ISS for PWA systems

Radu Cimpean, Alef Sterk and Stephan Trenn

***Abstract*— We offer a way to construct an ISS type bound for a PWA system. Starting from an ISS Lyapunov function in each region of the state space (called regional ISS Lyapunov function), and under suitable assumptions, we patch these functions and obtain initially a practical ISS type bound. After that, depending on the input size, we have to consider a distinction between large and small inputs. The large inputs dominate the dynamics and may induce region transitions, while the small inputs are sufficiently attenuated to direct and keep the trajectory within the region containing the equilibrium. The threshold between these regimes plays a key role in determining the point at which the system decays towards the equilibrium. By unifying the analysis across these input regimes, we provide a complete ISS-type bound valid for all admissible inputs. Consequently, the global ISS property is ensured, and further refined analysis can be applied.**

I. Introduction

Stability in the context of nonlinear systems has been a topic that has always received attention, mainly due to the presence of numerous real life situations requiring such a modeling framework. The study of this property becomes necessary in applications where increasing the complexity of the system can lead to unpredictable results, thus the unreliability of the model.

For this paper, the focus will be on a category of nonlinear systems, that belongs to the class of hybrid system, namely piecewise affine (PWA) systems, conceptually expounded in [1] and [2]. These systems are characterized by the existence of several regions where the dynamics have affine behavior and switching between these regions is caused either by a certain piecewise linear characteristic or as a consequence of the approximation of complex nonlinear dynamics [3]. Stability for PWA has been studied in numerous formats [4], [5], [6], [7], [8], [9], [10], [11], in this paper, we will investigate stability results within the input-to-state stability (ISS) framework. Since the introduction of the ISS concept in [12], there has been a lot of research done within this framework [12], [13], [14], [15]. Some work covers ISS for hybrid systems [16], [17], for switched systems [15], [18], [19]; and for PWA systems there are papers covering the discrete case [20], [21], [22] and continuous ones [23]. We emphasize that, in contrast to [23], the role of ISS in our framework is fundamentally different. In [23], ISS is used primarily as an underlying robustness concept that supports the convergence guarantees, rather than serving as the explicit focus of the analysis. In our work, by comparison, the ISS property constitutes the central analytical objective, while the PWA structure provides the essential framework through which regional Lyapunov functions are constructed and patched to establish global stability. The advantage of ISS systems is that it provides a way to asses and certify global attractivity when inputs are being fed to the system and once they are not, it reverts to global asymptotic stability (0-GAS, where the input is zero). Moreover, similarly to the implication that finding a global Lyapunov function means that the system is GAS, there is also a link between the existence of an ISS Lyapunov function and the system being ISS [24]. Moreover, obtaining the Lyapunov candidate can be achieved in several ways, out of which the cone copositive method [6], [4] was of particular interest for previous work [25] and hence elements of it were adapted to fit the ISS Lyapunov function requirements. Furthermore, the switching logic for the system constitutes a relevant feature for this paper. Some prominent cases found in literature use dwell time (DT) [26], [27], [22], [28], persistent dwell time (PDT) [29], [30] and of particular interest for this paper, the average dwell time (ADT) [15], [31], [32], [33]. The aim of this paper is to offer the necessary ingredients to combine ISS Lyapunov functions defined over regions of piecewise affine (PWA) systems and to obtain a global ISS type result. To achieve this outcome, we address two key notions: practical ISS (ISpS) and average dwell time (ADT). ISpS extends the classical ISS framework by incorporating a constant offset term that captures the influence of persistent disturbances or modeling uncertainties [34], [35]. The ADT property ensures that the system spends a finite (possibly brief) duration in each region and prevents Zeno behavior, i.e., infinitely many switches in finite time [15], [36].

The structure of the paper is as follows. Section II introduces the necessary preliminaries on PWA systems, the ISS framework and the ADT definition. In addition, it establishes the concept of regional ISS Lyapunov functions. Section III, outlines the problem statement with five key assumptions and research questions. Section IV contains the main results, split into two parts: IV.A in which an ISpS type bound is obtained by patching

R.Cimpean is with the Bernoulli Institute and Groningen Cognitive Systems and Materials Center (CogniGron), Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands (phone: +31 6 19721054, r.cimpean@rug.nl)

A.Sterk is with the Bernoulli Institute, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands, a.e.sterk@rug.nl

S.Trenn is with the Bernoulli Institute, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands, s.trenn@rug.nl

regional ISS Lyapunov functions and IV.B where the ISpS formulation is analyzed from the input perspective and reduced with suitable reasoning to a global ISS. Finally, conclusions about the current contribution are being offered, together with potential extensions.

Notation

We denote by $|x|$ the Euclidian norm of a vector $x \in \mathbb{R}^n$. For a bounded (measurable) function $u : [t_0,\infty) \to \mathbb{R}^m$ we let $\|u\|_\infty := \operatorname{ess\,sup}_{t\geq t_0} |u(t)|$. We use the well established function classes $\mathcal{K},\mathcal{KL},\mathcal{K}_\infty$, for details see e.g. [37, Def. 3.3, Def. 3.4, Lem. 3.2].

