Lyapunov Characterization for ISS of Impulsive Switched Systems

Saeed Ahmed, Patrick Bachmann, and Stephan Trenn

***Abstract*— In this study, we investigate input-to-state stability (ISS) of impulsive switched systems that have modes with both stable and unstable flows. We assume that the switching signal satisfies mode-dependent average dwell and leave time conditions. To establish ISS conditions, we propose two types of time-varying ISS-Lyapunov functions: one that is not necessarily decreasing along trajectories, which we call *generalized*, and another one that is decreasing. Our research proves that the existence of either of these ISS-Lyapunov functions is a necessary and sufficient condition for ISS. We also present a strictification technique for constructing a decreasing ISS-Lyapunov function from a large class of generalized ones, which is useful for its own sake. Our findings also have added value to previous research that only studied sufficient conditions for ISS, as our results apply to a broader class of systems. This is because we impose less restrictive dwell and leave time constraints on the switching signal and our ISS-Lyapunov functions are time-varying with general nonlinear conditions imposed on them. Moreover, we provide a method to guarantee ISS of a particular class of impulsive switched systems when the switching signal is unknown.**

***Index Terms*— ISS, switched systems, impulsive systems, Lyapunov methods, dwell time, leave time.**

I. Introduction

Impulsive and switched systems are two important classes of hybrid dynamical systems. Impulsive systems consist of a continuous behavior referred to as a *flow* and abrupt state changes referred to as *jumps* [1]. Switched systems, on the other hand, consist of a family of flows and a switching signal that determines which flow is active at any given time [2]. Systems involving impulsive and switching dynamics are ubiquitous in robotics, aircraft, the automotive industry, and network control [3].

When dealing with dynamical systems, it is vital to consider their sensitivity to external inputs or perturbations. Input-to-state stability (ISS) is a useful concept that ensures tolerance to these inputs and helps analyze the system's behavior. The ISS concept can also be applied to analyze the stability and synthesize controllers for dynamical systems with disturbance inputs and complex structures; see, e.g., [4], [5], and [6]. However, analyzing ISS of systems with impulsive and switching dynamics is a challenging problem due to the hybrid nature of these systems. This paper aims to focus on this problem and explore potential solutions.

The stability of switched systems can be categorized based on arbitrary and constrained switching. For a system with arbitrary switching, it is necessary to require that all of its flows are stable. However, even if all of its flows are stable, it is not true in general that the overall switched system is stable. Motivated by this, [7] and [8] provided several sufficient (and necessary) conditions for the stability of switched systems with arbitrary switching. To ensure the stability of switched systems, whose stability cannot be guaranteed under arbitrary switching, a constraint is imposed on the number of switches via a suitable bound referred to as a *dwell time* constraint. An interesting example of this is in switched systems with stable and unstable flows, whose stability can be guaranteed by quantifying stable and unstable flows in terms of dwell time conditions (cf. [9], [10]). A similar dwell time approach is used to guarantee the stability of impulsive systems, quantifying stable flow and unstable jumps, or unstable flow and stable jumps (cf. [11], [12]). Moreover, time-varying Lyapunov functions are proposed for a broad class of impulsive systems with simultaneous instability of flow and jumps in [13] and [14].

In this paper, we aim at deriving ISS conditions for impulsive switched systems having modes[1] with stable and unstable flows. Our approach is based on distinguishing between modes with stable and unstable flows and quantifying them to ensure ISS of the overall impulsive switched system. To accomplish this, we require the switching signal to satisfy two less restrictive constraints as compared to uniform dwell-time as established in [16]. These conditions are, namely, mode-dependent average dwell time (MDADT) and mode-dependent average leave time[2] (MDALT). MDADT indicates that less frequent jumps can stabilize modes with stable flows, while MDALT suggests that frequent jumps can stabilize modes with

Submitted on May 01, 2024.

*P. Bachmann is supported by grant MI 1886/5-1 of the German Research Foundation (DFG).

Saeed Ahmed is with the Jan C. Willems Center for Systems and Control and the Engineering, and Technology Institute Groningen (ENTEG), Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands (e-mail: s.ahmed@rug.nl).

Patrick Bachmann is with the Department of Mathematics, University of Bayreuth, Germany, and the Institute of Mathematics, University of Würzburg, Germany (e-mail: patrick.bachmann@uni-wuerzburg.de).

Stephen Trenn is with the Jan C. Willems Center for Systems and Control and the Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, Faculty of Science and Engineering, University of Groningen, The Netherlands (e-mail: s.trenn@rug.nl).

[1] It is worth noting that in this paper, the term *mode* has a different definition compared to most switched systems' literature. In switched systems, modes refer to the flows. Here, we expand the definition of the mode to include the succeeding jump as well. Alternatively, one could also consider a flow and its preceding jump as a mode, which is standard in, for example, switched DAEs [15]. Our results can be easily modified to cover this scenario as well.

[2] The mode-dependent average leave time is also referred to as mode-dependent reverse-average dwell time in the literature.

unstable flows.

To establish the ISS conditions, we introduce two classes of Lyapunov functions: (i) *generalized time-varying ISS-Lyapunov functions* by which we mean "ISS-Lyapunov function with a non-strictly decreasing dissipation rate"; (ii) *decreasing time-varying ISS-Lyapunov functions*, which are a subclass of the generalized ones. Then, we show that the existence of the generalized ones gives a sufficient condition for ISS and the decreasing ones provide a necessary condition. Therefore, it implies that the existence of both generalized and decreasing ISS Lyapunov functions is equivalent to ISS. Since a necessary condition for ISS implies existence of a Lyapunov function, this makes our definition of Lyapunov functions a preferable choice for verifying ISS of impulsive switched systems as compared to other formulation of Lyapunov functions [17]–[20]. Moreover, the necessary condition for ISS establish minimum requirements and offer theoretical boundaries for ISS.

To the best of the authors' knowledge, *a converse ISS-Lyapunov theorem* has not been provided before for impulsive switched systems. Nevertheless, the results [21] and [22] study converse Lyapunov functions for exponential stabilizability and practical stability of impulsive switched systems without inputs, respectively. Moreover, for a broad class of systems, we provide a technique for *constructing a decreasing ISS-Lyapunov function from a generalized one* referred to as *strictification* [23, Ch. 6], [24], which is important and useful in its own right. This construction is motivated by the facts that (i) the relation between decreasing and generalized ISS-Lyapunov functions is important because every decreasing Lyapunov function is also generalized but the converse does not hold true in general, and (ii) the level sets of a decreasing ISS-Lyapunov function directly indicate the reachable sets. Our results are summarized in the Fig. 1.

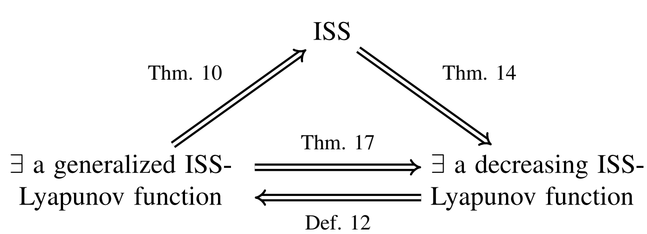

Fig. 1. Summary of our results. Note that Theorem 17 requires an additional restriction.

The results available in the literature for ISS of impulsive switched systems [17]–[20] only provide a *sufficient* condition of ISS. We even have added value to these results, which are summarized below (see Section VII below):

- Our results apply to systems in which some modes have stable flows, while others may have unstable flows. However, the results of [17] and [20] are limited to systems in which all modes have either stable or unstable flows.
- Our approach utilizes time-varying ISS-Lyapunov functions and thus can provide ISS conclusions for impulsive switched systems with simultaneous instability of flow and jumps, while results of [18] and [19] cannot be used to conclude ISS of this class of systems.
- Our dwell and leave time constraints are mode-dependent and we do not require them to be the same as in [18]. This allows us to consider a broader class of systems as compared to [18].
- We provide an approach to achieve robustness of our ISS results with respect to unknown switching signals for a class of impulsive switched systems with time-independent flow and jump maps.
- Our approach restricted to linear systems is constructive in the sense that a set of LMIs can be defined, whose feasibility guarantees ISS for a class of unknown switchings.
- We complement the approach in [25] by providing ISS and including impulsive dynamics and infinitely many modes.

This work presents several significant contributions to the study of ISS of impulse switched systems. Firstly, it introduces a framework that allows for an infinite number of modes. Moreover, it presents a method to construct a decreasing ISS-Lyapunov function from a generalized one. Finally, it provides a method to guarantee ISS of a class of impulsive switched systems with time-independent jump and flow maps when the switching signal is unknown. We additionally apply the aforementioned result to linear systems to obtain a set of LMIs guaranteeing ISS.

The rest of the paper is organized as follows. In Section II, we will introduce notation, system description, necessary definitions, and problem formulation. In Section III, we will provide a sufficient condition for ISS of the considered class of systems via a generalized ISS-Lyapunov function. In Section IV, we will provide a necessary and sufficient condition of ISS via a decreasing ISS-Lyapunov function. In Section V, we will suggest a method for constructing a decreasing ISS-Lyapunov function from the generalized ISS-Lyapunov function proposed in Section III for a large class of systems. Our theoretical results are illustrated in Section VII. In Section VI, we will provide a method to guarantee ISS for impulsive switched systems with time-independent flow and jump maps when the switching signal is unknown. We will then apply this robustness result to find sufficient conditions of ISS for linear systems in terms of LMIs in the special case when the ISS-Lyapunov functions are quadratic and time-independent. Finally, we will conclude the paper by summarizing our work in Section VIII.

II. Preliminaries and Problem Formulation

We denote the set of positive integers by $\mathbb{N}$, the set of nonnegative integers by $\mathbb{N}_0$, the set of real numbers by $\mathbb{R}$, the set of nonnegative real numbers by $\mathbb{R}_0^+$, the space of continuous functions from normed spaces X to Y by $\mathcal{C}(X,Y)$, the space of locally bounded piecewise continuous functions from X to Y by $\mathcal{PC}(X,Y)$, the identity function $\mathrm{id} : X \to X$, $x \mapsto x$, and the ball of radius $r > 0$ around 0 in X by $B_X(r)$. Let $X \subseteq \mathbb{R}^n$ with norm $\|\cdot\|_X$ and $U \subseteq \mathbb{R}^m$ with norm $\|\cdot\|_U$ represent the state space and input space, respectively. Let U_c

be the space of bounded functions from $[0,\infty)$ to U with norm $\|u\|_\infty := \sup_{t\in[0,\infty)}\{\|u(t)\|_U\}$. For a function $f\colon X \to Y$, we denote the image by $\mathrm{im}(f)$. We denote the left (right) limit of a function f at t as $f(t^-)$ $(f(t^+))$, and implicitly assume that the limit is well-defined when using this notation. For a continuous function $V : C \to \mathbb{R}$, $C \subseteq \mathbb{R}$, we denote the (upper) *Dini-derivative* by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(t) = \limsup_{s\searrow 0} \tfrac{1}{s}(V(t+s) - V(t)).$$

To introduce the notion of ISS-Lyapunov functions and corresponding ISS results later, we recall the definition of the *comparison function classes* from [26] as follows: *Class* $\mathcal{K}$ is the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$, which are strictly increasing and $\gamma(0) = 0$. *Class* $\mathcal{K}_\infty$ is the subset of class $\mathcal{K}$ for which additionally $\gamma(s) \to \infty$ as $s \to \infty$. *Class* $\mathcal{KL}$ is the set of all continuous functions $\beta : [0,\infty) \times [0,\infty) \to [0,\infty)$ for which $\beta(\,\cdot\,, r)$ is class $\mathcal{K}$ for every fixed $r \geq 0$, and for each fixed $s > 0$, the mapping $r \mapsto \beta(s,r)$ is strictly decreasing and converges to zero as $r \to \infty$. Moreover, we define *Class* $\mathcal{P}$ as the set of all continuous functions $\gamma : [0,\infty) \to [0,\infty)$, which satisfy $\gamma(0) = 0$ and $\gamma(r) > 0$ for all $r > 0$. For a function $\gamma\colon [0,\infty) \to \mathbb{R}$, we use the notation $\gamma \in \mathcal{P} \cup -\mathcal{P}$ to indicate that either $\gamma \in \mathcal{P}$ or $-\gamma \in \mathcal{P}$ holds true. We denote the disjoint union of two sets $\mathcal{S}$ and $\mathcal{U}$ by $\mathcal{S}\,\dot\cup\,\mathcal{U}$.

Definition 1: Let $t_0 \in \mathbb{R}$. A *switching signal* is a piecewise-constant, left-continuous function $\sigma : [t_0,\infty) \to \mathcal{M}$, where $\mathcal{M}$ is some (finite or infinite) index set which we call the set of *modes*. The set $S = \{t_i\}_{i\in\mathbb{N}} \subset (t_0,\infty)$ of discontinuities of σ is called the set of *switching instants* and it is assumed that the sequence $(t_i)_{i\in\mathbb{N}}$ is strictly increasing and unbounded, i.e., no accumulation towards a finite time (so-called Zeno behavior) is considered. The sequence $(p_i)_{i\in\mathbb{N}_0}$ such that $\sigma(t) = p_i$ on $[t_i, t_{i+1})$ is called *mode sequence* of σ. For the interval $I := [t_0,\infty)$ and $p \in \mathcal{M}$, we denote by I_p^σ the subset of I, on which mode p of a given switching signal σ is active, i.e. $I_p^\sigma := \{t \in I \mid \sigma(t) = p\}$.

In this paper, we consider *impulsive switched systems* of the form:

$$\Sigma\colon \begin{cases} \dot x(t) = f_{\sigma(t)}(t, x(t), u(t))\,, & t \in I \setminus S,\\ x(t_i^+) = g_{\sigma(t_i^-)}\big(t_i^-, x(t_i^-), u(t_i^-)\big)\,, & t_i \in S,\end{cases}$$

where $I = [t_0,\infty)$, σ is a switching signal with corresponding set of switching instants S, $x\colon I \to X$, and $u \in U_c$. We assume that for every $D > 0$, functions $f_p, g_p\colon I \times X \times U \to X$ are locally Lipschitz continuous in the second argument, uniformly for all $t \in I$, $p \in \mathcal{M}$ and $u \in B_U(D)$. We call $x : I \to X$ a solution of Σ for some given σ and some input $u \in U_c$, if x is locally absolutely continuous on $I \setminus S$ with well defined left- and right-limits at all $t_i \in S$ such that the equations of Σ hold for almost all $t \in I \setminus S$ and all $t_i \in S$. Without loss of generality, we assume that every solution x is right continuous and hence $x(t_i) = x(t_i^+)$. We furthermore assume that f_p and g_p are such that Σ has *bounded reachability sets*. That is, for every $(t, x_0, u) \in I \times X \times \mathcal{U}$ such that $t \in [t_0, t_0+\tau]$, $\|x_0\|_X \leq C$ and $\|u\|_{\mathcal{U}} \leq D$ for all parameters $\tau, C, D > 0$, the trajectory remains uniformly bounded [27, Def. 4]. This implies that

$$\begin{aligned} &K(C,D,\tau,\sigma)\\ &:= \sup_{x_0\in B_X(C),\, u\in B_{U_c}(D),\, t\in[t_0,t_0+\tau]} \|x(t;t_0,x_0,u,\sigma)\|_X \end{aligned} \tag{1}$$

is finite for each initial state bound $C > 0$, input bound $D > 0$, time duration $\tau > 0$ and switching signal σ; here $x(t;t_0,x_0,u,\sigma)$ denotes the (unique) solution of Σ. Note that by our definition, a mode of Σ consists of a flow between two consecutive switching instants and the succeeding jump.

In this work, we want to study input-to-state stability (ISS) of Σ, which is formally defined as follows.

