text{diff}}\tau}\Pi\| \le C_p \mathsf{e}^{-\lambda_p\tau}.$$

Consequently, we have

$$\|\mathsf{e}^{A_p^\varepsilon\tau}\| \le C_p\mathsf{e}^{-\lambda_p\tau} + P_p\mathsf{e}^{-\tau/\varepsilon},$$

where $P_p = \|\Pi_p - I\|$. Let $\lambda_{\min} := \min_p \lambda_p$ and choose $\varepsilon_{\max} = 1/\lambda_{\min}$. Then

$$\|\mathsf{e}^{A_p^\varepsilon\tau}\| \le C\mathsf{e}^{-\lambda_{\min}\tau} \quad \forall \varepsilon \in (0,\varepsilon_{\max}),$$

where $C := \max_p(C_p + P_p)$. Altogether we then have

$$\|\Phi_\sigma^\varepsilon(t,t_0)\| \le C^{\mathtt{m}}\mathsf{e}^{-\lambda_{\min}(t-t_0)},$$

where $\mathtt{m} := |\mathcal{M}|$ is the number of modes and where we used that $\sum_{p\in\mathcal{M}} \tau_p^\sigma(t) = t - t_0$. This shows that Σ^ε is uniformly exponentially stable for sufficiently small $\varepsilon > 0$. ∎

In the next section we present an example showing that just having a commuting switched DAE Σ^0 is not sufficient for deducing asymptotic equality between the Lyapunov exponents of Σ^ε and Σ^0 if the singular perturbation is *not* structurally aligned.

V. Example of gap in the case of commuting DAEs

Consider the singular perturbed switched system Σ^ε with $\mathcal{M} = \{1,2,3\}$ and

$$E_1^\varepsilon = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \quad A_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & -\omega^2 & -1 \end{bmatrix},$$

$$E_2^\varepsilon = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \quad A_2 = \begin{bmatrix} -1 & 0 & -\omega^2 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix},$$

$$E_3^\varepsilon = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} -1 & 1 & 0 \\ -\omega^2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

for some $\omega > 0$ to be fixed later. It is easily seen that the corresponding switched DAE Σ^0 is regular, index-one, and has flow matrices

$$A_1^{\text{diff}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_2^{\text{diff}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_3^{\text{diff}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

and consistency projectors $\Pi_i = -A_i^{\text{diff}}$, $i = 1,2,3$. In particular, the switched DAE Σ^0 is commuting and exponentially stable (with common quadratic Lyapunov function $V(x) = x^\top x$). Furthermore, it is easily seen that, for any choice of ω, Σ^ε satisfies the D-Hurwitz assumption from [3] (which guarantees the convergence of the flow of each mode of the ε-relaxation to the flow of the corresponding DAE).

Notice that

$$\begin{aligned}&\lim_{\varepsilon\to 0^+} \mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_3^\varepsilon}\mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_2^\varepsilon}\mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_1^\varepsilon} \\ &= \begin{bmatrix} 0 & \frac{\mathsf{e}^{-\frac{\pi}{2\omega}}}{\omega} & 0 \\ -\mathsf{e}^{-\frac{\pi}{2\omega}}\omega & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\mathsf{e}^{-\frac{\pi}{2\omega}}\omega \\ 0 & 1 & 0 \\ \frac{\mathsf{e}^{-\frac{\pi}{2\omega}}}{\omega} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{\mathsf{e}^{-\frac{\pi}{2\omega}}}{\omega} \\ 0 & -\mathsf{e}^{-\frac{\pi}{2\omega}}\omega & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & \mathsf{e}^{-\frac{\pi}{\omega}}\omega^{-2} \\ 0 & -\mathsf{e}^{-\frac{3\pi}{2\omega}}\omega^3 & 0 \\ \mathsf{e}^{-\frac{\pi}{2\omega}}\omega^{-1} & 0 & 0 \end{bmatrix}.\end{aligned}$$

Consider ω large enough so that $\mathsf{e}^{-\frac{3\pi}{2\omega}}\omega^3 > 1$. Then for ε small enough the matrix product $\mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_3^\varepsilon}\mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_2^\varepsilon}\mathsf{e}^{\frac{\varepsilon\pi}{2\omega}A_1^\varepsilon}$ has an eigenvalue of modulus larger than one, which implies that system Σ^ε is unstable for the corresponding periodic switching signal with period $\frac{3\varepsilon\pi}{2\omega}$ (see Figure 1).

VI. Conclusion

We have introduced the concept of structurally aligned singular perturbations which we believe to play an important role for the stability properties of singularly perturbed switched systems. We conjecture that the observed gap between the growth rate of the singularly perturbed switched system and the corresponding growth rate of the limiting system is partially due to a lack of structural alignment of the singular perturbations. We show that for the commuting case indeed the gap disappears for structurally aligned perturbations and a (commuting) example is provided for which non-structurally aligned perturbations lead to a gap. However, it is still an open research question when and how structural alignment of the singular perturbations prevents in general a gap in the growth bounds.

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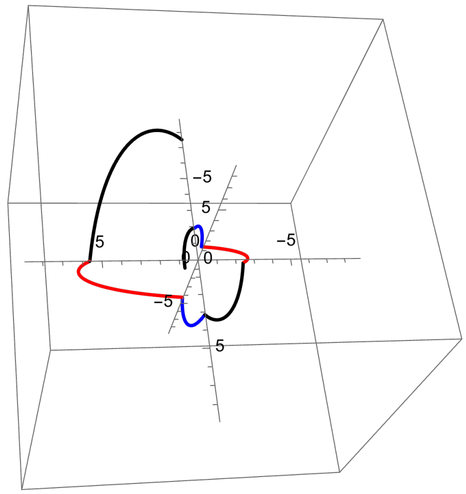
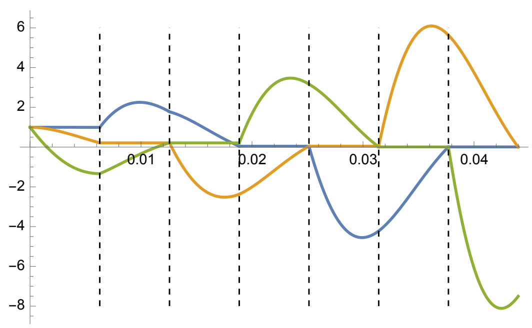


(a)



(b)

Fig. 1: A divergent trajectory of Σ^ε, with $\varepsilon = 10^{-2}$, $\omega = 2.5$, and initial condition $\begin{bmatrix}1 & 1 & 1\end{bmatrix}^\top$ corresponding to the periodic switching law obtained by applying sequentially the modes 1, 2, 3, each for a time $\frac{\varepsilon\pi}{2\omega}$. Subfigure (a) shows the curve in three-dimensional space, while subfigure (b) illustrates the time dependency of the first, second, and third coordinates in blue, orange, and green, respectively. The dashed vertical lines indicate the switching times. In contrast with the case $\varepsilon > 0$, note that the trajectories of Σ^0 either never switch or reach the origin after one switch, independently of the initial condition.

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