Funnel control

Origin and recent advances

Stephan Trenn

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

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Who am I?

Stephan Trenn https://stephantrenn.net

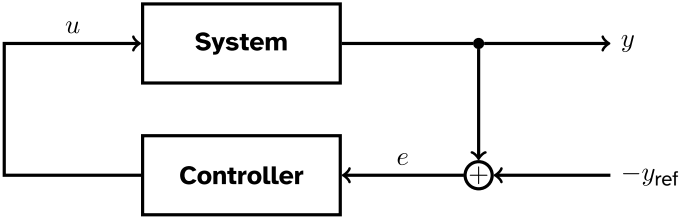
Associate Professor for Systems & Control, Bernoulli Institute
Programme Director for master degrees Math., Applied Math., Systems & Control

- studied Mathematics and Computer Science in Ilmenau, Germany
- six month Erasmus student in Southampton, UK
- PhD 2009 in mathematical control theory in Ilmenau
- Postdoc (9 month) at University of Illinois, Urbana-Champaign, USA
- Postdoc (17 month) at University of Würzburg, Germany
- Assistant Professor for Math. Control Theory (2011 - 2017), Kaiserslautern, Germany

Research:

- Switched systems
- Differential-algebraic equations (DAEs)
- Funnel control

Control Task

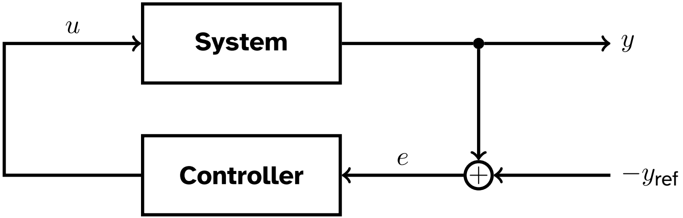


Goal: Output tracking $y(t) \approx y_{\text{ref}}(t)$

Applications

- Flying to the moon
- Robotics
- (Adaptive) cruise control in cars
- Chemical processes

Control Task

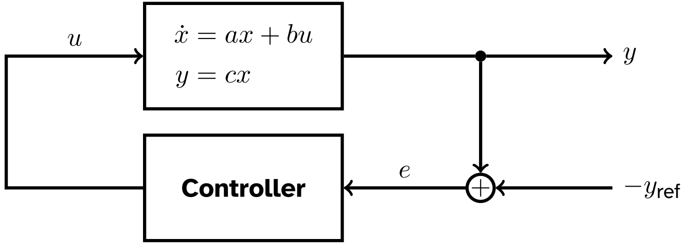


Goal: Output tracking $y(t) \approx y_{\text{ref}}(t)$

Challenge

- no exact knowledge of system model
- no future knowledge or model for reference signal

The scalar linear case with $y_{\text{ref}} = 0$

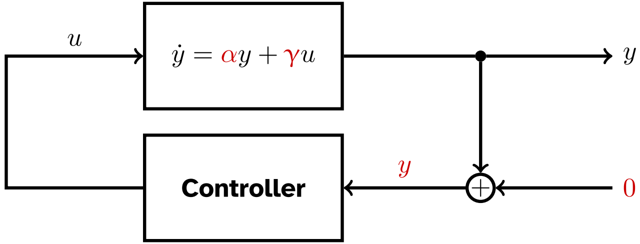


Assumptions

- Known model structure
- Known sign of *high frequency gain* $\gamma := cb$, assume $\gamma > 0$
- $y_{\text{ref}} = 0$

Unknown system parameters α and γ

The scalar linear case with $y_{\text{ref}} = 0$



Goal

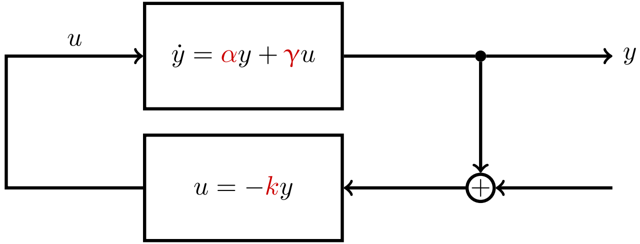
Design output feedback u such that $y(t) \to 0$ as $t \to \infty$

If we would know α, γ, how would we choose u?