II. Preliminaries

Definition 1. A piecewise affine (PWA) system is of the form

$$\dot{x} = A_s x + B_s u + b_s \quad x \in X_s,\ s = 0,1,\dots N, \tag{1}$$

where $A_s \in \mathbb{R}^{n\times n}$, $B_s \in \mathbb{R}^{n\times m}$, $b_s \in \mathbb{R}^n$ and $X = \bigcup_{s=0}^N X_s \subseteq \mathbb{R}^n$ is a polyhedral partition of the state space X, i.e. each region X_s is a finite intersection of closed half-spaces and each intersection $X_i \cap X_j$ is either empty or equal to a common boundary.

Note that in the above definition there is some non-uniqueness for the derivative of x on the boundaries of the regions. This can be resolved by either assuming that the dynamics are continuous on the boundaries (which we will do here) or by considering solutions in the sense of Filipov (cf. [4]).

Definition 2 (see e.g. [24])**.** Systems of the form

$$\dot{x} = f(x,u) \quad x(t_0) = x_0, \tag{2}$$

with $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ being locally Lipschitz are called input to state stable (ISS), if (2) for all bounded $u : [t_0,\infty) \to \mathbb{R}^m$ (with bound $\|u\|_\infty$) and all $x_0 \in \mathbb{R}^n$ there exists a unique solution $x : [t_0,\infty) \to \mathbb{R}^n$ of (2) and for all $t \geq t_0$ the following holds

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|u\|_\infty). \tag{3}$$

Note that PWA systems (1) which are continuously defined across boundaries can also be written in the form (2), where $f(x,u)$ is defined piecewise, hence ISS is also well defined for those PWA systems.

For checking ISS, the notion of an ISS Lyapunov function plays an important role:

Definition 3. A differentiable function $V : \mathbb{R}^n \to \mathbb{R}$ is called an ISS Lyapunov function for (2) if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \alpha \in \mathcal{K}_\infty$, and $\xi \in \mathcal{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n, \tag{4}$$

and for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, the following inequality holds:

$$\dot{V}(x,u) \leq -\alpha(V(x)) + \xi(|u|), \tag{5}$$

where $\dot{V}(x,u) := \nabla V(x) f(x,u)$.

We call V a *regional* ISS Lyapunov function on $R \subseteq \mathbb{R}^n$ if (4) and (5) only hold for all $x \in R$ instead of all $x \in \mathbb{R}^n$.

According to [24, Th. 2.12], the existence of an ISS Lyapunov function is equivalent to the system being ISS. Note that such an equivalence is *not* true anymore for the regional ISS Lyapunov function, in fact, it is not even required that the region R is positively invariant.

Next, we cover the definition of practical ISS, as in [13] and then subsequently in [34].

Definition 4. System (2) is said to be *input-to-state practically stable (ISpS)* if there exist a function $\beta \in \mathcal{KL}$, a function $\gamma \in \mathcal{K}$, and a nonnegative constant $c \geq 0$ such that all solutions satisfy

$$|x(t)| \leq \beta(|x_0|, t - t_0) + \gamma(\|u\|_\infty) + c \quad \text{for all } t \geq t_0. \tag{6}$$

Lastly, we recall the definition of the concept of average dwell time as in [15].

Definition 5. Let $\sigma : [t_0,\infty) \to \{0,1,\dots,N\}$ be a piecewise-constant switching signal, and let $\bar{t} > \underline{t} > t_0$. Denote by $N_\sigma(\underline{t},\bar{t})$ the number of discontinuities of σ in the interval $[\underline{t},\bar{t})$.

If there exist constants $N_0 \geq 1$, $\tau_a > 0$ such that for all $\bar{t} > \underline{t} \geq t_0$,

$$N_\sigma(\underline{t},\bar{t}) \leq N_0 + \frac{\bar{t} - \underline{t}}{\tau_a}, \tag{7}$$

then τ_a is called the *average dwell time (ADT)*, and N_0 is called the *chatter bound*.

Note that the above definition is usually considered for time-dependent switching signals, whereas a PWA induces a state-dependent switched system. By inserting one specific solution one arrives in a canonical way at a time-dependent switching signal and the above definition can be applied. Consequently, the property of having a certain average dwell time becomes a property of the specific solution of the considered PWA. We will say that a PWA has an average dwell time, if *all* solutions induce switching signals which have a common average dwell time (and a common chatter bound).

III. Problem statement

Our goal is to provide sufficient conditions to conclude ISS for a PWA system. Towards this goal we make the following assumptions:

- (A1) Continuity: $A_i x + b_i + B_i u = A_j x + b_j + B_j u$ for all $x \in X_i \cap X_j$, $i,j \in \{0,1,\dots,N\}$, and all $u \in \mathbb{R}^m$.
- (A2) Average dwell time: For every input bound $\|u\|_\infty \leq U_{\max}$ the corresponding solutions of (1) induce switching signals with an average dwell time $\tau_a > 0$ and chatter bound N_0.
- (A3) Existence of regional quadratic ISS Lyapunov functions: For each region X_s there exists regional ISS Lyapunov functions of the form $V_s(x) = x^\top P_s x_s + 2\nu_s^\top x + \omega_s$ with $P_s = P_s^\top \in \mathbb{R}^{n\times n}$, $\nu_s \in \mathbb{R}^n$, $\omega_s \in \mathbb{R}$. Furthermore, the $\mathcal{K}$ and $\mathcal{KL}$ bounds in (4) and (5) are also quadratic and/or exponential, i.e. there exists $0 < \varepsilon_{1,s} < \varepsilon_{2,s}$ and $\lambda_s > 0$, $\Gamma_s > 0$, $\Theta_s \geq 0$ such that

  $$\varepsilon_{1,s}|x|^2 \leq V_s(x) \leq \varepsilon_{2,s}(|x|^2 + 1) \qquad \forall x \in X_s,$$

  and, for all $u \in \mathbb{R}^m$,

  $$\dot{V}_s(x,u) \leq -\lambda_s V_s(x) + \Gamma_s |u|^2 + \Theta_s \quad \forall x \in X_s.$$

(A4) Crossings: For every solution of the PWA (1) which leaves region X_i and enters region X_j at some time $t_{ij} \geq t_0$ the values of the corresponding Lyapunov functions *do not increase*, i.e.

$$V_j(x(t_{ij})) \leq V_i(x(t_{ij})).$$

(A5) Origin: $0 \in X_0$ and $0 \notin X_s$ for all $s \in \{1,\dots,N\}$. Furthermore, the regional ISS Lyapunov function V_0 from Assumption (A3) takes the form $V_0(x) = x^\top P_0 x$ and satisfies the more strict inequalities, for all $x \in X_0$ and all $u \in \mathbb{R}^m$,

$$\varepsilon_{1,0}|x|^2 \leq V_0(x) \leq \varepsilon_{2,0}|x|^2,$$
$$\dot{V}_0(x,u) \leq -\lambda_0 V_0(x) + \Gamma_0 |u|^2.$$

Assumption (A1) avoids technicalities concerning existence and uniqueness of solutions because, in particular, sliding solutions (see e.g. [4]) cannot occur. Furthermore, our work is motivated to better understand the stability properties of the Wanner-Friedrich model for neural networks [25], [38], which satisfies this continuity assumption. Similiarly, Assumption (A2) avoids Zeno behavior and chattering; we believe that a large class of PWA systems satisfy this assumption, but providing a formal proof is outside the scope of this paper. Note that in general, the average dwell time τ_a and the chatter bound N_0 will depend on $U_{\max}$.

Assumption (A3) is the key assumption of our approach and follows a divide-and-conquer philosophy: We assume that it is not too difficult to find a regional ISS Lyapunov function in regions with linear dynamics. In fact, we have already provided suitable LMIs in [25] for finding such regional ISS Lyapunov functions. Note that we allow some additional constant terms in the inequalities, these constants enlarge the set of feasible ISS Lyapunov function candidates and at the same time do not prevent us from concluding overall ISpS. Note that the upper bounding $V_s(x)$ for $s \neq 0$ in terms of $|x|^2 + 1$ instead of just $|x|^2$ is motivated by the presence of a possible non-zero ω_s, which allows for Lyapynov functions with $V_s(0) > 0$; however, if $0 \notin X_s$ it would still be possible to bound $V_s(x)$ in terms of $|x|^2$ alone, but when setting up LMIs within the cone-copositive approach it is more straightforward to have an upper bound in terms of $|x|^2 + 1$. A similar remark also holds for the presence of $\Theta_s > 0$.

Assumption (A4) is the second key ingredient to conclude global ISpS; it allows us to patch together the regional ISS Lyapunov functions to obtain a global one. If no further knowledge about the crossing behavior is available, then we have to assume that solutions can cross in both directions, so that the two inequality constraints become one equality constraint. However, if some further knowledge, e.g. about possible input directions, is available, then requiring just the inequality provides much more freedom to find suitable ISS Lyapunov function candidates (c.f. the discussion in [4, Sec. V]).

Finally, Assumption (A5) is required to sharpen the ISpS property towards ISS. The assumption that the origin is contained in the interior of X_0 is satisfied for our motivating example of the Wanner-Friedrich model; however, in general, most of the interesting stability properties of PWA arise in cases where the origin is in the intersection of several regions (e.g. when considering homogeneous PWA systems); here we exclude these situations as it would add further technicalities and for now we want to focus on the broader perspective.

The research question, we now want to answer, is: Under which further quantitative assumptions (on the parameters $\varepsilon_{1,s}$, $\varepsilon_{2,s}$, λ_s, Γ_s and Θ_s) is it true that the overall PWA system (1) is ISS? Furthermore, we want to provide explicit β and γ functions in the ISS inequality (3), which in future work allows us then to utilize a network small-gain approach to conclude stability of a network of interconnected PWA systems.

IV. Main results

A. From regional ISS Lyapunov functions to ISpS

Our first main result shows how the existence of regional ISS Lyapunov function with suitable compatibility conditions on the region boundaries ensures ISpS in the sense of Definition 4.