Definition 2: For a given set of switching signals Ω, we call the system Σ *input-to-state stable (ISS)* if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$ such that for all switching signals $\sigma \in \Omega$, all initial conditions $x_0 \in X$ and every input function $u \in U_c$, the system has a global solution, which satisfies for all $t \in [t_0,\infty)$,

$$\|x(t;t_0,x_0,u,\sigma)\|_X \leq \beta(\|x_0\|_X, t-t_0) + \gamma(\|u\|_\infty)\,.$$

For a given switching signal σ, let $N_p^\sigma(s_1,s_2)$ denote the number of times that mode p is deactivated in the interval $(s_1,s_2]$ and $T_p^\sigma(s_1,s_2)$ denote the overall time duration that mode p is active in interval $(s_1,s_2]$. Note that the impulse associated to mode p is, by the definition of Σ, at the end of its respective interval. Therefore, we count deactivations.

We conclude this section by defining MDADT and MDALT for a switching signal; for this we assume that the mode set $\mathcal{M}$ is composed of two disjoint subsets $\mathcal{S},\mathcal{U} \subseteq \mathcal{M}$ and we define MDADT only for modes in $\mathcal{S}$ and MDALT for modes in $\mathcal{U}$. For the definition of MDADT and MDALT, the decomposition of $\mathcal{M}$ can be arbitrary, but based on the stability of the flows; see Definition 7.

Definition 3: Consider a subset of modes $\mathcal{S} \subseteq \mathcal{M}$ and let $\{\tau_p\}_{p\in\mathcal{S}}, \tau_p \geq \tau > 0$. If for the switching signal $\sigma\colon I \to \mathcal{M}$, there exists a constant $T_{\mathcal{S}} \geq 0$, such that for all $s_1, s_2 \in I$, $s_1 \leq s_2$, the inequality

$$\sum_{p\in\mathcal{S}} \big(N_p^\sigma(s_1,s_2)\tau_p - T_p^\sigma(s_1,s_2)\big) \leq T_{\mathcal{S}} \tag{2}$$

holds true, then we say that σ has MDADT $\{\tau_p\}_{p\in\mathcal{S}}$ (for all modes in $\mathcal{S}$), or short, σ satisfies the MDADT condition (2).

Definition 4: Consider a subset of modes $\mathcal{U} \subseteq \mathcal{M}$ and let $\{\tau_p\}_{p\in\mathcal{U}}, \tau_p \geq \tau > 0$. If for the switching signal $\sigma\colon I \to \mathcal{M}$, there exists a constant $T_{\mathcal{U}} \geq 0$, such that for all $s_1, s_2 \in I$, $s_1 \leq s_2$, the inequality

$$\sum_{p\in\mathcal{U}} \big(N_p^\sigma(s_1,s_2)\tau_p - T_p^\sigma(s_1,s_2)\big) \geq -T_{\mathcal{U}} \tag{3}$$

holds true, then we say that σ has the MDALT $\{\tau_p\}_{p\in\mathcal{U}}$ (for all modes in $\mathcal{U}$), or short, σ satisfies the MDALT condition (3).

Remark 5: We highlight here that Definitions 3 and 4 are formulated to allow switched systems with an infinite number of modes, which is one of our contributions. For switched systems with finitely many modes and without impulsive effects, these definitions reduce to the classical dwell time/leave time conditions as introduced in Definitions 4.1 and 4.2 of [25].

III. Sufficient condition for ISS

In this section, we provide one of our main results on ISS of system Σ. But before we proceed, we provide the notion of a candidate ISS-Lyapunov functions as follows:

Definition 6: Consider the system Σ with some given switching signal σ and let $\widetilde{V}_p \in \mathcal{C}(I_p^\sigma \times X, \mathbb{R}_0^+)$ for $p \in \mathcal{M}$. We call $V_\sigma \in \mathcal{PC}(I \times X, \mathbb{R}_0^+)$ given by $V_\sigma(t,x) := \widetilde{V}_{\sigma(t)}(t,x)$ a *candidate ISS-Lyapunov function (in implication form)* for the system Σ with switching signal σ, if it fulfills all of the following conditions:

1) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|_X) \le V_\sigma(t,x) \le \alpha_2(\|x\|_X) \tag{4}$$

holds true for all $t \in I$ and all $x \in X$.

2) There exist functions $\chi \in \mathcal{K}_\infty$ and $\underline{\varphi}, \overline{\varphi} \in \mathcal{P}$ such that for each $p \in \mathcal{M}$, there exist $\psi_p \in \mathcal{P}$ and $\varphi_p \in \mathcal{P} \cup -\mathcal{P}$ with $\underline{\varphi}(s) \le |\varphi_p(s)| \le \overline{\varphi}(s)\ \forall s \in \mathbb{R}_0^+$ such that for all inputs $u \in U_c$ and all solutions $x(t) = x(t; t_0, x_0, u, \sigma)$ of Σ, the inequality

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_\sigma(t, x(t)) \le \varphi_{\sigma(t)}(V_\sigma(t, x(t))), \tag{5}$$

holds for all $t \in I \setminus S$ whenever $V_\sigma(t, x(t)) \ge \chi(\|u\|_\infty)$, and

$$V_\sigma(t_i, x_i^+) \le \psi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x(t_i^-))\big), \ t_i \in S, \tag{6}$$

holds, whenever $V_\sigma(t_i^-, x(t_i^-)) \ge \chi(\|u\|_\infty)$, where $x_i^+ = g_{\sigma(t_i^-)}(t_i^-, x(t_i^-), u(t_i^-))$.

3) There exists a function $\alpha_3 \in \mathcal{K}$, such that for all $x \in X$, all $u \in U_c$, and all $i \in \mathbb{N}$, which satisfy $V_\sigma(t_i^-, x) < \chi(\|u\|_\infty)$, the jump rule satisfies

$$V_\sigma\Big(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u(t_i^-))\Big) \le \alpha_3(\|u\|_\infty). \tag{7}$$

Furthermore, we call V_σ a *candidate ISS-Lyapunov function in dissipation form* if 2) is replaced by

2') There exist functions $\chi \in \mathcal{K}_\infty$ and $\underline{\varphi}, \overline{\varphi} \in \mathcal{P}$, such that for each $p \in \mathcal{M}$, there exist $\psi_p \in \mathcal{P}$ and $\varphi_p \in \mathcal{P} \cup -\mathcal{P}$ with $\underline{\varphi}(s) \le |\varphi_p(s)| \le \overline{\varphi}(s)\ \forall s \in \mathbb{R}_0^+$, such that for all inputs $u \in U_c$ and all solutions $x(t) = x(t; t_0, x_0, u, \sigma)$ of Σ, the inequalities

$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_\sigma(t, x(t)) \le \varphi_{\sigma(t)}(V_\sigma(t, x(t))) + \chi(\|u\|_\infty), \quad t \in I \setminus S, \tag{8}$$

$$V_\sigma(t_i, x_i^+) \le \psi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x(t_i^-))\big) + \chi(\|u\|_\infty), \quad t_i \in S, \tag{9}$$

hold true, where $x_i^+ = g_{\sigma(t_i^-)}(t_i^-, x(t_i^-), u(t_i^-))$.

Let us define the functions $\Phi_p, \underline{\Phi}, \overline{\Phi} \colon [0,\infty) \to \mathbb{R} \cup \{-\infty\}$ for $p \in \mathcal{M}$ as

$$\Phi_p(v) := \int_1^v \frac{1}{|\varphi_p(s)|}\, \mathrm{d}s,$$

$$\underline{\Phi}(v) := \int_1^v \frac{1}{\underline{\varphi}(s)}\, \mathrm{d}s,$$

$$\overline{\Phi}(v) := \int_1^v \frac{1}{\overline{\varphi}(s)}\, \mathrm{d}s.$$

Note that the functions $\Phi_p, \underline{\Phi}, \overline{\Phi}$ are all strictly increasing. Therefore, their inverses $\Phi_p^{-1} : \mathrm{im}(\Phi_p) \to [0,\infty)$, $\underline{\Phi}^{-1} \colon \mathrm{im}(\underline{\Phi}) \to [0,\infty)$ and $\overline{\Phi}^{-1} \colon \mathrm{im}(\overline{\Phi}) \to [0,\infty)$ exist. Furthermore, note that if (5) holds with a linear bound, i.e. $\frac{\mathrm{d}}{\mathrm{d}t} V_\sigma \le \lambda_\sigma V_\sigma$, then $\Phi_p(v) = \frac{\ln v}{|\lambda_p|}$.

Definition 7: Let Σ be a switched system with switching signal σ. Let V_σ be a candidate ISS-Lyapunov function for the system Σ with corresponding functions φ_p, ψ_p as in Definition 6 and let there exist a partition $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$, such that $-\varphi_p \in \mathcal{P}$ for $p \in \mathcal{S}$ and $\varphi_p \in \mathcal{P}$ for all $p \in \mathcal{U}$. Furthermore, assume that σ satisfies the MDADT condition (2) for all modes in $\mathcal{S}$ and the MDALT condition (3) for all modes in $\mathcal{U}$ with corresponding dwell/leave times $\{\tau_p\}_{p \in \mathcal{S}}$ and $\{\tau_p\}_{p \in \mathcal{U}}$. If there exists $\delta \in (0,1)$ such that for all $a > 0$ and every switching time $t_i \in S$, one of the following two inequalities is satisfied:

$$\Phi_{\sigma(t_i)}(\psi_{\sigma(t_i^-)}(a)) - \Phi_{\sigma(t_i^-)}(a) \le \tau_{\sigma(t_i^-)}(1-\delta), \tag{10}$$

if $\sigma(t_i^-) \in \mathcal{S}$, or

$$-\Phi_{\sigma(t_i)}(\psi_{\sigma(t_i^-)}(a)) + \Phi_{\sigma(t_i^-)}(a) \ge \tau_{\sigma(t_i^-)}(1+\delta), \tag{11}$$

if $\sigma(t_i^-) \in \mathcal{U}$, then we call V_σ a *generalized ISS-Lyapunov function*.

Remark 8: Intuitively, we aim at partitioning $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$ such that the flow of modes in $\mathcal{S}$ is stable and the flow of modes in $\mathcal{U}$ is unstable. Stability here is with respect to a chosen, possibly time-varying, candidate ISS Lyapunov function; in particular, for a different choice of candidate ISS Lyapunov function, the stability classification of modes may be different.

Remark 9: In constrast to classical ISS Lyapunov functions, which are assumed to decrease along solutions for a zero input, the notion of generalized ISS Lyapunov does not require a decrease along solutions. Therefore an alternative name could be "non-decreasing" ISS Lyapunov function (cf. [28]). However, by definition every "classical" decreasing ISS Lyapunov function would also be a "non-decreasing" one, which is obviously a confusing a notation and hence we use "generalized" ISS function instead. The terminology "generalized" ISS-Lyapunov function is different from the one used in [29], in which the term "generalized" refers to removing the regularity assumptions on the Lyapunov function.

Now we are ready to provide our result on ISS of the impulsive switched system Σ.

Theorem 10: Consider the system Σ for a given switching signal σ. If there exists a generalized ISS-Lyapunov function as given in Definition 7, then Σ is ISS.

Proof: For an arbitrary but fixed input signal $u \in U_c$, consider the set

$$A_1(t) := \{x \in X \mid V_\sigma(t,x) < \chi(\|u\|_\infty)\}.$$

Our proof consists of two steps. At first, we show that for all initial conditions x_0 outside $A_1(t_0)$, we have a convergent behavior towards $A_1(t)$, i.e., there exists a $\mathcal{KL}$-function β, such that the inequality

$$\|x(t; t_0, x_0, u, \sigma)\|_X \le \beta(\|x_0\|_X, t - t_0) \tag{12}$$

holds true for all $t \in [t_0, t_*)$, where $t_* > t_0$ is such that $x(t) \notin A_1(t)$ for all $t \in [t_0, t_*)$. Second, we show that trajectories, once they have reached the set A_1, will stay bounded.

Step 1: Let $x_0 \in X$ be fixed and consider its trajectory x. Let $t_* = \inf\{t \in [t_0, \infty] \mid x(t) \in A_1(t)\}$, i.e., $V_\sigma(t, x(t)) \geq \chi(\|u\|_\infty)$ holds for all $t \in [t_0, t_*)$ and hence (5) and (6) are satisfied on $[t_0, t_*)$. For brevity, define $v(t) := V_\sigma(t, x(t))$ and denote $v_i := v(t_i)$ and $v_i^- := v(t_i^-)$. Let

$$t' := \begin{cases} \min\{t \in [t_0, t_*) \mid v(t) = 0\}, & \text{if such a } t \text{ exists.,} \\ t_*, & \text{else.} \end{cases}$$

In the following, we will show that (12) holds for all $t \in [t_0, t')$. If $t' < t_*$, then it follows from positive definiteness of V_σ that $x(t') = 0$. Furthermore, from $0 = v(t') = V_\sigma(t', x(t')) \geq \xi(\|u\|_\infty)$, it follows that u must be identically zero, which then implies that $x(t) = 0$ for all $t \geq t'$. Consequently, any extension of a $\mathcal{KL}$-function β in (12) onto the interval $[t_0, t_*)$ also makes (12) true on the whole interval $[t_0, t_*)$. Now we consider, the behavior on $[t_0, t')$ on which $v(t) \neq 0$. Then for $p \in \mathcal{M}$, the inequality (5) becomes

$$\frac{\frac{\mathrm{d}}{\mathrm{d}t} v(t)}{|\varphi_p(v(t))|} \leq \frac{\varphi_p(v(t))}{|\varphi_p(v(t))|} = \begin{cases} 1, & \text{if } \varphi_p(v(t)) > 0, \\ -1, & \text{if } \varphi_p(v(t)) < 0, \end{cases} \tag{13}$$

for all $t \in I_p^\sigma \cap [t_0, t')$.

Now, we will estimate $\Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0)$ to conclude that it is bounded by a class $\mathcal{KL}$ function. To this aim, we first estimate the behavior between the switching instants and at the switching instants, separately.

Integrating (13) over the interval $[t_i, \hat{t}]$, $i \in \mathbb{N}_0$, for some $\hat{t} \in [t_i, t_{i+1}) \cap [t_0, t')$, we obtain

$$\int_{v_i}^{v(\hat{t})} \frac{1}{|\varphi_p(s)|}\, \mathrm{d}s = \int_{t_i}^{\hat{t}} \frac{\frac{\mathrm{d}}{\mathrm{d}t} v(t)}{|\varphi_p(v(t))|}\, \mathrm{d}t \leq \begin{cases} -(\hat{t} - t_i), & \text{if } p \in \mathcal{S}, \\ \hat{t} - t_i, & \text{if } p \in \mathcal{U}, \end{cases}$$

where we used the integration parameter change $s \to v(t)$. It follows that

$$\Phi_p(v(\hat{t})) - \Phi_p(v_i) \leq \begin{cases} -T_p^\sigma(t_i, \hat{t}), & \text{if } p \in \mathcal{S}, \\ T_p^\sigma(t_i, \hat{t}), & \text{if } p \in \mathcal{U}. \end{cases} \tag{14}$$

For the switching instants $t_i \in [t_0, t')$, the inequality

$$\begin{aligned} &\Phi_{\sigma(t_i)}(v_i) - \Phi_{\sigma(t_i^-)}(v_i^-) \\ &\leq \Phi_{\sigma(t_i)}(\psi_{\sigma(t_i^-)}(v_i^-)) - \Phi_{\sigma(t_i^-)}(v_i^-) \leq \tau_{\sigma(t_i^-)}(1 - \delta) \end{aligned} \tag{15}$$

holds for $\sigma(t_i^-) \in \mathcal{S}$. Here, we used (6) in the first inequality and (10) in the second. Analogously, from (11), we obtain

$$\begin{aligned} &\Phi_{\sigma(t_i)}(v_i) - \Phi_{\sigma(t_i^-)}(v_i^-) \\ &\leq \Phi_{\sigma(t_i)}(\psi_{\sigma(t_i^-)}(v_i^-)) - \Phi_{\sigma(t_i^-)}(v_i^-) \leq -\tau_{\sigma(t_i^-)}(1 + \delta) \end{aligned} \tag{16}$$

for $\sigma(t_i^-) \in \mathcal{U}$.