Goal: $\dot{y} \overset{!}{=} -\lambda y \quad \rightsquigarrow \quad$ achievable with $u = -ky$ and $k := \frac{\alpha+\lambda}{\gamma}$

In general, with $u = -ky$ we have $\dot{y} = (\alpha - \gamma k)y$

The scalar linear case with $y_{\text{ref}} = 0$



Hence we have arrived at our first high gain control result:

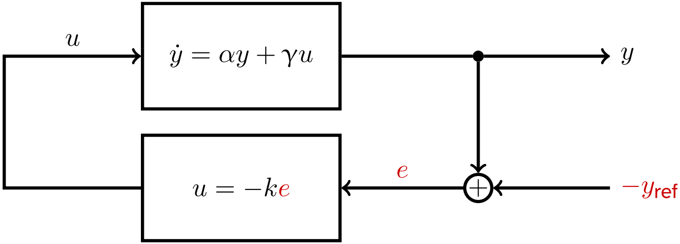
Theorem

The proportional negative feedback

$$u = -ky$$

achieves convergence for all $k > \frac{\alpha}{\gamma}$.

What happens for $y_{\text{ref}} \neq 0$?



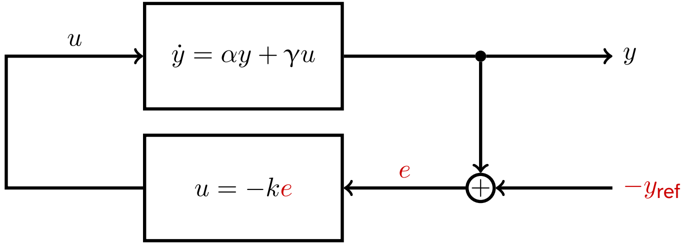
Error dynamics: $\dot{e} = \ldots = (\alpha - \gamma k)e + \alpha y_{\text{ref}} - \dot{y}_{\text{ref}}$

Equilibrium for constant y_{ref}:

$$0 = (\alpha - \gamma k)e + \alpha y_{\text{ref}} \quad \iff \quad e = \frac{\alpha}{\gamma k - \alpha} y_{\text{ref}} \neq 0$$

$\rightsquigarrow$ no convergence to zero anymore

What happens for $y_{\text{ref}} \neq 0$?



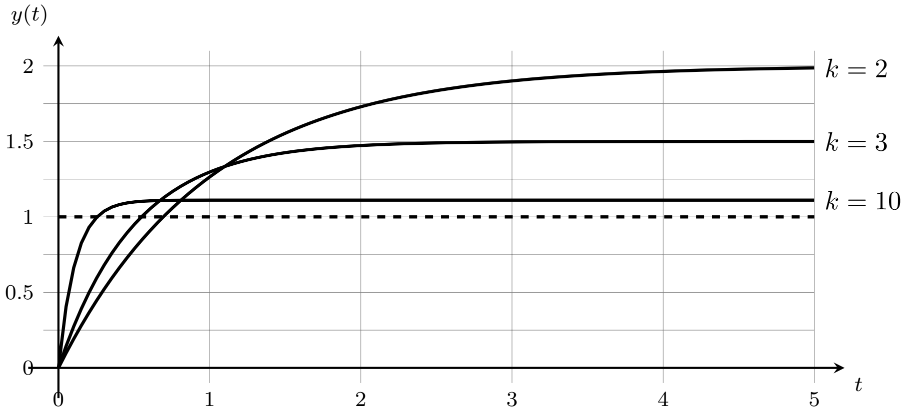
In general: Practical tracking with high gain control:

Theorem

If y_{ref} and $\dot{y}_{\text{ref}}$ are bounded, then

$$\forall y_0 \ \forall \varepsilon > 0 \ \exists K_\varepsilon > 0 \ \forall k > K_\varepsilon \ \exists T_{k,\varepsilon,y_0} > 0 : \quad |e(t)| < \varepsilon \quad \forall t \geq T_{k,\varepsilon,y_0}$$

Example $\alpha = 1$, $\gamma = 1$, $y_{\text{ref}} \equiv 1$



Relative degree

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \qquad A \in \mathbb{R}^{n\times n}, b, c^\top \in \mathbb{R}^n \qquad (*)$$

Definition (Relative degree)

Write $g(s) := c(sI - A)^{-1}b$ as $g(s) = \frac{p(s)}{q(s)}$.

Then $r := \deg q(s) - \deg p(s)$ is called relative degree of $(*)$.

Remarks:

- For $p(s) \not\equiv 0$: $0 \leq \deg p(s) < \deg q(s) \leq n \quad \rightsquigarrow \quad r \in \{1, 2, \ldots, n\}$.
- If $g(s) = 0 = p(s)$: $r = \infty$
- If $(*)$ has feedthrough term, i.e. $y = cx + du$ with $d \neq 0$, then $r := 0$
- For *descriptor systems* the relative degree can also be *negative*

Relative degree and Markov parameters

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx \end{aligned} \qquad A \in \mathbb{R}^{n\times n}, b, c^\top \in \mathbb{R}^n \qquad (*)$$

Definition (Markov parameters)

The numbers $M_k := cA^k b$, $k \in \mathbb{N}$, are called Markov parameters of $(*)$.