Theorem 6. Consider the PWA system (1) satisfying assumptions (A1)–(A4) and with inputs bounded by some $U_{\max} > 0$. Then there exist positive constants C_1, C_2, C_3 and Λ such that for all solutions of (1) and all $t \geq t_0$:

$$|x(t)| \leq C_1 e^{-\Lambda(t-t_0)}|x(t_0)| + C_2\|u\|_\infty + C_3. \quad (8)$$

Proof. Let $x : [t_0,\infty) \to \mathbb{R}^n$ be a solution of (1) for some $u : [t_0,\infty) \to \mathbb{R}^m$ and let $t > t_0$ be arbitrary. Furthermore, let $\sigma : [0,t) \to \{0,\dots,N\}$ be the corresponding induced switching signal, i.e. $x(\tau) \in X_{\sigma(\tau)}$ for all $\tau \in [t_0,t)$, with switching times $t_0 < t_1 < t_2 < \ldots < t_m < t$; for notational convenience, we define $t_{m+1} := t$, even if t is not a switching time. Finally, let $i_k \in \{0,\dots,N\}$ denote the mode on the k-th interval $[t_{k-1},t_k)$ for $k = 1,\dots,m+1$, i.e. $x(\tau) \in X_{i_k}$ for all $\tau \in [t_{k-1},t_k)$.

For each $\tau \in [t_{k-1},t_k)$ and each $k = 1,2,\dots,m+1$ we have by assumption (A3) that (with $\Theta_0 := 0$)

$$\dot{V}_{i_k}(x(\tau),u(\tau)) \leq -\lambda_{i_k} V_{i_k}(x(\tau)) + \Gamma_{i_k}|u(\tau)|^2 + \Theta_{i_k}.$$

From $\dot{V}_{i_k}(x(\tau),u(\tau)) = \frac{\mathrm{d}}{\mathrm{d}\tau} V_{i_k}(x(\tau))$ we can conclude that

$$V_{i_k}(x(t_k)) \leq e^{-\lambda_{i_k}\Delta_k} V_{i_k}(x(t_{k-1})) + \int_{t_{k-1}}^{t_k} e^{-\lambda_{i_k}(t_k-\tau)} \left(\Gamma_{i_k}|u(\tau)|^2 + \Theta_{i_k}\right) d\tau, \quad (9)$$

where $\Delta_k := t_k - t_{k-1}$. From

$$\int_{t_{k-1}}^{t_k} e^{-\lambda_{i_k}(t_k-\tau)} d\tau = \frac{1}{\lambda_{i_k}}(1 - e^{-\lambda_{i_k}\Delta_k}) \leq \frac{1}{\lambda_{i_k}},$$

we can conclude that

$$V_{i_k}(x(t_k)) \leq e^{-\lambda_{i_k}\Delta_k} V_{i_k}(x(t_{k-1})) + \frac{\Gamma_{i_k}}{\lambda_{i_k}}\|u\|_\infty^2 + \frac{\Theta_{i_k}}{\lambda_{i_k}}. \quad (10)$$

Utilizing Assumption (A4) we obtain recursively from (10) that

$$V_{i_{m+1}}(x(t)) \le \left(\prod_{k=1}^{m} e^{-\lambda_{i_k}\Delta_k}\right) V_{i_1}(x(t_0)) + \sum_{k=1}^{m}\left(\prod_{j=k+1}^{m} e^{-\lambda_{i_j}\Delta_j}\right)\left(\frac{\Gamma_{i_k}}{\lambda_{i_k}}\|u\|_\infty^2 + \frac{\Theta_{i_k}}{\lambda_{i_k}}\right). \quad (11)$$

Let $\Lambda_{\min} := \min_s \lambda_s$, $\Gamma_{\max} := \max_s \Gamma_s$ and $\Theta_{\max} := \max_s \Theta_s$, then we can conclude from (11) that

$$V_{i_{m+1}}(x(t)) \le e^{-\Lambda_{\min}(t-t_0)} V_{i_1}(x(t_0)) + \left(\frac{\Gamma_{\max}}{\Lambda_{\min}}\|u\|_\infty^2 + \frac{\Theta_{\max}}{\Lambda_{\min}}\right)\sum_{k=1}^{m} e^{-\Lambda_{\min}(t_m-t_k)}. \quad (12)$$

Because of (A2), we can utilize the average dwell time condition (7) for each interval $[t_k, t_m)$:

$$m-k = N_\sigma(t_k,t_m) \le N_0 + \frac{t_m - t_k}{\tau_a},$$

which implies that

$$t_m - t_k \ge \begin{cases} 0, & m-k-N_0 \le 0 \\ \tau_a(m-k-N_0), & \text{otherwise} \end{cases} \quad (13)$$

Therefore, we have

$$\sum_{k=1}^{m} e^{-\Lambda_{\min}(t_m-t_k)} \le \sum_{k=1}^{m-N_0} e^{-\Lambda_{\min}(m-N_0-k)\tau_a} + \sum_{k=m-N_0+1}^{m} e^0$$
$$= \sum_{\ell=0}^{m-N_0-1} e^{-\Lambda_{\min}\ell\tau_a} + N_0$$
$$\le \sum_{\ell=0}^{\infty} (e^{-\Lambda_{\min}\tau_a})^\ell + N_0 = \frac{1}{1-e^{\Lambda_{\min}\tau_a}} + N_0$$