Let $n = n(t) := \sum_{p \in M} N_p^\sigma(t_0, t)$. With the estimates (14), (15), and (16) at hand, we obtain

$$\begin{aligned} &\Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0) \\ &= \Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(t_n)}(v_n) \\ &\quad + \sum_{i=1}^{n} \Big(\Phi_{\sigma(t_i)}(v_i) - \Phi_{\sigma(t_i^-)}(v_i^-) \\ &\qquad + \Phi_{\sigma(t_i^-)}(v_i^-) - \Phi_{\sigma(t_{i-1})}(v_{i-1}) \Big) \\ &\leq \sum_{p \in \mathcal{S}} \left(N_p^\sigma(t_0, t) \tau_p (1 - \delta) - T_p^\sigma(t_0, t) \right) \\ &\quad + \sum_{p \in \mathcal{U}} \left(-N_p^\sigma(t_0, t) \tau_p (1 + \delta) + T_p^\sigma(t_0, t) \right), \end{aligned} \tag{17}$$

where we separated the stable and the unstable modes, and took into account that $\sigma(t_{i-1}) = \sigma(t_i^-)$. The first sum can be bounded by the MDADT condition given in Definition 3 as

$$\begin{aligned} &\sum_{p \in \mathcal{S}} \left(N_p^\sigma(t_0, t) \tau_p (1 - \delta) - T_p^\sigma(t_0, t) \right) \\ &= (1 - \delta) \left(\sum_{p \in \mathcal{S}} \left(N_p^\sigma(t_0, t) \tau_p - T_p^\sigma(t_0, t) \right) \right) - \delta \sum_{p \in \mathcal{S}} T_p^\sigma(t_0, t) \\ &\leq (1 - \delta) T_{\mathcal{S}} - \delta \sum_{p \in \mathcal{S}} T_p^\sigma(t_0, t) \end{aligned} \tag{18}$$

for $p \in \mathcal{S}$ and a constant $T_{\mathcal{S}} \geq 0$; and for $p \in \mathcal{U}$ the MDALT condition in Definition 4 implies for a constant $T_{\mathcal{U}} \geq 0$ that

$$\begin{aligned} &\sum_{p \in \mathcal{U}} \left(-N_p^\sigma(t_0, t) \tau_p (1 + \delta) + T_p^\sigma(t_0, t) \right) \\ &= (1 + \delta) \left(\sum_{p \in \mathcal{U}} \left(-N_p^\sigma(t_0, t) \tau_p + T_p^\sigma(t_0, t) \right) \right) - \delta \sum_{p \in \mathcal{U}} T_p^\sigma(t_0, t) \\ &\leq (1 + \delta) T_{\mathcal{U}} - \delta \sum_{p \in \mathcal{U}} T_p^\sigma(t_0, t). \end{aligned} \tag{19}$$

From (17), (18), and (19), it follows that

$$\begin{aligned} &\Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0) \\ &\leq (1 - \delta) T_{\mathcal{S}} - \delta \sum_{p \in \mathcal{S}} T_p^\sigma(t_0, t) + (1 + \delta) T_{\mathcal{U}} - \delta \sum_{p \in \mathcal{U}} T_p^\sigma(t_0, t) \\ &= -\delta(t - t_0) + (1 - \delta) T_{\mathcal{S}} + (1 + \delta) T_{\mathcal{U}}. \end{aligned} \tag{20}$$

This means that $\Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0)$ is linearly decreasing in t for $t \in [t_0, t')$.

As $\underline{\varphi}(x) \leq |\varphi_p(x)| \leq \overline{\varphi}(x)$, it holds that

$$\underline{\Phi}(k) - \underline{\Phi}(l) \geq \Phi_p(k) - \Phi_q(l) \geq \overline{\Phi}(k) - \overline{\Phi}(l) \tag{21}$$

for all $p, q \in \mathcal{M}$ and all $k \geq l$, $k, l \in \mathbb{R}_0^+$. Then from (20), we obtain

$$\begin{aligned} \underline{\Phi}(v(t)) - \underline{\Phi}(v_0) &\leq \Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0) \\ &\leq -\delta(t - t_0) + C \qquad \text{if } v(t) \leq v_0 \end{aligned} \tag{22}$$

$$\begin{aligned} \overline{\Phi}(v(t)) - \overline{\Phi}(v_0) &\leq \Phi_{\sigma(t)}(v(t)) - \Phi_{\sigma(0)}(v_0) \\ &\leq -\delta(t - t_0) + C \qquad \text{if } v(t) \geq v_0, \end{aligned} \tag{23}$$

where $C := (1 - \delta) T_{\mathcal{S}} + (1 + \delta) T_{\mathcal{U}}$.

Next, we distinguish between two cases: Case 1: For the case $\inf(\operatorname{im} \underline{\Phi}) = m > -\infty$, we set $\overline{\Phi}^{-1}(s) = 0$ if $s \leq \inf(\operatorname{im} \overline{\Phi})$, and define

$$\tilde{\beta}(r, s) := \max\left\{ \underline{\Phi}^{-1}(\Gamma(\underline{\Phi}(r), \delta s)), \overline{\Phi}^{-1}\left(\overline{\Phi}(r) + C - \delta s\right) \right\},$$

where

$$\Gamma(u,v) := u + C - (u+C-m)\left(1-\exp\left(\frac{-v}{u+C-m}\right)\right) \geq u + C - v.$$

Case 2: If $\inf(\operatorname{im}\underline{\Phi}) = -\infty$, we define

$$\tilde{\beta}(r,s) := \max\Big\{\underline{\Phi}^{-1}(\underline{\Phi}(r)+C-\delta s), \overline{\Phi}^{-1}(\overline{\Phi}(r)+C-\delta s)\Big\}.$$

We made this case distinction because we want $\tilde{\beta}$ to be strictly falling to zero in the second argument. Note that $\tilde{\beta}$ is strictly increasing in the first argument.
Thus, from (22) and (23), it follows that

$$v(t) \leq \tilde{\beta}(v_0, t-t_0)$$

for all $t \in [t_0, t')$.
Obviously, $\tilde{\beta}$ is continuous and strictly decreasing to zero for $s \to \infty$ by the definition of $\overline{\Phi}$ and $\underline{\Phi}$. However, it is a priori not clear if $\tilde{\beta}$ is bounded, and therefore, we cannot conclude that $\tilde{\beta} \in \mathcal{KL}$. Note that for $t - t_0 \geq \tau := \frac{C}{\delta}$, it follows that $\tilde{\beta}(r, t-t_0) \leq r$. It remains to show that we can bound $v(t) = V_\sigma(t, x(t))$ for $t - t_0 \leq \tau$.
To this end, from Lemma 27, for $V_{\sigma(t_0)}(t_0, x(t_0)) \geq \chi(\|u\|_\infty)$, i.e.,

$$\|u\|_\infty \leq \chi^{-1}(V_{\sigma(t_0)}(t_0, x(t_0))) \leq \chi^{-1}(\alpha_2(\|x_0\|_X)),$$

it follows that

$$\max_{t\in[t_0,\tau]} \|x(t)\|_X \leq L(\|x_0\|_X, \chi^{-1}(\alpha_2(\|x_0\|_X)), \tau, \sigma)\,\|x_0\|_X.$$

Therefore, for $t \in [t_0, t')$ and $t - t_0 \leq \tau$, $\|x(t;t_0,x_0,u,\sigma)\|$ can be bounded by a $\mathcal{K}_\infty$-function in x_0, uniformly in $t - t_0$. Hence, there exists $\beta \in \mathcal{KL}$ defined by $\beta(r,s) := \alpha_1^{-1}\Big(\tilde{\beta}(\alpha_2(r), s)\Big)$, such that

$$\|x(t;t_0,x_0,u,\sigma)\|_X \leq \beta(\|x_0\|_X, t-t_0) \tag{24}$$

for $t \in [t_0, t')$ and $t - t_0 \geq \tau$ (and by the above argument then also on $[t_0, t_*]$). Note that the constructed β is actually independent of t_0, i.e. the same bound can be used also on a later time interval $[t_1, t_2)$ with $t_2 > t_1 > t_0$, such that $x(t) \notin A_1(t)$ for all $t \in [t_1, t_2)$.
Step 2: Next, we show that trajectories that reach to $A_1(t)$ at any time $t \in I$, stay bounded for all times. We define the sets

$$\begin{aligned} A_2(t) &:= \{x \in X \mid V_\sigma(t,x) \leq \gamma_2(\|u\|_\infty)\}, \\ A_3(t) &:= \{x \in X \mid V_\sigma(t,x) \leq \gamma_3(\|u\|_\infty)\}, \end{aligned}$$

where $\gamma_2, \gamma_3 \in \mathcal{K}_\infty$ and are defined by

$$\begin{aligned} \gamma_2(s) &= \max\{\alpha_3(s), \chi(s)\}, \\ \gamma_3(s) &= \max\big\{\gamma_2(s), \alpha_2\big(\beta(\alpha_1^{-1}(\gamma_2(s)), 0)\big)\big\}. \end{aligned}$$

Obviously, $A_1(t) \subseteq A_2(t) \subseteq A_3(t)$ for all $t \in [t_0, \infty)$, $p \in \mathcal{M}$. Trajectories leaving $A_1(t)$ by flow have to cross the boundary $\partial A_1(t)$ and trajectories leaving $A_1(t)$ by jump only reach to $A_2(t)$ due to condition (6). In both cases, there exists a time $\tilde{t} \in [t, \infty)$, such that $x\big(\tilde{t}\big) \subseteq A_2\big(\tilde{t}\big) \setminus A_1\big(\tilde{t}\big)$. Therefore, we can apply (24) combined with (4), where $t = t_0 = \tilde{t}$. As a consequence, all the trajectories that leave $A_1(t)$ will stay in $A_3(t)$.
Next, we define $\gamma \in \mathcal{K}_\infty$, $\gamma := \alpha_1^{-1} \circ \gamma_3$. Then $\|x(t;t_0,x_0,u,\sigma)\|_X \leq \gamma(\|u\|_\infty)$ holds for all $t > t_*$. From this equation and (24), we can conclude

$$\|x(t;t_0,x_0,u,\sigma)\|_X \leq \beta(\|x_0\|_X, t-t_0) + \gamma(\|u\|_\infty),$$

just as desired. ∎

Remark 11: Note that in the case of an impulsive system with only one stable flow and one unstable jump, condition (10) becomes

$$\int_a^{\psi(a)} \frac{1}{-\varphi(s)}\,\mathrm{d}s \leq \tau(1-\delta),$$

where $\varphi_p = \varphi$, $\psi_p = \psi$ as there is only one mode $p \in \mathcal{S}$. Conversely, for the case of an impulsive system with only one unstable flow and one stable jump, i.e., a single mode $p \in \mathcal{U}$, condition (11) reduces to

$$\int_a^{\psi(a)} \frac{1}{-\varphi(s)}\,\mathrm{d}s \geq \tau(1+\delta).$$

Thus, the conditions (10) and (11), for the case of an impulsive system, boils down to the dwell time conditions in [12].
Finally, let us discuss the case that the rate functions $\varphi_p(s)$ and $\psi_p(s)$ are linear, i.e., $\varphi_p(s) = \eta_p \cdot s$, $\eta_p \in \mathbb{R} \setminus \{0\}$, and $\psi_p(s) = \mu_p \cdot s$, $\mu_p > 0$. Then, it follows that

$$\Phi_p(v) = \int_1^v \frac{1}{|\varphi_p(s)|}\,\mathrm{d}s = \int_1^v \frac{1}{|\eta_p|\,s}\,\mathrm{d}s = \frac{1}{|\eta_p|}\ln v.$$

Therefore, in the case $p \in \mathcal{S}$, (10) reduces to

$$\begin{aligned} \frac{\ln(\tilde{\mu}_{\sigma(t_i^-)})}{\left|\eta_{\sigma(t_i^-)}\right|} &= \frac{1}{\left|\eta_{\sigma(t_i)}\right|}\ln(\mu_{\sigma(t_i^-)}\cdot a) - \frac{1}{\left|\eta_{\sigma(t_i^-)}\right|}\ln(a) \\ &\leq \tau_{\sigma(t_i^-)}(1-\delta) < \tau_{\sigma(t_i^-)}, \end{aligned}$$

where we define $\tilde{\mu}_{\sigma(t_i^-)} := \mu_{\sigma(t_i^-)} e^{\left|\eta_{\sigma(t_i^-)}\right| - \left|\eta_{\sigma(t_i)}\right|}$. Conversely for $p \in \mathcal{U}$, from (11), it follows that

$$\begin{aligned} -\frac{\ln(\tilde{\mu}_{\sigma(t_i^-)})}{\left|\eta_{\sigma(t_i^-)}\right|} &= -\frac{1}{\left|\eta_{\sigma(t_i)}\right|}\ln(\mu_{\sigma(t_i^-)}\cdot a) + \frac{1}{\left|\eta_{\sigma(t_i^-)}\right|}\ln(a) \\ &\geq \tau_{\sigma(t_i^-)}(1+\delta) > \tau_{\sigma(t_i^-)}. \end{aligned}$$

Thus, the conditions (10) and (11), for the case of a switched system and generalized ISS-Lyapunov functions with linear rates, boils down to the dwell time conditions in [25].

IV. Sufficient and necessary condition for ISS

In this section, we first introduce a more restrictive characterization of ISS-Lyapunov functions, i.e., a decreasing ISS-Lyapunov function as defined below:
Definition 12: Let V_σ be a candidate ISS-Lyapunov function. If for each $p \in \mathcal{M}$, it holds that $\varphi_p \in -\mathcal{P}$ and $\psi_p \leq \mathrm{id}$, we call V_σ a *(decreasing) ISS-Lyapunov function*, which we denote by W_σ.
Then, in the following, we prove that the existence of such a time-varying decreasing ISS-Lyapunov function is not only a

sufficient but also a necessary condition for ISS of the system Σ.

Corollary 13: If there exists an ISS-Lyapunov function for the system Σ for a given switching signal σ as in Definition 12, then Σ is ISS.

Proof: By the definition of generalized ISS-Lyapunov functions in Definition 7, it holds that $\mathcal{S} = \mathcal{M}$. Inequality (10) is trivially fulfilled because $\psi_{\sigma(t_i^-)}(a) \leq a$ and $\Phi_{\sigma(t_i)}$ is increasing for each t_i, and therefore, the left-hand side of (10) is less than or equal to 0. So, we can choose $\tau_p = 0$ for each $p \in \mathcal{M}$ and as a consequence, (2) is always fulfilled. This completes the proof. ■

Next, we will provide a converse Lyapunov theorem as follows:

Theorem 14: Let the system Σ with a given switching signal σ be ISS. Then, there exists a decreasing ISS-Lyapunov function for the system Σ.

Proof: Considering a fixed switching signal σ, we define

$$\begin{aligned} \widetilde{f}(t,x,u) &:= f_{\sigma(t)}(t,x,u), \qquad && t \in I \setminus S, \\ \widetilde{g}_i(x,u) &:= g_{\sigma(t_i^-)}(t_i^-,x,u), \qquad && t_i \in S,\ i \in \mathbb{N} \end{aligned}$$

for all $(x,u) \in X \times U$. Note that for every constants $C, D > 0$, functions $\widetilde{f}$ and $\widetilde{g}$ are locally Lipschitz continuous with respect to x for $u \in B_{U_c}(D), t \in I \setminus S$, and locally Lipschitz continuous with respect to u for $x \in B_X(C), i \in \mathbb{N}$. Then, we can treat the impulsive switched system Σ as impulsive system

$$\widetilde{\Sigma}\colon \begin{cases} \dot{x}(t) = \widetilde{f}(t,x(t),u(t)), & t \in I \setminus S, \\ x(t_i^+) = \widetilde{g}_i(x(t_i^-),u(t_i^-)), & t_i \in S,\ i \in \mathbb{N}. \end{cases}$$

Then, we can employ [14, Theorem 2] to conclude that ISS of Σ implies the existence of a decreasing ISS-Lyapunov function as given in Definition 12. ■

Lemma 15: Consider the system Σ for a given switching signal σ. Then, every decreasing ISS-Lyapunov function as given in Definition 12 for Σ is also a generalized ISS-Lyapunov function as in Definition 7.