Altogether, we therefore have

$$V_{i_{m+1}}(x(t)) \le e^{-\Lambda_{\min}(t-t_0)} V_{i_1}(x(t_0)) + \tilde{C}_2\|u\|_\infty^2 + \tilde{C}_3$$

with

$$\tilde{C}_2 := \frac{\Gamma_{\max}}{\Lambda_{\min}}\left(\frac{1}{1-e^{-\Lambda_{\min}\tau_a}} + N_0\right),$$
$$\tilde{C}_3 := \frac{\Theta_{\max}}{\Lambda_{\min}}\left(\frac{1}{1-e^{-\Lambda_{\min}\tau_a}} + N_0\right).$$

Using the bounds on the quadratic Lyapunov function $V_{i_1}(x(t_0)) \le \varepsilon_{2,i_1}(|x(t_0)|^2+1)$ and $V_{i_{m+1}}(x(t)) \ge \varepsilon_{1,i_{m+1}}|x(t)|^2$, we deduce that

$$|x(t_m)|^2 \le \tfrac{\varepsilon_{2,\max}}{\varepsilon_{1,\min}} e^{-\Lambda_{\min}(t-t_0)}(|x(t_0)|^2+1) + \frac{\tilde{C}_2}{\varepsilon_{1,\min}}\|u\|_\infty^2 + \frac{\tilde{C}_3}{\varepsilon_{1,\min}}, \quad (14)$$

where $\varepsilon_{2,\max} := \max_s \varepsilon_{2,s}$ and $\varepsilon_{1,\min} := \min_s \varepsilon_{1,s}$; together with the subadditivity property of the square root function we get the desired inequality

$$|x(t_m)| \le C_1 e^{-\Lambda(t-t_0)}|x(t_0)| + C_2\|u\|_\infty + C_3,$$

where $\Lambda := \Lambda_{\min}/2$ and

$$C_1 := \sqrt{\frac{\varepsilon_{2,\max}}{\varepsilon_{1,\min}}} \ge 1,$$
$$C_2 := \sqrt{\frac{\tilde{C}_2}{\varepsilon_{1,\min}}} > 0,$$
$$C_3 := \sqrt{\frac{\tilde{C}_3 + \varepsilon_{2,\max}}{\varepsilon_{1,\min}}} > 0.$$

□

Some similarities of formulation and assumptions can be also seen in [18, Th. III.1], but the end goal and result, in the mentioned paper, focus on proving global ISS under the ADT condition together with the $e^{\lambda t}$-weighted input-to-state stable ($e^{\lambda t}$-weighted ISS) format, whereas we focus on patching the regions and obtaining an ISpS type bound regardless of how many regions we cross.

Remark 7. Note that the proof of Theorem 6 is constructive and provides specific formulas for the constants C_1, C_2, C_3 and Λ. In particular, the freedom one has in the choice of regional Lyapunov functions can be utilized to minimize the constants C_1, C_2, C_3 or gives guidance on the tradeoff between the size of certain constants. For example, when considering the inequality $\dot{V}_s(x,u) \le -\lambda_s V_s(x) + \Gamma_s|u| + \Theta_s$a for $s \neq 0$, then $V_s(x)$ is lower bounded on X_s and hence we can decrease Θ_s (and hence C_3) if we also decrease λ_s (slower exponential decay). This is particularly useful if the worst case constant $\Theta_{\max}$ and worst case decay $\Lambda_{\min}$ are attained for different modes.

Remark 8. The special role of the region X_0 as required by Assumption (A5) is actually not needed for the conclusion of Theorem 6, in particular, ISpS can also be included if the bounds for V_0 take the same form as for $s \neq 0$ (in particular, the upper bound for $V_0(x)$ is in terms of $|x|^2+1$ and the upper bound for $\dot{V}_0(x,u)$ contains an extra term $\Theta_0 > 0$.

B. From ISpS to global ISS

While ISpS already shows that solutions of the PWA systems cannot blow up, it is not sufficient to conclude that solutions come to rest once the input vanishes, in particular, when C_3 in (8) is large. We now want to utilize Assumption (A3) for region X_0, which guarantees (non-practical) ISS around the origin. The idea is to utilize ISpS to ensure that solutions enter X_0 when the input is sufficiently small and then indeed converge to zero if the input vanishes. For large inputs solutions will in general *not* enter region X_0, but for (non-practical) ISS this is in fact not required.

Our main result about concluding ISS from ISpS is as follows.

Theorem 9. Consider a PWA system (1) which is ISpS and all solutions satisfy (8). Based on Assumption (A5), choose $r_{\max} > 0$ such that $B(r_{\max}) :=$

$\{x \in \mathbb{R}^n \mid |x| \le r_{\max}\} \subseteq X_0$. If

$$C_3 < r_{\max}, \tag{15}$$

then the PWA system (1) is ISS.

Before proving Theorem 9, we would like to provide some intuition behind condition (15): For very small inputs, the ISpS property (8) ensures that $|x(t)|$ for sufficiently large t is basically bounded by C_3 and if C_3 is sufficiently small, $x(t)$ will be in X_0 for all sufficiently large t. Because the PWA system is ISS in X_0, we can now conclude that $x(t)$ shrinks further for very small inputs (and actually converges to zero if the input is zero).

The above argument requires us to distinguish whether the input is "sufficiently small" or not. Based on the strict inequality (15) together with (8) we can now quantify "sufficiently small" by the following input bound

$$\bar{u} := \frac{r_{\max} - C_3 - \Omega}{C_2} \tag{16}$$

where $\Omega \in (0, C_3 - r_{\max})$ is a free parameter which can be used later to optimize the resulting ISS bounds. A visual representation of this idea can be seen in Figure 1.

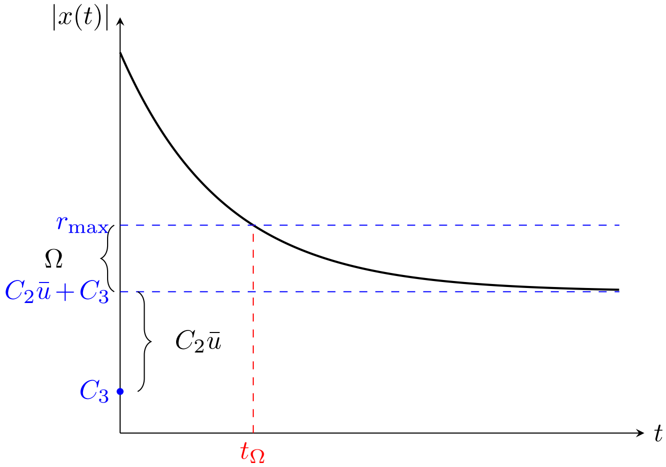

Fig. 1. ISpS-type decay bound for small inputs with steady-state levels reaching X_0.

For the proof of Theorem 9 we will use the following two lemmas.

Lemma 10. Consider the setup of Theorem 9 and assume that the input u satisfies $\|u\|_\infty \ge \bar{u}$. Then all solutions satisfy for all $t \ge t_0$

$$|x(t)| \le C_1 e^{-\Lambda(t-t_0)}|x_0| + C_2^{\bar{u}}\|u\|_\infty,$$

where $C_2^{\bar{u}} := C_2 + \frac{C_3}{\bar{u}}$

Proof. This follows straightforwardly from

$$C_2\|u\|_\infty + C_3 = \left(C_2 + \frac{C_3}{\|u\|_\infty}\right)\|u\|_\infty \le \left(C_2 + \frac{C_3}{\bar{u}}\right)\|u\|_\infty.$$

□

Lemma 11. Consider the setup of Theorem 9 and assume $\|u\|_\infty \le \bar{u}$. Let $t_\Omega := t_0 + \frac{\ln \frac{C_1|x_0|}{\Omega}}{\Lambda}$ if $|x_0| > \frac{\Omega}{C_1}$ and $t_\Omega = t_0$ otherwise. Then, for all $t \in [t_0, t_\Omega)$, we have

$$|x(t)| \le C_1^\Omega e^{-\Lambda(t-t_0)}|x_0| + C_2\|u\|_\infty,$$

where $C_1^\Omega := C_1(1 + \frac{C_3}{\Omega})$, and for all $t \ge t_\Omega$, there exist $C_1^0 > 0$ and $C_2^0 > 0$.

$$|x(t)| \le C_1^0 e^{-\lambda_0(t-t_\Omega)}|x(t_\Omega)| + C_2^0\|u\|_\infty. \tag{17}$$

Proof. For $t \in [t_0, t_\Omega)$ (and in particular $|x_0| > 0$) we have that

$$\frac{C_1|x_0|}{\Omega} e^{-\Lambda(t-t_0)} \ge \frac{C_1|x_0|}{\Omega} e^{-\Lambda(t_\Omega - t_0)} = 1,$$

hence we can conclude from (8) that

$$\begin{aligned}|x(t)| &\le C_1 e^{-\Lambda(t-t_0)}|x_0| + C_2\|u\|_\infty + C_3\frac{C_1|x_0|}{\Omega}e^{-\Lambda(t-t_0)} \\ &= C_1^\Omega e^{-\Lambda(t-t_0)}|x_0| + C_2\|u\|_\infty.\end{aligned}$$

For $t \ge t_\Omega$ we conclude from (8) that

$$\begin{aligned}|x(t)| &\le C_1 e^{-\Lambda(t_\Omega-t_0)}|x_0| + C_2\bar{u} + C_3 \\ &= \Omega + C_2\frac{r_{\max} - C_3 - \Omega}{C_2} + C_3 = r_{\max}.\end{aligned}$$

In particular, $x(t) \in X_0$ for all $t \ge t_\Omega$. Utilizing the existence of an ISS Lyapunov function $V_0(x)$ on X_0 and following similar arguments as in the proof of Theorem 6 leading to (10) we conclude that

$$V_0(x(t)) \le e^{-\lambda_0(t-t_\Omega)}V_0(x(t_\Omega)) + \frac{\Gamma_0}{\lambda_0}\|u\|_\infty^2,$$

and with $\varepsilon_{1,0}\|x\|^2 \le V(x) \le \varepsilon_{2,0}\|x\|^2$ we can choose $C_1^0 := \sqrt{\frac{\varepsilon_{2,0}}{\varepsilon_{1,0}}}$ and $C_2^0 := \sqrt{\frac{\Gamma_0}{\lambda_0\varepsilon_{1,0}}}$ such that inequality (17) holds. □

We actually can combine both cases of Lemma 11 into one inequality:

Corollary 12. Under the assumptions of Lemma 11, in particular, $\|u\|_\infty \le \bar{u}$, we have for all $t \ge t_0$

$$|x(t)| \le C_1^{0,\Omega} e^{-\Lambda(t-t_0)}|x_0| + C_{1,2}^0\|u\|_\infty,$$

where $C_1^{0,\Omega} := C_1^0 C_1^\Omega$ and $C_{1,2}^0 := C_1^0 C_2 + C_2^0$.