Proof: Let V_σ be a decreasing Lyapunov function. By Definition 12, $\mathcal{M} = \mathcal{S}$, and we can choose $\varphi_p = \underline{\varphi}$ for all $p \in \mathcal{M}$. Then, by using Definition 12 and $\Phi_p = \underline{\Phi}$ for all $p \in \mathcal{M}$, where $\underline{\Phi}$ is increasing, it holds that

$$\begin{aligned} \Phi_{\sigma(t_i)}(\psi_{\sigma(t_i^-)}(a)) - \Phi_{\sigma(t_i^-)}(a) &\leq \underline{\Phi}(a) - \underline{\Phi}(a) \\ &= 0 \leq \tau_{\sigma(t_i^-)}(1-\delta), \end{aligned}$$

i.e., (10) is satisfied for each $(\tau_p)_{p \in \mathcal{M}}$ and $\delta \in (0,1)$. Hence, V_σ is a generalized ISS-Lyapunov function. ■

Remark 16: Since ISS implies the existence of a decreasing ISS-Lyapunov, it follows that ISS implies the existence of a generalized ISS-Lyapunov.

V. Construction of ISS-Lyapunov functions

Now, let us assume that the generalized ISS-Lyapunov function as given in Definition 7 is in fact decreasing along trajectories. We will show that, in contrast to Theorem 10, it is not necessary to impose the dwell/leave time conditions (2) and (3) in order to conclude ISS from the existence of an ISS-Lyapunov function. That is why we will show, in the following, how to construct a (decreasing) ISS-Lyapunov function from a large class of generalized ones, provided the switching signal satisfies the corresponding dwell/leave time conditions of Theorem 10. This construction is motivated by the facts that (i) the existence of such a decreasing ISS-Lyapunov function is a necessary and sufficient condition for ISS, (ii) the level sets of decreasing ISS-Lyapunov functions directly indicate the reachable sets, and (iii) it facilitates the computation of β which, in turn, determines the convergence rate.

Theorem 17: Let V_σ be a generalized ISS-Lyapunov functions for the system Σ with switching signal σ and φ_p, ψ_p and τ_p for $p \in \mathcal{M}$ defined as in Definition 7. Let $\operatorname{im}(\overline{\Phi}) = \mathbb{R}$.

Then, an ISS-Lyapunov function for the system Σ is given by

$$W_\sigma(t,x) = \exp\big(\Phi_{\sigma(t)}(V_\sigma(t,x)) + h_\sigma(\sigma(t),t)\big),$$

where

$$\begin{aligned} &h_\sigma(p,t) \\ &= \min_{j \in \{0,\dots,i\}} \Bigg\{ 0, \left(\sum_{p \in \mathcal{S}} \big(T_p^\sigma(t_j,t) - \tau_p N_p^\sigma(t_j^-,t)\big) \right)(1-\delta) \\ &\qquad - \left(\sum_{p \in \mathcal{U}} \big(T_p^\sigma(t_j,t) - \tau_p N_p^\sigma(t_j,t)\big) \right)(1+\delta) \Bigg\} \end{aligned}$$

for $t \in [t_i, t_{i+1})$, $i \in \mathbb{N}_0$. Even more, W_σ has linear and mode-independent decay rate $\varphi_p = -\delta \cdot \mathrm{id}$ for all $p \in \mathcal{M}$ and some $\delta > 0$.

Before we prove Theorem 17, we analyze the properties of the function h_σ:

Lemma 18: Consider the system Σ with switching signal σ and let h_σ be the function defined in Theorem 17. Then, the map $H\colon I \to \mathbb{R}, t \mapsto h_\sigma(\sigma(t),t)$ has the following properties:

1) Range: For all $t \in I$, it holds that
$$H(t) \in [-T_{\mathcal{S}}(1-\delta) - T_{\mathcal{U}}(1+\delta), 0].$$
2) Continuity from the right: The map H is continuous on the time interval $[t_i, t_{i+1})$ for all $i \in \mathbb{N}_0$.
3) Differentiability: On the interval $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$, H is almost everywhere differentiable, i.e.,
$$\tfrac{\mathrm{d}}{\mathrm{d}t} H(t) \in \begin{cases} \{0, 1-\delta\}, & \text{if } \sigma(t) \in \mathcal{S}, \\ \{-(1+\delta)\}, & \text{if } \sigma(t) \in \mathcal{U}. \end{cases}$$
4) Discontinuities: At the time instant t_i, $i \in \mathbb{N}$, H has a jump of size
$$H(t_i) - H(t_i^-) \in \begin{cases} \left\{-\tau_{\sigma(t_i^-)}(1-\delta)\right\}, & \text{if } \sigma(t_i^-) \in \mathcal{S}, \\ \left[0, \tau_{\sigma(t_i^-)}(1+\delta)\right], & \text{if } \sigma(t_i^-) \in \mathcal{U}. \end{cases}$$

Proof: See Appendix IX-B for the proof. ■

Before providing a formal proof of Theorem 17, we first highlight that the construction of W is based on the following three steps: (i) Translate the convergence of V into a linear growth/decay by applying Φ_p, (ii) rectifying a potentially positive growth with a negative linear decay encoded by h, and (iii) transforming to the exponential growth/decay domain with

exp. Next, we discuss the intuition behind the construction of the decreasing Lyapunov function from a generalized one.
To this end, we consider a system with uniform fixed switching instants $(t_i)_{i\in\mathbb{N}}$ as given in the example below.
Example 19: Let us consider the following system with two modes and state space $X = \mathbb{R}$ given by $\dot{x} = (-0.5 + (-0.7)^{\sigma(t)})x$, $x(t_i^+) = \exp\big(-0.35 \cdot (-1)^{\sigma(t)}\big)$ for modes $p = 1 \in \mathcal{S}$, $p = 2 \in \mathcal{U}$, $S = \{0.5, 1, 1.5, 2, 2.5, 3, \dots\}$, $\sigma(t) = 2$ on $[t_0, t_2) \cup [t_5, t_6)$ and $\sigma(t) = 1$ on $[t_2, t_5)$. The trajectory for initial condition $x_0 = 1$, $V_\sigma(t, x) = |x|$ and $W_\sigma(t, x) = \exp(\ln|x| + h_\sigma(\sigma(t), t))$ with a piecewise linear function h_σ is depicted in Figure 2.

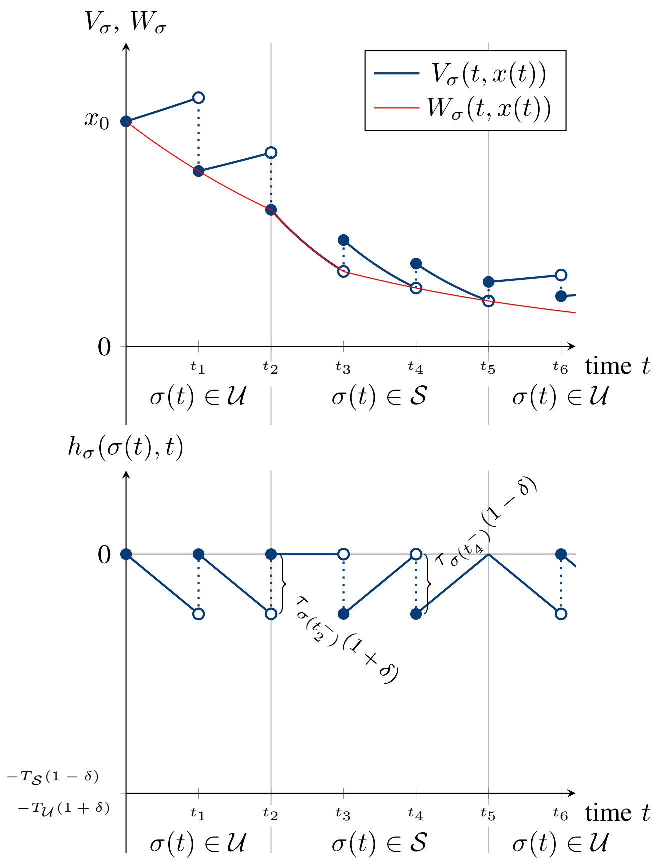

Fig. 2. Construction of a decreasing ISS-Lyapunov function $W_\sigma(t, x(t))$ from a generalized ISS-Lyapunov function $V_\sigma(t, x(t))$ and their relation and according plot of function $h_\sigma(p, t)$ for the system given in Example 19.

Remark 20: Note that for an unstable mode $p \in \mathcal{U}$, the Lyapunov value of the generalized ISS-Lyapunov function V_σ might be increasing during the flow, but in general, the concluding jumps are stabilizing the overall behavior. Therefore, when the trajectory is a priori known, one can connect the point at the beginning of the unstable interval with the point at the next jump instant to obtain a strictly decreasing ISS-Lyapunov function W_σ. As the trajectory is not a priori known, this effect is implicitly determined by the MDALT Condition (3) and Condition (11), which V_σ has to satisfy. The interpolation is encrypted in h_σ falling during the flow times.
For a stable mode $p \in \mathcal{S}$, the scenario is analogous. Note that the decreasing ISS-Lyapunov function W_σ is always smaller than the generalized ISS-Lyapunov function V_σ. For a stable mode $p \in \mathcal{S}$, the Lyapunov value of the generalized ISS-Lyapunov function V_σ is falling faster than the successive (unstable) jump. Therefore, when the trajectory is a priori known, one can interpolate the minimum points before the stable interval with the point directly before the succeeding jump to obtain a strictly decreasing ISS-Lyapunov function W_σ. As the trajectory is, in general, not a priori known, the next jump height has to be estimated, which is implicitly done by the MDADT Condition (2) and Condition (10), which V_σ has to satisfy. The balance of the jumps is encrypted in a function h_σ which is falling at the jump instants.
Observe that specifically for Example 19 on the interval $[t_5, t_6)$, W_σ is strictly smaller than V_σ. This is due to the delayed effect that W_σ has to be smaller than the generalized ISS-Lyapunov function V_σ on interval $[t_2, t_3)$.
Now, to consider a more general system which satisfies an average dwell-time condition.
Example 21: Let us consider the following system with two modes and state space $X = \mathbb{R}$ given by $\dot{x} = (-0.5 + (-1)^{\sigma(t)})x$, $x(t_i^+) = \exp\big(-0.1 - 0.3 \cdot (-1)^{\sigma(t)}\big)$ for modes $p = 1 \in \mathcal{S}$, $p = 2 \in \mathcal{U}$, $S = \{0.2, 0.3, 0.9, 1.0, 1.2, 1.3, 1.9, 2.0, 2.3, \dots\}$. The trajectory for initial condition $x_0 = 1$, $V_\sigma(t, x) = |x|$ and $W_\sigma(t, x) = \exp(\ln|x| + h_\sigma(\sigma(t), t))$ with h_σ is depicted in Figure 3.
As can be seen in Figure 3, additionally to estimating the jump heights, we also have to estimate the switching times. For this, a correction term h as depicted in Figure 3 has to be introduced, which measures if the next jump is already overdue or already too many jumps have occurred as compared to the τ_p in the MDADT and MDALT conditions (2) and (3), respectively. These terms appear piecewise linearly in W_σ because Φ_σ maps the Lyapunov values to sub-linear functions.

Proof of Theorem 17: Let us verify the conditions in Definition 6 for W_σ to be an ISS-Lyapunov function.
Condition 1: From Lemma 18, 1), it follows for $(t, x, u) \in (I \setminus S) \times X \times U$ that

$$\begin{aligned} &W_\sigma(t, x) \\ &\geq \exp\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) - T_{\mathcal{S}}(1-\delta) - T_{\mathcal{U}}(1+\delta)\big) \\ &\geq \exp\big(\Phi_{\sigma(t)}(\alpha_1(\|x\|_X)) - T_{\mathcal{S}}(1-\delta) - T_{\mathcal{U}}(1+\delta)\big) \\ &\geq \exp\big(\min\{\overline{\Phi}, \underline{\Phi}\}(\alpha_1(\|x\|_X)) - T_{\mathcal{S}}(1-\delta) - T_{\mathcal{U}}(1+\delta)\big) \\ &=: \tilde{\alpha}_1(\|x\|_X)\,. \end{aligned}$$

From (21) together with $\operatorname{im}(\overline{\Phi}) = \mathbb{R}$ it follows that $\tilde{\alpha}_1$ is a $\mathcal{K}_\infty$-function. Furthermore, we have

$$\begin{aligned} W_\sigma(t, x) &\leq \exp\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) + 0\big) \\ &\leq \exp\big(\max\{\overline{\Phi}, \underline{\Phi}\}(V_\sigma(t, x))\big) \\ &\leq \exp\big(\max\{\overline{\Phi}, \underline{\Phi}\}(\alpha_2(\|x\|_X))\big) =: \tilde{\alpha}_2(\|x\|_X)\,. \end{aligned}$$

Hence, since h is piecewise continuous by Lemma 18, 2), Condition 1 is shown if t is not a switching instant.
If t is a switching instant, then we have to consider two cases. If $\sigma(t^-) \in \mathcal{S}$, then we have

$$\begin{aligned} &W_\sigma(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u)) \\ &= \exp\Big(\Phi_{\sigma(t_i)}\Big(V_\sigma\Big(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u)\Big)\Big) + h_\sigma(\sigma(t_i), t_i)\Big) \end{aligned}$$

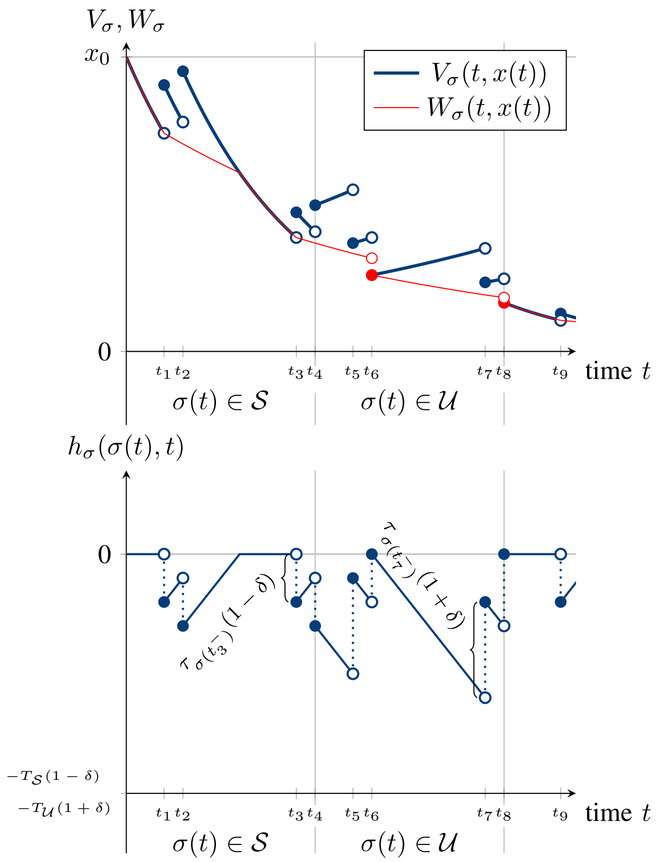

Fig. 3. Construction of a decreasing ISS-Lyapunov function $\boldsymbol{W_\sigma(t, x(t))}$ from a generalized ISS-Lyapunov function $\boldsymbol{V_\sigma(t, x(t))}$ and their relation for the system given in Example 21.

$$\begin{aligned}
&\le \exp\Big(\Phi_{\sigma(t_i)}\Big(\psi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x)\big)\Big) \\
&\qquad\qquad + h_\sigma(\sigma(t_i^-), t_i^-) - \tau_{\sigma(t_i^-)}(1-\delta)\Big) \\
&\le \exp\Big(\Phi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x)\big) + h_\sigma(\sigma(t_i^-), t_i^-)\Big) \\
&= W_\sigma(t_i^-, x), \qquad (25)
\end{aligned}$$

where we used Lemma 18, 4) in the second step and (10) in the third step.

Now, if $\sigma(t^-) \in \mathcal{U}$, then we have

$$\begin{aligned}
&W_\sigma(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u)) \\
&= \exp\Big(\Phi_{\sigma(t_i)}\Big(V_\sigma\Big(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u)\Big)\Big) + h_\sigma(\sigma(t_i), t_i)\Big) \\
&\le \exp\Big(\Phi_{\sigma(t_i)}\Big(\psi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x)\big)\Big) \\
&\qquad\qquad + h_\sigma(\sigma(t_i^-), t_i^-) + \tau_{\sigma(t_i^-)}(1+\delta)\Big) \\
&\le \exp\Big(\Phi_{\sigma(t_i^-)}\big(V_\sigma(t_i^-, x)\big) + h_\sigma(\sigma(t_i^-), t_i^-)\Big) \\
&= W_\sigma(t_i^-, x), \qquad (26)
\end{aligned}$$

where we used Lemma 18, 4) in the second step and the inequality (11) in the third step.

Condition 2: Let $\widetilde{\chi} = \chi$ and $W_\sigma(t, x) \ge \widetilde{\chi}(\|u\|_\infty)$. Then it holds that $V_\sigma(t, x) \ge W_\sigma(t, x) \ge \widetilde{\chi}(\|u\|_\infty)$.

Part a: We bound the flow behavior for $t \in I \setminus S$.

Let us first consider the case $\sigma(t) \in \mathcal{S}$. It follows

$$\begin{aligned}
\dot W_\sigma(t, x) &\le \exp\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) + h_\sigma(\sigma(t), t)\big) \\
&\qquad \times \Big(\tfrac{\dot V_\sigma(t,x)}{-\varphi_{\sigma(t)}(V_\sigma(t,x))} + 1\cdot(1-\delta)\Big) \\
&\le -\delta \cdot \exp\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) + h_\sigma(\sigma(t), t)\big) \\
&= -\delta \cdot W_\sigma(t, x),
\end{aligned}$$

Here, we used Lemma 18, 2) in the first step, and (13), and $\sigma(t^-) = \sigma(t)$ in the second step.

In the case $\sigma(t) \in \mathcal{U}$, we obtain with the help of Lemma 18, 2) that

$$\begin{aligned}
\dot W_\sigma(t, x) &\le (\exp)'\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) + h_\sigma(\sigma(t), t)\big) \\
&\qquad \times \Big(\tfrac{\dot V_\sigma(t,x)}{\varphi_{\sigma(t)}(V_\sigma(t,x))} - 1\cdot(1+\delta)\Big) \\
&\le -\delta \cdot \exp\big(\Phi_{\sigma(t)}(V_\sigma(t, x)) + h_\sigma(\sigma(t), t)\big) \\
&= -\delta \cdot W_\sigma(t, x).
\end{aligned}$$

Therefore, for $\widetilde{\varphi}_p := -\delta \cdot \mathrm{id}$ for $p \in M$, the inequality $\dot W_\sigma(t, x) \le \widetilde{\varphi}_{\sigma(t)}(W_\sigma(t, x))$ holds for all $t \in I \setminus S$, i.e., $\widetilde{\varphi}_p$ can even be chosen to be linear.

Part b: Inequalities (25) and (26) verify for all $t_i \in S$ the desired jump behavior

$$W_\sigma(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u(t_i^-))) \le W_\sigma(t_i^-, x). \qquad (27)$$

Condition 3: For $W_\sigma(t_i^-, x) < \widetilde{\chi}(\|u\|_\infty)$, either $V_\sigma(t_i^-, x) < \chi(\|u\|_\infty)$ or $V_\sigma(t_i^-, x) \ge \chi(\|u\|_\infty)$ holds. In the former case, we have

$$\begin{aligned}
W_\sigma(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u(t_i^-))) &\le V_\sigma(t_i^-, g_{\sigma(t_i^-)}(t_i^-, x, u(t_i^-))) \\
&\le \alpha_3(\|u\|_\infty),
\end{aligned}$$

while in the latter case the estimate (27) implies

$$W_\sigma(t_i, g_{\sigma(t_i^-)}(t_i^-, x, u(t_i^-))) \le W(t_i^-, x) \le \widetilde{\chi}(\|u\|_\infty).$$

Hence, we can choose $\widetilde{\alpha}_3 \in \mathcal{K}$ as

$$\widetilde{\alpha}_3(a) := \max\{\alpha_3(a), \widetilde{\chi}(a)\}.$$

This concludes the proof. ■

VI. Robustness with Respect to the Switching Signal

The ISS result in Theorem 10 applies to a system Σ under a predefined switching signal, where a generalized ISS-Lyapunov function tailored to that specific sequence imposes a sufficient condition for ISS. In practice, however, switching signals are rarely known in advance but are often uncertain or subject to perturbations. Consequently, results that rely on a fixed, predetermined signal have limited practical relevance. To address this, we aim to establish ISS guarantees that hold uniformly over an entire class of admissible switching signals. In particular, we consider sets of switching signals Ω in which only the allowable mode transitions are prescribed, while the exact switching instants and the order of modes remain unspecified. To formalize this, we define $\mathcal{Q} \subseteq \mathcal{M} \times \mathcal{M}$ as the set of pairs (p, q) corresponding to the allowed mode switchings, i.e., $p = \sigma(t_i)$ and $q = \sigma(t_i^-)$ for any $\sigma \in \Omega$.

More formally, let a *set of mode switchings* be given by $\mathcal{Q} \subseteq \mathcal{M} \times \mathcal{M}$. We define

$$\Omega(\mathcal{Q}) := \left\{\text{switching signal } \sigma \,|\, \forall t \in S\colon (\sigma(t), \sigma(t^-)) \in \mathcal{Q}\right\}. \tag{28}$$

Using the same notation as for system Σ but with time-invariant f_p and g_p, we now restrict the class of switching signals to $\sigma \in \Omega(\mathcal{Q})$ and denote the resulting impulsive switched system as

$$\widehat{\Sigma}\colon \begin{cases} \dot{x}(t) = f_{\sigma(t)}(x(t), u(t)), & t \in I \setminus S, \\ x(t_i^+) = g_{\sigma(t_i^-)}\big(x(t_i^-), u(t_i^-)\big), & t_i \in S. \end{cases}$$

Note that to obtain a robust ISS result with respect to sets of switching signals associated with $\mathcal{Q}$, we cannot rely on the generalized ISS-Lyapunov functions V_σ from Definition 7. These functions are tailored to a single switching sequence and are built from components $(\widetilde{V}_p)_{p\in\mathcal{M}}$ that are only defined on the intervals I_p^σ where the corresponding mode p is active. To overcome this limitation, we now restrict the functions $(\widetilde{V}_p)_{p\in\mathcal{M}}$ to be time-invariant, ensuring that each $\widetilde{V}_p$ remains valid beyond its active interval and can be applied consistently across all admissible switching signals. This modification is essential: it allows the construction of Lyapunov functions that are independent of the precise switching instants or ordering, thereby enabling ISS conclusions for all signals in $\Omega(\mathcal{Q})$. In this sense, the following corollary provides a robust extension of Theorem 10 and explicitly captures robustness with respect to the set of switching signals in $\Omega(\mathcal{Q})$.

Corollary 22: Let $\mathcal{Q}$ be a set of mode switchings, and consider the impulsive switched system $\widehat{\Sigma}$. Let $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$. Let $\widetilde{V}_p \in \mathcal{C}(X, \mathbb{R}_0^+)$ for $p \in M$, such that all of the following conditions hold true:

1) There exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that
$$\alpha_1(\|x\|_X) \le \widetilde{V}_p(x) \le \alpha_2(\|x\|_X)$$
holds true for all $t \in I$, all $x \in X$ and for each $p \in \mathcal{M}$.
2) There exist functions $\chi \in \mathcal{K}_\infty$ and $\underline{\varphi}, \overline{\varphi} \in \mathcal{P}$, such that for each $(p,q) \in \mathcal{Q}$, there exist $\psi_p \in \mathcal{P}$ and φ_p, such that $-\varphi_p \in \mathcal{P}$ for $p \in \mathcal{S}$, and $\varphi_p \in \mathcal{P}$ for all $p \in \mathcal{U}$ with $\underline{\varphi}(s) \le |\varphi_p(s)| \le \overline{\varphi}(s)$ $\forall s \in \mathbb{R}_0^+$, such that for all inputs $u \in U$ and all $x \in X$, the inequality
$$\left(\tfrac{\mathrm{d}}{\mathrm{d}x}\widetilde{V}_p(x)\right)^T \cdot f_p(x,u) \le \varphi_p\left(\widetilde{V}_p(x)\right)$$
holds true, whenever $\widetilde{V}_p(x) \ge \chi(\|u\|_\infty)$, and
$$\widetilde{V}_p(x^+) \le \psi_q\left(\widetilde{V}_p(x)\right)$$
holds true, whenever $\widetilde{V}_q(x) \ge \chi(\|u\|_\infty)$, where $x^+ = g_q(x,u)$.
3) There exists a function $\alpha_3 \in \mathcal{K}$, such that for all $x \in X$, all $u \in U_c$, which satisfy $\widetilde{V}_q(x) < \chi(\|u\|_\infty)$, the jump rule satisfies
$$\widetilde{V}_p(x^+) \le \alpha_3(\|u\|_\infty).$$

Let every switching signal $\sigma \in \Omega(\mathcal{Q})$ satisfy the MDADT condition (2) for all modes in $\mathcal{S}$ and the MDALT condition (3) for all modes in $\mathcal{U}$ with corresponding dwell/leave times $\{\tau_p\}_{p\in\mathcal{S}}$ and $\{\tau_p\}_{p\in\mathcal{U}}$. Let there exist a $\delta > 0$ such that for all $a > 0$, one of the following two inequalities is satisfied:

$$\Phi_p(\psi_q(a)) - \Phi_q(a) \le \tau_q(1-\delta),$$

if $q \in \mathcal{S}$, or

$$-\Phi_p(\psi_q(a)) + \Phi_q(a) \ge \tau_q(1+\delta),$$

if $q \in \mathcal{U}$.

Then, $\widehat{\Sigma}$ is ISS, and the function $V : I \times X \to \mathbb{R}_0^+ : V_\sigma(t,x) := \widetilde{V}_{\sigma(t)}(x)$ for all $(t,x) \in I \times X$ defines a candidate ISS-Lyapunov function.

Proof: For fixed σ, V_σ is a generalized ISS-Lyapunov function. The claim then follows from Theorem 10. ■

Definition 23: Let Ω be a set of switching signals, and $\widehat{\Sigma}$ be a switched system. Let $V : \Omega \times I \times X \to \mathbb{R}_0^+$ be a function, such that for each switching signal $\sigma \in \Omega$, we have that $V_\sigma(t,x) := V(\sigma,t,x) = \widetilde{V}_{\sigma(t)}(x)$ for all $(\sigma,t,x) \in \Omega \times I \times X$. Then, we call $\{V_\sigma\}_{\sigma\in\Omega}$ a *family of candidate ISS-Lyapunov functions*.

Remark 24: Note that, if the constants for forward-completeness $K = K(C,D,\tau,\sigma)$ and δ are independent of σ, then β and γ can be chosen independently of σ, i.e., Σ is ISS uniformly with respect to $\sigma \in \Omega$.

The result in Corollary 22 provides a general sufficient condition for ISS under sets of switching signals. Since these conditions may be difficult to check directly in practice, we now specialize them to the case of linear impulsive switched systems. In this setting, we consider a family of candidate ISS-Lyapunov functions as in Definition 23, restricted to the quadratic form $\widetilde{V}_p(x) := x^T M_p x$ for some $M_p \in \mathbb{R}^{n\times n}$, $p \in \mathcal{M}$. This choice allows the inequalities in Corollary 22 to be expressed as LMIs, which are computationally tractable and therefore yield practically verifiable ISS conditions. In order to verify the LMIs numerically in practice, we only consider systems with finitely many modes p. We formalize this in the following theorem.

Theorem 25: Consider a linear impulsive switched system $\widehat{\Sigma}$:

$$\begin{aligned} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t), && t \in I \setminus S, \\ x(t_i) &= J_{\sigma(t_i^-)}x(t_i^-) + H_{\sigma(t_i^-)}u(t_i^-), && t_i \in S, \end{aligned}$$

where $A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, $J_p \in \mathbb{R}^{n\times n}$ and $H_p \in \mathbb{R}^{n\times m}$ for $p \in \mathcal{M}$ and let σ satisfy the mode switchings $\mathcal{Q}$.

If there exist symmetric positive definite matrices $M_p \in \mathbb{R}^{n\times n}, Q_p \in \mathbb{R}^{m\times m}$, $p \in \mathcal{M}$, and constants $\eta_p \in \mathbb{R}$ and $\mu_p > 0$, such that the LMIs

$$\begin{pmatrix} A_p^T M_p + M_p A_p - \eta_p M_p & M_p B_p \\ B_p^T M_p & -Q_p \end{pmatrix} \le 0, \tag{29}$$

$$\begin{pmatrix} J_q^T M_p J_q - \mu_q M_q & J_q^T M_p H_q \\ H_q^T M_p J_q & H_q^T M_p H_q - Q_q \end{pmatrix} \le 0 \tag{30}$$

are satisfied for all $(p,q) \in \mathcal{Q}$, then $\widehat{\Sigma}$ is ISS for each switching signals σ satisfying the MDADT condition (2) for all modes in $\mathcal{S}$ and the MDALT condition (3) for all modes in $\mathcal{U}$ with corresponding dwell/leave times $\{\tau_p\}_{p\in\mathcal{S}}$ and $\{\tau_p\}_{p\in\mathcal{U}}$ with

$$\eta_q < 0 \quad \wedge \quad \frac{\ln \mu_q}{|\eta_p|} \le \tau_q(1-\delta), \qquad \forall q \in \mathcal{S}. \tag{31}$$

$$\eta_q \geq 0 \quad \wedge \quad -\frac{\ln \mu_q}{|\eta_p|} \geq \tau_q(1+\delta), \qquad \forall q \in \mathcal{U}. \quad (32)$$

Moreover, a generalized ISS-Lyapunov function for $\widehat{\Sigma}$ is given by $\widetilde{V}_\sigma(x) = x^T M_{\sigma(t)} x$.

Remark 26: Note that (29)–(30) becomes (analytically) feasible for large enough η_p's and large enough μ_p's, $p \in \mathcal{M}$ and infeasible for small η_p's and small μ_p's, $p \in \mathcal{M}$. Now with intervals for which (29)–(30) turns feasible at hand, one can then optimize the admissible dwell/leave times $(\tau_p)_{p\in\mathcal{M}}$ in (31)–(32) by approximating the parameters η_p, μ_p, $p \in \mathcal{M}$ with a bisectional approach. For a more systematic approach, see [30]. Since we are optimizing for multiple dwell/leave-times $(\tau_p)_{p\in\mathcal{M}}$, it might not be possible to optimize all of them simultaneously. In general, one might get "better" τ_p for some $p \in \mathcal{M}$ when in turn τ_q, $q \neq p$ gets more conservative.