Proof. This just follows by plugging the bound from the first case for $|x(t_\Omega)|$ into the second case and taking into account that $\lambda_0 > \Lambda = \Lambda_{\min}/2$ and the fact that $C_1^{0,\Omega} \ge C_1^\Omega$ and $C_{1,2}^0 \ge C_2$. □

Now we can easily proof the main result:

Proof of Theorem 9. Combining Lemma 10 with Corollary 12 we immediately obtain

$$|x(t)| \le C_1^* e^{-\Lambda(t-t_0)}|x_0| + C_2^*\|u\|_\infty$$

where $C_1^* := \max\{C_1, C_1^{0,\Omega}\}$ and $C_2^* := \max\{C_2^{\bar{u}}, C_{1,2}^0\}$. This shows that the PWA system (1) is ISS in the sense of Definition 2 with $\beta(r,\tau) = C_1^* e^{-\Lambda\tau} r$ and $\gamma(r) = C_2^* r$. □

V. Conclusions

In this paper we offered a recipe for obtaining a global ISS type bound on the trajectory that operates in a PWA system setting.

We started in [25], where we set up LMIs that lead to producing regional ISS Lyapunov functions. Now, with these regional outcomes, we focused in this paper on properly patching them and concluding global ISS.

To achieve the patching, we have provided assumptions that highlight the recipe for obtaining an ISpS result first. Within these assumptions, one pertains to the *behavior within each region* (A_3), two ensure *well-posed and stable transitions across region boundaries* (A_1,A_4), and one concerns the *average dwell time (ADT)* condition that regulates the *bounding of trajectories during successive crossings* (A_2). Based on these ingredients, we offered the first key result of the paper, namely Theorem 6, that patches the regional ISS Lyapunov functions into the general ISpS bound for the trajectory. The result is constructive and the obtained constants and terms that comprise the ISpS bound are independent of the number of regions crossed; however, they currently depend on the average dwell time which in general will depend on the a priori fixed input bound $U_{\max}$.

Next, since the input is explicitly considered, we refined the global analysis by separating it into distinct regimes. For *large inputs*, the affine term is absorbed into the input-dependent part of the bound, leading directly to the inequality of Lemma 10. For *small inputs*, we established in Lemma 11 that the trajectories first exhibit exponential decay until they reach a threshold influenced by Ω into the equilibrium region X_0, after which the decay continues inside the ball of radius $r_{\max}$, effectively entering a regime where decay is stronger than the input effect. The corresponding bounds were then unified in Corollary 12, which compactly summarizes both phases into a single inequality. The above mentioned results quantify how the ISpS constants (C_1,C_2,C_3) reorganize across input regimes, and this reasoning, together with assumption A_5, culminates in Theorem 9, which formalizes the transition from the regional ISpS framework to a fully global ISS result.

Future directions include establishing results for more general cases (e.g. not necessarily continuous vector fields or other boundary types). Also, we plan to establish the theoretical framework and then test the results on the neural network cases using small-gain theory to conclude stability of such networks.

VI. Acknowledgements

This work has been conducted as part of a PhD project funded by Groningen Cognitive Systems and Materials Center (CogniGron).

References

[1] E. D. Sontag, "Nonlinear regulation: The piecewise linear approach," *IEEE Trans. Autom. Control*, vol. 26, pp. 346–358, 1981.

[2] F. J. Christophersen, *Piecewise affine systems.* Springer, 2007.

[3] N. van de Wouw and A. Pavlov, "Tracking and synchronisation for a class of PWA systems," *Automatica*, vol. 44, no. 11, pp. 2909–2915, 2008.

[4] R. Iervolino, S. Trenn, and F. Vasca, "Asymptotic stability of piecewise affine systems with Filippov solutions via discontinuous piecewise Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1513–1528, 2020.

[5] J. M. Gonçalves, "Regions of stability for limit cycles of piecewise linear systems," in *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, vol. 1. IEEE, 2003, pp. 651–656.

[6] R. Iervolino, D. Tangredi, and F. Vasca, "Lyapunov stability for piecewise affine systems via cone-copositivity," *Automatica*, vol. 81, pp. 22–29, 2017.

[7] P. Petsagkourakis, W. P. Heath, and C. Theodoropoulos, "Stability analysis of piecewise affine systems with multi-model predictive control," *Automatica*, vol. 111, p. 108539, 2020.

[8] M. Kantner, "Robust stability of piecewise linear discrete time systems," in *Proceedings of the 1997 American Control Conference (Cat. No. 97CH36041)*, vol. 2. IEEE, 1997, pp. 1241–1245.