Proof: Let (31) and (32) be satisfied. We show that $\widetilde{V}_p(x) = x^T M_p x$ defines a generalized ISS-Lyapunov function in dissipation form. Let $\chi(s) := \lambda s^2$ for $s \in \mathbb{R}_0^+$, where λ is the maximum eigenvalue of Q_p for all $p \in \mathcal{M}$. Then, the time derivative of $\widetilde{V}_p$ along the solutions of the continuous dynamics of $\widehat{\Sigma}$ satisfies

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\widetilde{V}_p(x) = \dot{x}^T M_p x + x^T M_p \dot{x}\\
&= (A_p x + B_p u)^T M_p x + x^T M_p (A_p x + B_p u)\\
&= x^T A_p^T M_p x + x^T M_p A_p x + u^T B_p^T M_p x + x^T M_p B_p u\\
&\leq \eta_p x^T M_p x + u^T Q_p u\\
&\leq \eta_p \widetilde{V}_p(x) + \chi(\|u\|_\infty)
\end{aligned}$$

for all $x \in X$, $u \in U$, where we used (29) in the fourth step. For the jump dynamics of $\widehat{\Sigma}$, we have

$$\begin{aligned}
\widetilde{V}_p(x) &= x^T M_p x\\
&= (J_q x^- + H_q u^-)^T M_p (J_q x^- + H_q u^-)\\
&= (x^-)^T J_q^T M_p J_q x^- + (u^-)^T H_q^T M_p J_q x^-\\
&\quad + (x^-)^T J_q^T M_p H_q u^- + (u^-)^T H_q^T M_p H_q u^-\\
&\leq \mu_q (x^-)^T M_q x^- + (u^-)^T Q_q u^-\\
&\leq \mu_q \widetilde{V}_q(x^-) + \chi(\|u\|_\infty)
\end{aligned}$$

for all $x^- \in X$, $u^- \in U$, such that $x = J_q x^- + H_q u^-$. Note that we applied (30) in the fourth step.

Therefore, $\widetilde{V}_\sigma$ defines a generalized Lyapunov function in dissipation form with rates $\varphi_p(s) = \eta_p s$ and $\psi_p(s) = \mu_p s$ for $s \in \mathbb{R}_0^+$. Note that (31) and (32) correspond to (10) and (11) (cf. Remark 11). Then, by Lemma 28 and Corollary 22, $\widehat{\Sigma}$ is ISS. ∎

VII. Illustration

This section showcases the effectiveness of our results. The intuitive and easy-to-follow linear system example compares our results with the relevant state of the art. The second example applies our results to a nonlinear system.

A. Comparison with the relevant state of the art

Let us investigate the two-dimensional linear impulsive switched system on $X = U = \mathbb{R}^2$ with Euclidean norm with two modes $p \in \mathcal{M} = \{1, 2\}$,

$$\Sigma_{\text{lin}}: \begin{cases} \dot{x}(t) = A_{\sigma(t)} x(t) + u(t), & \sigma(t) = p,\ t \in \mathbb{R}_0^+ \setminus S,\\ x(t) = J x(t^-) + u(t^-), & t \in S, \end{cases}$$

where $A_p := \left(\begin{smallmatrix} p & 0\\ 0 & -p\end{smallmatrix}\right)$, $J := \left(\begin{smallmatrix} \frac{1}{2} & 0\\ 0 & \frac{3}{2}\end{smallmatrix}\right)$.

First, we consider a zero input for comparative analysis of our results with the literature and later on we add the input to verify ISS. Let us consider a single mode $\sigma(t) \equiv p \in \mathcal{M}$. In direction of the first unit vector, the system with $u \equiv 0$ reduces to

$$\begin{aligned} \dot{x}_1(t) &= p x_1(t), && t \in \mathbb{R}_0^+ \setminus S,\\ x_1(t) &= \tfrac{1}{2} x_1(t^-), && t \in S. \end{aligned} \qquad (33)$$

Obviously, the flow is unstable and the jumps are stable. The solution of the system is given by

$$x_1(t) = \left(\tfrac{1}{2}\right)^{N_p^\sigma(0,t)} e^{pt} x_0.$$

By analysis of the limit and considering (11), one can see that system (33) is stable as long as $\tau_p < \frac{1}{p}\ln 2$.

Analogously, we investigate the system with zero input induced by the second unit vector

$$\begin{aligned} \dot{x}_2(t) &= -p x_2(t), && t \in \mathbb{R}_0^+ \setminus S,\\ x_2(t) &= \tfrac{3}{2} x_2(t^-), && t \in S. \end{aligned} \qquad (34)$$

The solution here is given by

$$x_2(t) = \left(\tfrac{3}{2}\right)^{N_p^\sigma(0,t)} e^{-pt} x_0.$$

Therefore, the jumps are unstable, but the flow stabilizes the entire system for $\tau_p > \frac{1}{p}\ln\frac{3}{2}$.

This means system Σ_1 with zero input is in fact asymptotically stable for average dwell-time

$$\tau_p \in \left(\tfrac{1}{p}\ln\left(\tfrac{3}{2}\right), \tfrac{1}{p}\ln 2\right). \qquad (35)$$

Note that for $p \in \{1, 2\}$, we obtain the condition $\tau_1 \in \left(\ln\left(\frac{3}{2}\right), \ln 2\right)$ and $\tau_2 \in \left(\frac{1}{2}\ln\left(\frac{3}{2}\right), \frac{1}{2}\ln 2\right)$, respectively, where $\frac{1}{2}\ln 2 < \ln\left(\frac{3}{2}\right)$. As a consequence, there does not exist a uniform mode-independent dwell-time condition and the results of [18] cannot be applied to conclude ISS of system Σ_{lin} for switching signals in which both modes $p = 1$ and $p = 2$ appear. Hence, it is inevitable to consider *mode-dependent average dwell-time/leave-time* to conclude stability of this system.

Note also that the approaches in [18] and [19] are not able to conclude ISS of system Σ_{lin} as both the flow and the jumps are simultaneously unstable. This is due to the fact that the approaches therein rely on time-invariant Lyapunov functions, which is a restriction to conclude ISS of this class of systems. However, we propose time-varying Lyapunov functions in the spirit of [13] and [14] to cover impulsive switched systems with simultaneous instability of flow and jumps.

Let us consider a time-varying candidate Lyapunov function for system Σ_{lin} given by, for $p = 1 \in \mathcal{S}$,

$$V_1(t, x) := x^T J^{4(t-t_i)} x,\ t \in I_1^\sigma, \qquad (36)$$

and, for $p = 2 \in \mathcal{U}$ and some $\varepsilon > 0$,

$$V_2(t, x) = x^T (J + \varepsilon I)^{2\frac{t-t_i}{t_{i+1}-t_i}} x,\ t \in I_2^\sigma, \qquad (37)$$

where in both cases t_i (and t_{i+1}) are the corresponding switching times of σ such that $t \in [t_i, t_{i+1})$ and $(J+\varepsilon I)^\theta$ is defined for $\theta \in \mathbb{R}$ as $\exp(\ln(J+\varepsilon I)\cdot\theta)$, where $\ln(J+\varepsilon I)$ is well defined because $J+\varepsilon I$ is a diagonal matrix with positive entries. Here we assume additional dwell and leave time conditions of the form $T_p^{\min} \le t_{i+1}-t_i \le T_p^{\max}$, $p=1,2$, which are required in order to find uniform bounds $\alpha_1, \alpha_2, \varphi_p$, and ψ_p according to Definition 6. These restrictions do not follow from the dwell- and leave-time constraints (10) and (11) but are consequences of our choice of the Lyapunov function. Furthermore, we conjecture that for system Σ_{lin}, Lyapunov functions with average dwell- and leave-time conditions require knowledge about the history of the switching signal. However, this requires further investigation.

It can be shown (for the details see Appendix IX-C) that the Lyapunov function defined by (36) and (37) indeed satisfies Definition 6 with

$$\alpha_1(s) = C_1 s^2, \qquad C_1 = \min\Big\{\big(\tfrac{1}{2}\big)^{4T_1^{\max}}, \big(\tfrac{1}{2}+\varepsilon\big)^2\Big\},$$
$$\alpha_2(s) = C_2 s^2, \qquad C_2 = \max\Big\{\big(\tfrac{3}{2}\big)^{4T_1^{\max}}, \big(\tfrac{3}{2}+\varepsilon\big)^2\Big\},$$
$$\alpha_3(s) = \left(\tfrac{3}{2\lambda}\sqrt{\tfrac{C_2}{C_1}}+1\right)^2 s^2,$$
$$\psi_1(s) = \mu_1 s,$$
$$\mu_1 = \left(\max\Big\{\big(\tfrac{1}{2}\big)^{-2T_1^{\max}+1}, \big(\tfrac{3}{2}\big)^{-2T_1^{\min}+1}\Big\}+\lambda\right)^2,$$
$$\varphi_1(s) = \eta_1 s, \qquad \eta_1 = 4\ln\big(\tfrac{3}{2}\big)-2+\lambda,$$
$$\psi_2(s) = \mu_2 s, \qquad \mu_2 = \big(1+\tfrac{2}{3}(\varepsilon-\lambda)\big)^{-2},$$
$$\varphi_2(s) = \eta_2 s,$$
$$\eta_2 = \max\Big\{\tfrac{2}{T_2^{\max}}\ln\big(\tfrac{1}{2}+\varepsilon\big)+4, \tfrac{2}{T_2^{\min}}\ln\big(\tfrac{3}{2}+\varepsilon\big)-4\Big\}+\lambda,$$
$$\chi(s) = \tfrac{1}{\lambda^2}\alpha_2(s).$$

Choosing $T_1^{\min}=0.4$, $T_1^{\max}=0.55$, $T_2^{\min}=0.21$, $T_2^{\max}=0.22$ and $\varepsilon=0.25$, we see (taking Remark 11 into account) that the dwell-time condition (10) become $\tau_1 > \frac{\ln(\mu_1)}{|\eta_1|} \approx 0.435$ and $\tau_2 < -\frac{\ln(\mu_2)}{|\eta_2|} \approx 0.222$.

Note that the results of [17] and [20] can only treat impulsive switched systems for which either all modes have stable flows, or all modes have unstable flows. In the present case, we have a mode with stable flow $p=1$ and a mode with unstable flow $p=2$. Therefore, our results cover cases where some modes can have stable flows, and others can have unstable flows.

B. Application to a nonlinear impulsive switched system

We consider the nonlinear system in $X=U=\mathbb{R}^2$ with Euclidean norm $\|\cdot\|$:

$$\Sigma_{\text{nl}}: \begin{cases} \dot{x}(t) = c_1(-x_1^4-x_2^2+1)\cdot x(t)+u(t), \\ \qquad\qquad \sigma(t)=1,\ t\in\mathbb{R}_0^+\setminus S, \\ \dot{x}(t) = c_2(-\|x\|^8-\|x\|^4+1)x+Ax(t)+u(t), \\ \qquad\qquad \sigma(t)=2,\ t\in\mathbb{R}_0^+\setminus S, \\ x(t) = \frac{x(t^-)}{\sqrt{\|x(t^-)\|}}, \qquad \sigma(t)=1,\ t\in S, \\ x(t) = c_3\begin{pmatrix} x_1(t^-)\cdot\|x(t^-)\| \\ (x_2(t^-)\cdot\|x(t^-)\|)^4\end{pmatrix}, \quad \sigma(t)=2,\ t\in S,\end{cases}$$

where $A=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$ and $c_1,c_2,c_3\in(0,1]$ are constants to be defined later. Note that none of the flow and jump dynamics is globally stable.

For simplicity, we consider an alternating mode sequence $(p_i)_{i\in\mathbb{N}_0}=(1,2,1,2,\dots)$ with constant dwell-times $\tau_2=\frac{\pi}{2}$ and τ_1 to be defined later.

A generalized ISS-Lyapunov function is given by

$$V_1(t,x) := (x_1^4+x_2^2)e^{-4c_1(t-t_i)},$$
$$V_2(t,x) := c_4\Big(\big(\cos(t-t_i)x_1+\sin(t-t_i)x_2\big)^4\|x\|^4 + \big(-\sin(t-t_i)x_1+\cos(t-t_i)x_2\big)^2\|x\|^2\Big)^2 e^{-4c_2(t-t_i)},$$

for $t\in[t_i,t_{i+1})$, $x\in X$.

Note that for

$$\alpha_1(s) := \min\Big\{e^{-4c_1\tau_1}\min\{s,s^2\}^2, c_4e^{-4c_2\tau_2}\min\{s,s^2\}^8\Big\},$$
$$\alpha_2(s) := \max\Big\{e^{-4c_1\tau_1}\max\{s,s^2\}^2, c_4e^{-4c_2\tau_2}\max\{s,s^2\}^8\Big\},$$

(4) is satisfied.

Next, we define $\chi\in\mathcal{K}_\infty$,

$$\chi(s) := \alpha_2\Big(\max\Big\{\tfrac{2s}{\lambda}, \big(\tfrac{s}{2\lambda}\big)^{\frac{1}{13}}\Big\}\Big) \tag{38}$$

for some sufficiently small $\lambda\in(0,\min\{1,c_1,c_2\})$. Then, it holds that

$$\dot{V}_1(t,x) \le -2(c_1-\lambda)e^{4c_1(t-t_i)}V_1^2(t,x),$$
$$\dot{V}_2(t,x) \le -4\tfrac{c_2-\lambda}{c_4}e^{4c_2(t-t_i)}V_2^2(t,x),$$

for $t\in\mathbb{R}_0^+\setminus S$, $x\in X$ and some $c_4>0$, where we denote $\theta=t-t_i$. For the detailed computation of $\dot{V}_1,\dot{V}_2$, see Appendix IX-D. Furthermore, with $\tau_2=\theta=\frac{\pi}{2}$, we obtain by Appendix IX-D

$$V_1(t_i,x^+) \le \tfrac{c_3^2}{c_4}e^{2c_2\pi}V_2(t_i^-,x),$$

for $t=t_i\in S$, $x\in X$, $x_1^+=\frac{1}{2}x_1(t^-)\cdot\|x(t^-)\|$ and $x_2^+=\frac{1}{2}(x_2(t^-)\cdot\|x(t^-)\|)^4$, and

$$V_2(t_i,x^+) = c_4e^{8c_1\tau_1}V_1^2(t_i^-,x)$$

holds for $t=t_i\in S$, $x\in X$ and $x^+=\frac{x(t^-)}{\sqrt{\|x(t^-)\|}}\in X$.

Therefore, the functions φ_p,ψ_p as required by Definition 6 can be chosen as

$$\varphi_1(s) = -2(c_1-\lambda)s^2, \qquad \varphi_2(s) = -4\tfrac{c_2-\lambda}{c_4}s^2,$$
$$\psi_1(s) = c_4e^{8c_1\tau_1}s^2, \qquad \psi_2(s) = \tfrac{c_3^2}{c_4}e^{2c_2\pi}s,$$

and

$$\Phi_1(s) = -t\tfrac{1}{2c_1s}+\tfrac{1}{2c_1}, \qquad \Phi_2(s) = -\tfrac{c_4}{4c_2s}+\tfrac{c_4}{4c_2}.$$

We choose $c_4=2\frac{c_2-\lambda}{c_1-\lambda}$. Then, by (10), the system is stable, if there exists some $\delta>0$ such that for all $a>0$, it holds that

$$-\tfrac{1}{2(c_1-\lambda)c_4e^{8c_1\tau_1}a^2}+\tfrac{1}{2(c_1-\lambda)a} \le \tau_1(1-\delta),$$
$$\tfrac{1}{2(c_1-\lambda)a}\Big(-\tfrac{c_4}{c_3^2e^{2c_2\pi}}+1\Big) \le \tfrac{\pi}{2}(1-\delta).$$

Hence, as

$$\max_{a>0}\Big\{-\tfrac{1}{2(c_1-\lambda)c_4e^{8c_1\tau_1}a^2}+\tfrac{1}{2(c_1-\lambda)a}\Big\} = \tfrac{c_2-\lambda}{4(c_1-\lambda)^2}e^{8c_1\tau_1}$$

and the second equation can only be satisfied, if the term in brackets is non-positive, we have

$$c_2 = 4\frac{(c_1-\lambda)^2}{e^{8c_1\tau_1}} \max_{a>0}\left\{-\frac{1}{2(c_1-\lambda)c_4 e^{8c_1\tau_1}a^2} + \frac{1}{2(c_1-\lambda)a}\right\} + \lambda$$
$$< 4(c_1-\lambda)^2 e^{-8c_1\tau_1}\tau_1 + \lambda,$$
$$c_3 \le \sqrt{2\frac{c_2-\lambda}{c_1-\lambda}}e^{-2c_2\pi}.$$

For appropriate parameters c_1, c_2, c_3, τ_1 and sufficiently small $\lambda \in (0, \min\{1, c_1, c_2\})$, system Σ_{nl} is ISS.

VIII. Conclusion

We provided necessary and sufficient ISS conditions for impulsive switched systems that have modes with both stable and unstable flows. To achieve this, we used time-varying ISS-Lyapunov functions with nonlinear rate functions, along with MDADT and MDALT conditions. We also presented a method for constructing decreasing ISS-Lyapunov functions from a large class of generalized ones, which is an important and useful result in its own right. Additionally, we provided a method to guarantee ISS for a particular class of impulsive switched systems with unknown switching signals.

One promising direction for future research is the development of an ISS-Lyapunov function-based control design for impulsive switched systems, particularly in the presence of external disturbances. Moreover, extending the analysis presented in this work to impulsive switched systems with random time-varying delays or stochastic noise represents another valuable research avenue. Furthermore, as parameter δ can be chosen smaller, it is possible to relax the condition on τ_p slightly. Therefore, analogously to [31], it is interesting to investigate in which cases the system's asymptotic stability and stability (in Lyapunov sense) is robust with respect to a violation of the dwell-time condition. Another important direction for future work is the application of the proposed theoretical framework to realistic system models and practical engineering case studies.

IX. Appendix

A. Technical Lemmas

Lemma 27: Consider the system Σ for a given switching signal σ. Let f_p be locally Lipschitz continuous in the second argument, uniformly for all $t \in I$ and $p \in \mathcal{M}$, and $u \in B_{U_c}(D)$ for some constant $D > 0$. Let the function g_p be locally Lipschitz continuous in the second variable, uniformly with respect to $t \in I$ and $p \in \mathcal{M}$, and $u, v \in B_{U_c}(D)$. Then, on compact time intervals, uniform for all switching sequences, the solutions of Σ are locally Lipschitz continuous with respect to the initial values, i.e., for all $C > 0$, $x_0, y_0 \in B_X(C)$, and for $\tau > t_0$, there exists a constant $L = L(C, D, \tau, \sigma)$, such that for $x(t) = x(t; t_0, x_0, u, \sigma)$ and $y(t) = x(t; t_0, y_0, v, \sigma)$, it holds that

$$\max_{t\in[t_0,\tau]} \|x(t) - y(t)\|_X \le L(C, D, \tau, \sigma)\|x_0 - y_0\|_X.$$

Proof: Let $C > 0$, $\tau > t_0$, $u \in B_{U_c}(D)$, and $x_0, y_0 \in B_X(C)$. Let us denote the Lipschitz constant for $f_p = f_p(t, x, u)$ and $g_p = g_p(t, x, u)$, for $\|x\| \le C$ and $\|u\| \le C$, by $L_f = L_f(C)$ and $L_g(C)$, respectively.

Let $t \in [t_0, t_1)$ be such that $t \le \tau$. Then, we have

$$x(t) = x_0 + \int_{t_0}^{t} f_{\sigma(s)}(s, x(s), u(s))\, \mathrm{d}s.$$

Let $x_i^- := x(t_i^-)$ and $x_i := x(t_i) = g_{\sigma(t_i^-)}(t_i^-, x_i^-, u(t_i^-))$ be such that

$$x(t) = x_i + \int_{t_i}^{t} f_{\sigma(s)}(s, x(s), u(s))\, \mathrm{d}s$$

holds true for all $t \in [t_i, t_{i+1})$, $t \le \tau$. From this, for any two solutions x and y of Σ with initial values x_0 and y_0, respectively, it follows that

$$\|x(t) - y(t)\|_X$$
$$\le \|x_i - y_i\|_X + \int_{t_i}^{t} L_f(K(C, D, \tau, \sigma))\,\|x(s) - y(s)\|_X\, \mathrm{d}s,$$

where $t \in [t_i, t_{i+1})$, $t \le \tau$, $y_i^- := y(t_i^-)$, $y_i := y(t_i) = g_{\sigma(t_i^-)}(t_i^-, y_i^-, v(t_i^-))$, and $K = K(C, D, \tau, \sigma)$, as introduced in (1). Using Gronwall's inequality, it follows that

$$\|x(t) - y(t)\|_X \le \|x_i - y_i\|_X\, e^{L_f(K)(t-t_i)}$$

for all $t \in [t_i, t_{i+1}), t \le \tau$. Moreover,

$$\|x(t_{i+1}) - y(t_{i+1})\|_X \le L_g(K)\left\|x(t_{i+1}^-) - y(t_{i+1}^-)\right\|_X$$
$$\le L_g(K)\,\|x_i - y_i\|_X e^{L_f(K)(t_{i+1}-t_i)},$$

where L_g is the Lipschitz constant. By induction, we obtain

$$\|x(t) - y(t)\|_X \le L_g^n(K)\, e^{(L_f(K))(t-t_0)}\,\|x_0 - y_0\|_X$$

for $t \in [t_n, t_{n+1})$, $t \le \tau$ and $n \in \mathbb{N}_0$.

As the switching sequence does not exhibit Zeno behavior, it follows that

$$\max_{t\in[t_0,\tau]} \|x(t) - y(t)\|_X$$
$$\le \|x_0 - y_0\|_X \max_{t\in[t_0,\tau]}\left\{L_g^{N(t_0,t)}(K)\, e^{L_f(K)(t-t_0)}\right\}.$$

By the extreme value theorem, the bound

$$L(C, D, \tau, \sigma)$$
$$:= \max_{t\in[t_0,\tau]}\left\{L_g^{N(t_0,t)}(K(C, D, \tau, \sigma))\, e^{L_f(K(C,D,\tau,\sigma))(t-t_0)}\right\}$$

exists and gives us a Lipschitz constant for the solutions of Σ with respect to the initial conditions. ■

Lemma 28: If a system Σ has a generalized ISS-Lyapunov function in dissipation form with linear rates $\widetilde{\varphi}_p(s) = \widetilde{\eta}_p s$, $\widetilde{\psi}_p(s) = \widetilde{\mu}_p s$, $s \in \mathbb{R}_0^+$ and $\widetilde{\chi}_p \in \mathcal{K}_\infty$, then it has a generalized ISS-Lyapunov function (in implication form) with linear rate function.

We can choose $\varphi_p(s) := \eta_p s$, $\psi_p(s) = \mu_p s$ for $s \in \mathbb{R}_0^+$, where $\eta_p = \frac{1-\delta}{1-\frac{3}{4}\delta}\widetilde{\eta}_p$ in case $p \in \mathcal{S}$, i.e., $\widetilde{\eta}_p < 0$, and $\eta_p = \frac{1+\delta}{1+\frac{3}{4}\delta}\widetilde{\eta}_p$ else, as well as $\mu_p = e^{-\frac{\delta\tau_p\eta_p}{4}}\widetilde{\mu}_p$ and $\chi_p(s) := \min\left\{\frac{1}{\eta_p-\widetilde{\eta}_p}, \frac{1}{\mu_p-\widetilde{\mu}_p}\right\} \cdot \widetilde{\chi}_p(s)$.

Proof: With the choices of $\varphi_p, \psi, \chi, \eta$, and μ from above follows that $V_p(x) \ge \chi_p(\|u\|_\infty)$. Then, it follows for each $p \in \mathcal{M}$ that

$$\dot{V}_p(x) \le \widetilde{\varphi}_p(V_p(x)) + \widetilde{\chi}_p(\|u\|_\infty)$$

$$\leq \widetilde{\eta}_p V_p(x) + (\eta_p - \widetilde{\eta}_p) V_p(x) = \eta_p V_p(x) = \varphi_p(V_p(x)),$$
$$V_p(x) \leq \widetilde{\psi}_p(V_p(x)) + \widetilde{\chi}_p(\|u\|_\infty)$$
$$\leq \widetilde{\mu}_p V_p(x) + (\mu_p - \widetilde{\mu}_p) V_p(x) = \mu_p V_p(x) = \psi_p(V_p(x)).$$

Furthermore, for $p \in \mathcal{S}$ and $\frac{\ln \tilde{\mu}_p}{|\tilde{\eta}_p|} \leq \tau_p(1-\delta)$, it holds that

$$\frac{\ln \mu_p}{|\eta_p|} = \frac{\ln\left(e^{\frac{\delta \tau_p |\eta_p|}{4}} \widetilde{\mu}_p\right)}{|\eta_p|} = \frac{\frac{\delta \tau_p |\eta_p|}{4}}{|\eta_p|} + \frac{\ln \widetilde{\mu}_p}{\frac{1-\delta}{1-\frac{3}{4}\delta} |\widetilde{\eta}_p|}$$
$$= \frac{\delta \tau_p}{4} + \tau_p \left(1 - \tfrac{3}{4}\delta\right) \leq \tau_p \left(1 - \delta'\right),$$

where $\delta' := \frac{\delta}{2}$. From this, it follows immediately that V_p is a generalized ISS-Lyapunov function in implication form. A similar procedure achieves the result for $p \in \mathcal{U}$. ■

B. Proof of Lemma 18

Property 1): By (2) and (3), it holds that

$$h_\sigma(\sigma(t), t) \geq \min_{j \in \{0,\ldots,i\}} \{0, -T_{\mathcal{S}}(1-\delta) - T_{\mathcal{U}}(1+\delta)\}.$$

Property 2): Except for σ and N_p^σ, all component functions of H are continuous. Therefore, on the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}_0$, where N_p^σ is constant, the function H is continuous.
Property 3): We fix $i \in \mathbb{N}_0$ and $t \in [t_i, t_{i+1})$ and assume that $H(t) < 0$. Let $j = j^*$ be such that the minimum in the definition of $h_\sigma(p, t)$ for $p = \sigma(t)$ is attained. Then, as $N_p^\sigma(t_{j^*}, \cdot)$ is constant on $[t_i, t_{i+1})$, it holds that $\frac{\mathrm{d}}{\mathrm{d}t} T_{\sigma(t)}^\sigma(t_{j^*}, t) = 1$ and if $\sigma(t) \in \mathcal{S}$, the equation $\frac{\mathrm{d}}{\mathrm{d}t} H(t) = 1 - \delta$ is true. Moreover, if $\sigma(t) \in \mathcal{U}$, we have $\frac{\mathrm{d}}{\mathrm{d}t} H(t) = -(1+\delta) < 0$ and since

$$H(t) \leq \left(T_p^\sigma(t_i, t) - \tau_p N_p^\sigma(t_i, t)\right)(1+\delta) \leq 0$$

holds, the case $H(\widetilde{t}) = 0$ for $\widetilde{t} \in [t, t+\varepsilon)$ for some $\varepsilon > 0$ can only be attained for $\sigma(t) \in \mathcal{S}$. Hence, $\frac{\mathrm{d}}{\mathrm{d}t} H(t) \in \{0, 1-\delta\}$ if $\sigma(t) \in \mathcal{S}$.
Property 4): We fix $i \in \mathbb{N}_0$ and consider $H(t_i) < 0$. Let $j = j^*$ be such that the minimum in the definition of $h_\sigma(p, t)$ for $p = \sigma(t)$ is attained. Then, for all $p \in \mathcal{M}$, $T_p^\sigma(t_{j^*}, \cdot)$ is continuous in t_i and for all $p \in \mathcal{M} \setminus \{\sigma(t)\}$, $N_p^\sigma(t_{j^*}, \cdot)$ is continuous in t_i. Thus, for $p = \sigma(t_{j^*}^-)$, we have

$$H(t_i) - H(t_i^-) = \begin{cases} -\tau_{\sigma(t_i^-)}(1-\delta), & \text{if } \sigma(t_i^-) \in \mathcal{S}, \\ \tau_{\sigma(t_i^-)}(1+\delta), & \text{if } \sigma(t_i^-) \in \mathcal{U}. \end{cases}$$

In the case $H(t_i) = 0$, which can only happen if $\sigma(t_i^-) \in \mathcal{U}$ (otherwise $H(t_i) - H(t_i^-) < 0$), the jump may be smaller, i.e., $0 \leq H(t_i) - H(t_i^-) \leq \tau_{\sigma(t_i^-)}(1+\delta)$. This completes the proof of the lemma.

C. Supplementary computations for Subsection VII-A

We show that (36) satisfies the requirements of Definition 6.

1) $V_1(t,x) = \left(\frac{1}{2}\right)^{4(t-t_i)} x_1^2 + \left(\frac{3}{2}\right)^{4(t-t_i)} x_2^2$, and therefore
$$\left(\tfrac{1}{2}\right)^{4T_1^{\max}} \|x\|^2 \leq V_1(t,x) \leq \left(\tfrac{3}{2}\right)^{4T_1^{\max}} \|x\|^2.$$
2) We choose $\lambda = \frac{1}{1000}$ and χ as given in Definition 6 such that $V_1(t,x) \geq \chi(\|u\|_\infty) = \frac{1}{\lambda^2}\left(\frac{3}{2}\right)^{4T_1^{\max}} \|u\|_\infty^2$, i.e., $\|u\| \leq \lambda \|x\|$. Then, it holds that
$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_1(t,x)$$
$$= 4x^T J^{4(t-t_i)} \ln(J) x + 2x^T J^{4(t-t_i)}(Ax+u)$$
$$= x^T J^{4(t-t_i)} \left[4\ln(J) + 2A + \lambda I\right] x.$$
Then, if
$$4\ln(J) + 2A + \lambda = \begin{pmatrix} -4\ln(2)+2+\lambda & 0 \\ 0 & 4\ln\left(\frac{3}{2}\right)-2+\lambda \end{pmatrix} \leq 0$$
is satisfied, we have
$$\tfrac{\mathrm{d}}{\mathrm{d}t} V_1(t,x) = x^T J^{4(t-t_i)} \left[4\ln(J) + 2A + \lambda I\right] x$$
$$\leq \left(4\ln\left(\tfrac{3}{2}\right) - 2 + \lambda\right) x^T J^{4(t-t_i)} x$$
$$= \eta_1 V_1(t,x) = \varphi_1(V_1(t,x))$$
for all initial conditions $x_0 \in X$ and all $t \in I \setminus S$, where $\eta_1 = 4\ln\left(\frac{3}{2}\right) - 2 + \lambda$ and $\varphi_1(s) := \eta_1 s$. Moreover, we have to calculate ψ_1 as defined in (6). We demonstrate the case of switching from mode 1 to mode $p = 1, 2$: V_1, V_2 fulfill
$$V_p(t_i, Jx+u) = (Jx+u)^T J^0 (Jx+u)$$
$$\leq x^T (J + \lambda I)^2 x$$
$$= x^T J^{2(t-t_i)} (J^{-2(t-t_i)+1} + \lambda I) J^{2(t-t_i)} x$$
$$\leq \mu_1 x^T J^{4(t-t_i)} x = \mu_1 V_1(t_i^-, x) = \psi_1(V_1(t_i^-, x)),$$
where $\mu_1 = \left(\max\left\{\left(\frac{1}{2}\right)^{-2T_1^{\max}+1}, \left(\frac{3}{2}\right)^{-2T_1^{\min}+1}\right\} + \lambda\right)^2$ and $\psi_1(s) := \mu_1 s$ which is exactly (6).
3) $V_1(t,x) < \chi(\|u\|_\infty)$ implies
$$\|x\| < \tfrac{1}{\lambda} \alpha_1^{-1} \circ \alpha_2(\|u\|_\infty) = C_3 \|u\|_\infty,$$
where $C_3 := \frac{1}{\lambda}\sqrt{\frac{C_2}{C_1}}$. Therefore, it holds that
$$V_p(t_i, Jx+u) = (Jx+u)^T J^0 (Jx+u)$$
$$\leq u^T (JC_1 + I)^2 u \leq \left(\tfrac{3C_3}{2} + 1\right)^2 \|u\|_\infty^2 =: \alpha_3(\|u\|_\infty).$$

We choose $T_1^{\min} = 0.4$, $T_1^{\max} = 0.55$. Then, with the considerations from Remark 11, dwell-time condition (10) become $\frac{\ln(\mu_p)}{|\eta_q|} < \tau_p$, where $p = \sigma(t_i^-)$, $q = \sigma(t_i)$, $i \in \mathbb{N}$. For switchings from mode $p = 1$ to itself, we then obtain $\tau_1 > \frac{\ln(\mu_1)}{|\eta_1|} \approx 0.435$.
Analogously, it can be verified that (37) fulfills the requirements of Definition 6 with $\mu_2 := \left(1 + \frac{2}{3}(\varepsilon - \lambda)\right)^{-2}$ and $\eta_2 := \max\left\{\frac{2}{T_2^{\max}} \ln\left(\frac{1}{2} + \varepsilon\right) + 4, \frac{2}{T_2^{\min}} \ln\left(\frac{3}{2} + \varepsilon\right) - 4\right\} + \lambda$. Choosing $\varepsilon = 0.25$, $T_2^{\min} = 0.21$ and $T_2^{\max} = 0.22$, we obtain $\tau_2 < -\frac{\ln(\mu_2)}{|\eta_2|} \approx 0.222$.