[9] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An LMI approach," in *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)*, vol. 1. IEEE, 2000, pp. 504–509.

[10] L. B. Groff, G. Valmórbida, and J. M. G. da Silva, "Stability analysis of piecewise affine discrete-time systems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 8172–8177.

[11] L. Rodrigues, "Stability analysis of piecewise-affine systems using controlled invariant sets," *Systems & control letters*, vol. 53, no. 2, pp. 157–169, 2004.

[12] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Trans. Autom. Control*, vol. 34, no. 4, pp. 435–443, 1989.

[13] Z.-P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Math. Control Signals Systems*, vol. 7, no. 2, pp. 95–120, 1994.

[14] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang, "A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems," *Automatica*, vol. 32, no. 8, pp. 1211–1215, 1996.

[15] J. P. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th IEEE Conf. Decis. Control*, 1999, pp. 2655–2660.

[16] C. Cai and A. R. Teel, "Results on input-to-state stability for hybrid systems," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 5403–5408.

[17] ——, "Characterizations of input-to-state stability for hybrid systems," *Syst. Control Lett.*, vol. 58, no. 1, pp. 47–53, 2009. [Online]. Available: http://dx.doi.org/10.1016/j.sysconle.2008.07.009

[18] L. Vu, D. Chatterjee, and D. Liberzon, "ISS of switched systems and applications to switching adaptive control," in *Proceedings of the 44th IEEE Conference on Decision and Control*. IEEE, 2005, pp. 120–125.

[19] L. Vu and D. Liberzon, "Invertibility of switched linear systems," *Automatica*, vol. 44, no. 4, pp. 949–958, 2008.

[20] M. Lazar and M. Heemels, "Global input-to-state stability and stabilization of discrete-time PWA systems," in *Analysis and Design of Hybrid Systems 2006: A Proceedings volume from the 2nd IFAC Conference, Alghero, Italy, 7-9 June 2006*. Elsevier, 2006, p. 296.

[21] M. Hovd and S. Olaru, "Relaxing PWQ Lyapunov stability criteria for PWA systems," *Automatica*, vol. 49, no. 2, pp. 667–670, 2013.

[22] Y. Zhu, Z. Zhong, M. V. Basin, and D. Zhou, "A descriptor system approach to stability and stabilization of discrete-time switched PWA systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 10, pp. 3456–3463, 2018.

[23] N. van den Wouw, A. Pavlov, and H. Nijmeijer, "Controlled synchronisation of continuous PWA systems," in *Group coordination and cooperative control*. Springer, 2006, pp. 271–289.

[24] A. Mironchenko, *Input-to-State Stability: Theory and Applications*. Springer, 2023.

[25] R. Cimpean and S. Trenn, "Utilizing discontinuous piecewise Lyapunov function for a biological PWA system and extending

the analysis via input-to-state type stability," in *Proc. 23rd European Control Conference (ECC 2025)*, EUCA. Thessaloniki, Greece: IEEE, 2025, pp. 194–200.

[26] W.-H. Chen and W. X. Zheng, "Delay-independent minimum dwell time for exponential stability of uncertain switched delay systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 10, pp. 2406–2413, 2010.

[27] C. Briat, "Dwell-time stability and stabilization conditions for linear positive impulsive and switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 24, pp. 198–226, 2017.

[28] Y. Li and H. Zhang, "Dwell time stability and stabilization of interval discrete-time switched positive linear systems," *Nonlinear Analysis: Hybrid Systems*, vol. 33, pp. 116–129, 2019.

[29] J. P. Hespanha, "Uniform stability of switched linear systems: Extensions of lasalle's invariance principle," *IEEE transactions on Automatic Control*, vol. 49, no. 4, pp. 470–482, 2004.

[30] S. Zhuang, X. Yu, J. Qiu, Y. Shi, and H. Gao, "Stability analysis of switched linear systems under persistent dwell-time constraints," *IEEE Transactions on Automatic Control*, vol. 67, no. 12, pp. 6739–6744, 2021.

[31] C. Yuan and F. Wu, "Hybrid control for switched linear systems with average dwell time," *IEEE Transactions on Automatic Control*, vol. 60, no. 1, pp. 240–245, 2014.

[32] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: an average dwell time approach," *International Journal of Systems Science*, vol. 32, no. 8, pp. 1055–1061, 2001.

[33] L. Zhang and H. Gao, "Asynchronously switched control of switched linear systems with average dwell time," *Automatica*, vol. 46, no. 5, pp. 953–958, 2010.

[34] E. D. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Trans. Autom. Control*, vol. 41, no. 9, pp. 1283–1294, 1996.

[35] Z.-P. Jiang and Y. Wang, "Input-to-state stability for discrete-time nonlinear systems," *Automatica*, vol. 37, no. 6, pp. 857–869, 2001.

[36] D. Liberzon, *Switching in Systems and Control*, ser. Systems and Control: Foundations and Applications. Boston: Birkhäuser, 2003.

[37] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[38] A. A. Wanner and R. W. Friedrich, "Whitening of odor representations by the wiring diagram of the olfactory bulb," *Nature neuroscience*, vol. 23, no. 3, pp. 433–442, 2020.