The restrictions on τ_1 and τ_2 in case of a mode change, i.e. $\tau_1 > \frac{\ln(\mu_1)}{|\eta_2|} \approx 0.118$, $\tau_2 < \frac{\ln(\mu_2)}{|\eta_1|} \approx 0.814$ are more relaxed than the ones calculated above. Therefore, we can conclude stability of Σ_1 via the existence of time-varying candidate Lyapunov functions (36) and (37) for every switching signal fulfilling the above restrictions.

D. Supplementary computations for Subsection VII-B

With (38) at hand, $V_i(t,x) \geq \chi(\|u\|_\infty)$ implies

$$4x_1^3 u_1 + 2x_2 u_2 \leq \lambda(4x_1^8 + 2x_2^4), \quad (39)$$
$$4y_1^3 \|x\|^4 u_1 + 2y_2 \|x\|^2 u_2 \leq \lambda(4y_1^4 \|x\|^{12} + 2y_2^2 \|x\|^6), \quad (40)$$

where

$$y_1 := \cos(\theta)x_1 + \sin(\theta)x_2,$$
$$y_2 := -\sin(\theta)x_1 + \cos(\theta)x_2.$$

Then, by using (39) and (40), it holds that

$$\begin{aligned}
&\dot{V}_1(t,x)\\
&= \left(4x_1^3\left(c_1(-x_1^4-x_2^2+1)x_1+u_1\right)\right.\\
&\quad +2x_2\left(c_1(-x_1^4-x_2^2+1)x_2+u_2\right)\\
&\quad \left.-4c_1x_1^4-4c_1x_2^2\right)e^{-4c_1(t-t_i)}\\
&\leq -2(c_1-\lambda)e^{4c_1(t-t_i)}V_1^2(t,x),\\
&\dot{V}_2(t,x)\\
&= 2c_4\Big((\cos(\theta)x_1+\sin(\theta)x_2)^4\left\|x\right\|^4\\
&\quad +(-\sin(\theta)x_1+\cos(\theta)x_2)^2\left\|x\right\|^2\Big)\\
&\times\Big(4(\cos(\theta)x_1+\sin(\theta)x_2)^3\left\|x\right\|^4\\
&\quad\times\Big(-\sin(\theta)x_1+\cos(\theta)x_2\\
&\qquad -\cos(\theta)\left(c_2(\left\|x\right\|^8+\left\|x\right\|^4-1)x_1-u_1\right)\\
&\qquad -\sin(\theta)\left(c_2(\left\|x\right\|^8+\left\|x\right\|^4-1)x_2-u_2\right)\\
&\qquad -\cos(\theta)x_2+\sin(\theta)x_1\Big)\\
&\quad +(\cos(\theta)x_1+\sin(\theta)x_2)^4 2x^Tx\cdot 2(-x_2,x_1)x\\
&\quad +2(-\sin(\theta)x_1+\cos(\theta)x_2)\left\|x\right\|^2\\
&\quad\times\Big(-\cos(\theta)x_1-\sin(\theta)x_2\\
&\qquad +\sin(\theta)\left(c_2(\left\|x\right\|^8+\left\|x\right\|^4-1)x_1-u_1\right)\\
&\qquad -\cos(\theta)\left(c_2(\left\|x\right\|^8+\left\|x\right\|^4-1)x_2-u_2\right)\\
&\qquad +\sin(\theta)x_2+\cos(\theta)x_1\Big)\\
&\quad +(-\sin(\theta)x_1+\cos(\theta)x_2)^2\cdot 2(-x_2,x_1)\times x\Big)\\
&\times e^{-4c_2(t-t_i)}-4c_2V_2(t,x)\\
&= -2(c_2-\lambda)c_4e^{-4c_2(t-t_i)}\Big(4(\cos(\theta)x_1+\sin(\theta)x_2)^4\left\|x\right\|^4\\
&\quad +2(-\sin(\theta)x_1+\cos(\theta)x_2)^2\left\|x\right\|^2\Big)^2\left(\left\|x\right\|^8+\left\|x\right\|^4\right)\\
&\leq -4\tfrac{c_2-\lambda}{c_4}e^{4c_2(t-t_i)}V_2^2(t,x),
\end{aligned}$$

for $t\in\mathbb{R}_0^+\setminus S$, $x\in X$ and some $c_4>0$, where we denote $\theta=t-t_i$. Furthermore, with $\tau_2=\theta=\frac{\pi}{2}$, we obtain

$$\begin{aligned}
&V_1(t_i,x^+)\\
&= ((x_1^+)^4+(x_2^+)^2)e^{-4c_1(t_i-t_i)}\\
&= (c_3x_1\cdot\left\|x\right\|)^4+(c_3x_2^4\cdot\left\|x\right\|^4)^2\\
&= c_3^4x_1^4\left\|x\right\|^4+c_3^2x_2^8\left\|x\right\|^8\\
&\leq c_3^2e^{4c_2(t_i-t_{i-1})}\left(x_1^2\left\|x\right\|^2+x_2^4\left\|x\right\|^4\right)^2e^{-4c_2(t_i-t_{i-1})}\\
&= \tfrac{c_3^2}{c_4}e^{2c_2\pi}V_2(t_i^-,x),
\end{aligned}$$

for $t=t_i\in S$, $x\in X$, $x_1^+=\frac{1}{2}x_1(t^-)\cdot\left\|x(t^-)\right\|$ and $x_2^+=\frac{1}{2}(x_2(t^-)\cdot\left\|x(t^-)\right\|)^4$, and

$$\begin{aligned}
V_2(t_i,x^+) &= c_4\left((x_1^+\left\|x^+\right\|)^4+(x_2^+\left\|x^+\right\|)^2\right)^2e^{-4c_2(t_i-t_i)}\\
&= c_4\left(x_1^4+x_2^2\right)^2\\
&= c_4e^{8c_1(t_i-t_{i-1})}\left(x_1^4+x_2^2\right)^2e^{-8c_1(t_i-t_{i-1})}\\
&= c_4e^{8c_1\tau_1}V_1^2(t_i^-,x)
\end{aligned}$$

holds for $t=t_i\in S$, $x\in X$ and $x^+=\frac{x(t^-)}{\sqrt{\left\|x(t^-)\right\|}}\in X$.

References

[1] P. Simeonov and D. Bainov, "Stability with respect to part of the variables in systems with impulse effect," *Journal of Mathematical Analysis and Applications*, vol. 117, no. 1, pp. 247–263, Jul. 1986.

[2] D. Liberzon, *Switching in Systems and Control*. Birkhäuser Boston, MA, 2003.

[3] Z. Li, Y. Soh, and C. Wen, *Switched and Impulsive Systems: Analysis, Design and Applications*. Springer-Verlag Berlin, Heidelberg, 2005.

[4] Z. P. Jiang, A. R. Teel, and L. Praly, "Small-gain theorem for ISS systems and applications," *Mathematics of Control, Signals and Systems*, vol. 7, pp. 95–120, 1994.

[5] W. Heemels, S. Weiland, and A. L. Juloski, "Input-to-state stability of discontinuous dynamical systems with an observer-based control application," in *International Workshop on Hybrid Systems: Computation and Control*. Springer, 2007, pp. 259–272.

[6] W. Heemels and S. Weiland, "Input-to-state stability and interconnections of discontinuous dynamical systems," *Automatica*, vol. 44, no. 12, pp. 3079–3086, 2008.

[7] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Transactions on Automatic control*, vol. 44, no. 4, pp. 751–760, 1999.

[8] H. Lin and P. J. Antsaklis, "Stability and stabilizability of switched linear systems: A survey of recent results," *IEEE Transactions on Automatic control*, vol. 54, no. 2, pp. 308–322, 2009.

[9] S. Ahmed, F. Mazenc, and H. Özbay, "Dynamic output feedback stabilization of switched linear systems with delay via a trajectory based approach," *Automatica*, vol. 93, pp. 92–97, Jul. 2018.

[10] G. Zhai, B. Hu, K. Yasuda, and A. N. Michel, "Stability analysis of switched systems with stable and unstable subsystems: An average dwell time approach," *International Journal of Systems Science*, vol. 32, no. 8, pp. 1055–1061, 2001.

[11] J. P. Hespanha, D. Liberzon, and A. R. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *Automatica*, vol. 44, no. 11, pp. 2735–2744, 2008.

[12] S. Dashkovskiy and A. Mironchenko, "Input-to-state stability of nonlinear impulsive systems," *SIAM Journal on Control and Optimization*, vol. 51, no. 3, pp. 1962–1987, Jan. 2013.

[13] P. Bachmann and S. Ahmed, "Construction of time-varying iss-lyapunov functions for impulsive systems," *IFAC-PapersOnLine*, vol. 56, no. 1, pp. 1–6, 2023.

[14] P. Bachmann, S. Ahmed, and N. Bajcinca, "Lyapunov characterization of input-to-state stability for a class of impulsive systems," *IEEE Transactions on Automatic Control*, vol. 69, no. 10, pp. 6996–7003, Oct. 2024.

[15] S. Trenn, "Switched Differential Algebraic Equations," in *Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters*, F. Vasca and L. Iannelli, Eds. London: Springer, 2012, ch. 6, pp. 189–216.

[16] X. Zhao, P. Shi, Y. Yin, and S. K. Nguang, "New results on stability of slowly switched systems: a multiple discontinuous Lyapunov function approach," *IEEE Transactions on Automatic Control*, vol. 62, no. 7, pp. 3502–3509, Jul. 2017.

[17] J. Liu, X. Liu, and W.-C. Xie, "Class-$\mathcal{KL}$ estimates and input-to-state stability analysis of impulsive switched systems," *Systems & Control Letters*, vol. 61, no. 6, pp. 738–746, 2012.

[18] X. Li, P. Li, and Q.-g. Wang, "Input/output-to-state stability of impulsive switched systems," *Systems & Control Letters*, vol. 116, pp. 1–7, 2018.

[19] H. Zhu, P. Li, X. Li, and H. Akca, "Input-to-state stability for impulsive switched systems with incommensurate impulsive switching signals," *Communications in Nonlinear Science and Numerical Simulation*, vol. 80, p. 104969, 2020.

[20] J. L. Mancilla-Aguilar and H. Haimovich, "Uniform input-to-state stability for switched and time-varying impulsive systems," *IEEE Transactions on Automatic Control*, vol. 65, no. 12, pp. 5028–5042, 2020.

[21] M. Cao and Z. Ai, "Periodic stabilization of continuous-time multi-module impulsive switched linear systems," *IEEE Access*, vol. 7, pp. 16 648–16 654, 2019.

[22] B. Ghanmi, M. Dlala, and M. A. Hammami, "Converse theorem for practical stability of nonlinear impulsive systems and applications," *Kybernetika*, pp. 496–521, Jul. 2018.

[23] M. Malisoff and F. Mazenc, *Constructions of strict Lyapunov functions*. Springer London, 2009.

[24] T. Zhou, G. Cai, and B. Zhou, "Further results on the construction of strict Lyapunov–Krasovskii functionals for time-varying time-delay systems," *Journal of the Franklin Institute*, vol. 357, no. 12, pp. 8118–8136, Aug. 2020.

[25] H. Yin, B. Jayawardhana, and S. Trenn, "On contraction analysis of switched systems with mixed contracting-noncontracting modes via mode-dependent average dwell time," *IEEE Transactions on Automatic Control*, vol. 68, no. 10, pp. 6409–6416, 2023.

[26] E. D. Sontag, "Smooth stabilization implies coprime factorization," *IEEE Transactions on Automatic Control*, vol. 34, no. 4, pp. 435–443, Apr. 1989.

[27] A. Mironchenko and F. Wirth, "Characterizations of input-to-state stability for infinite-dimensional systems," *IEEE Transactions on Automatic Control*, vol. 63, no. 6, pp. 1692–1707, Jun. 2018.

[28] M. Defoort, M. Djemai, and S. Trenn, "Nondecreasing Lyapunov functions," in *Proceedings of the 21st International Symposium on Mathematical Theory of Networks and Systems*, 2014, pp. 1038–1043.

[29] V. Chellaboina, A. Leonessa, and W. Haddad, "Generalized Lyapunov and invariant set theorems for nonlinear dynamical systems," in *Proceedings of the American Control Conference San Diego, California June 1999*, vol. 5. IEEE, 1999, pp. 3028–3032.

[30] S. Liu, S. Martínez, and J. Cortés, "Average dwell-time minimization of switched systems via sequential convex programming," *IEEE Control Systems Letters*, vol. 6, pp. 1076–1081, 2022.

[31] P. Feketa and N. Bajcinca, "On robustness of impulsive stabilization," *Automatica*, vol. 104, pp. 48–56, Jun. 2019.

Saeed Ahmed is an Assistant Professor of Systems and Control at the University of Groningen, The Netherlands, where he is affiliated with the Engineering and Technology Institute Groningen and the Jan C. Willems Center for Systems and Control. Prior to joining this position, he held postdoctoral positions at the University of Groningen and at the Technical University of Kaiserslautern (now RPTU), Germany. He completed his Ph.D. at Bilkent University, Turkey. During his Ph.D., he also served as a research assistant at Inria, CentraleSupélec, University of Paris-Saclay, France. His research interests span various topics in systems and control engineering. From a theoretical point of view, he is interested in stability and control, feedback optimization, observer design, and nonlinear and hybrid (switched and impulsive) systems. From an application point of view, he is interested in designing intelligent control algorithms for autonomous vehicles and energy systems. He received the best presentation award in the Control/Robotics/Communications/Network category at the IEEE Graduate Research Conference 2018 held in Bilkent University, Turkey, the outstanding reviewer award from the European Journal of Control in 2017, and the Investments in Practical Innovations (IPI) Award 2025 from the University of Groningen, The Netherlands. He is an associate editor of Systems and Control Letters and the IEEE Technology Conference Editorial Board (TCEB). He is a member of the IFAC Technical Committees on Networked Systems, Non-linear Control Systems, and Distributed Parameter Systems.

Patrick Bachmann received his Bachelor's degree in business mathematics from the University of Mannheim, Germany, in 2015 and his Master's degree in Mathematics from Karlsruhe Institute of Technology in 2018. He worked as a research assistant at the Technical University of Kaiserslautern, Germany, and the University of Würzburg, Germany. Currently, he is working at the University of Bayreuth, Germany, while pursuing his PhD degree in Mathematics under the supervision of Sergey Dashkovskiy and Andrii Mironchenko. His research interests include impulsive systems, stability and control theory, Lyapunov functions, and infinite-dimensional systems.

Stephan Trenn received his Ph.D. (Dr. rer. nat.) within the field of differential algebraic systems and distribution theory at the Ilmenau University of Technology, Germany, in 2009. Afterwards, he held Postdoc positions at the University of Illinois at Urbana-Champaign, USA (2009–2010) and at the University of Würzburg, Germany (2010–2011). After being an Assistant Professor (Juniorprofessor) at the University of Kaiserslautern, Germany, he became Associate Professor for Systems and Control at the University of Groningen, Netherlands, in 2017. He is an Associate Editor for the journals Systems and Control Letters, Nonlinear Analysis: Hybrid Systems, IEEE Control Systems Letters, and DAE Panel.