Lemma (Transfer function and Markov parameters)

$$g(s) = c(sI - A)^{-1}b = c\sum_{k=0}^{\infty} \frac{A^k}{s^{k+1}} b = \sum_{k=0}^{\infty} \frac{M_k}{s^{k+1}}$$

Lemma (Markov parameters and relative degree)

$$r = \min\{k \in \mathbb{N}_{>0} \mid M_{k-1} \neq 0\}$$

Intuition for relative degree

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \qquad A \in \mathbb{R}^{n\times n}, b, c^\top \in \mathbb{R}^n \qquad (*)$$

$d \neq 0 \iff$ r.d. 0
$\rightsquigarrow$ input u directly influences $y = cx + d\,u$

$d = 0$ and $cb \neq 0 \iff$ r.d. 1
$\rightsquigarrow$ y not directly influenced by u, but $\dot{y} = c\dot{x} = cAx + cb\,u$ directly influenced by u

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$d = 0$, $cb = 0$ and $cAb \neq 0 \iff$ r.d. 2
$\rightsquigarrow$ y, $\dot{y}$ not directly influence by u, but $\ddot{y} = cA\dot{x} = cA^2x + cAb\,u$ directly influenced by u

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$\vdots$

$d = 0$, $cb = 0$, ..., $cA^{r-2}b = 0$ and $cA^{r-1}b \neq 0 \iff$ r.d. r
$\rightsquigarrow$ $y, \ldots, y^{(r-1)}$ not influenced by u, but $y^{(r)} = cA^rx + cA^{r-1}b\,u$ directly influence by u

Intuition behind relative degree

Relative degree $=$ lowest derivative of y which is directly influence by input u

Zero dynamics

Zero dynamics

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \qquad A \in \mathbb{R}^{n\times n}, b, c^\top \in \mathbb{R}^n \qquad (*)$$

Question

What input is needed to keep the output identically zero?

Relative degree $r \in \{1, 2, \ldots, n\}$ $\rightsquigarrow$ $0 \overset{!}{=} y^{(r)}(t) = cA^r x(t) + \underbrace{cA^{r-1}b}_{=:\gamma} u(t)\ \forall t$

$\rightsquigarrow$ $u(t) = -\frac{1}{\gamma} cA^r x(t)$ keeps output identically zero

$\rightsquigarrow$ $\dot{x} = (A - \frac{1}{\gamma} bcA^r)x$ is called zero dynamics[1] (ZD)

Problem

Unstable ZD $\rightsquigarrow$ unbounded state x

$\rightsquigarrow$ unbounded input needed to keep output bounded

[1]when considered on the subspace $\ker[c/cA/\ldots/cA^{r-1}]$, which results from the conditions $0 = y^{(k)}(t) = cA^k x(t)$, $k = 0, 1, \ldots, r-1$, i.e. $x(t) \in \ker[c/cA/\ldots/cA^{r-1}]$

Stable zero dynamics

$$\begin{aligned}\dot{x} &= Ax + bu \\ y &= cx\end{aligned} \qquad A \in \mathbb{R}^{n\times n}, b, c^\top \in \mathbb{R}^n, \quad g(s) = c(sI - A)b = \frac{p(s)}{q(s)} \qquad (*)$$

Theorem

$(*)$ has stable ZD $\iff \operatorname{rank}\begin{bmatrix}\lambda I - A & b \\ c & 0\end{bmatrix} = n+1$ *for all* $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$.

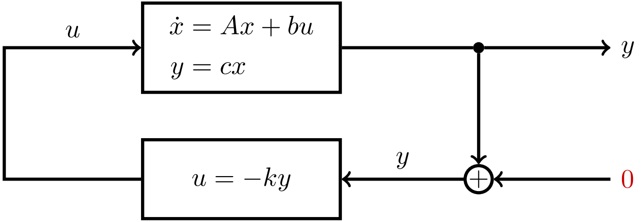
If $(*)$ is controllable and observable, then it has stable ZD $\iff$ $p(s)$ *is stable*

Remarks

- The property of having stable ZD is related to the notion minimum phase[2]:
  $|g_1(\mathrm{i}\omega)| = |g_2(\mathrm{i}\omega)|$ and the first has stable ZD $\implies \arg g_1(\mathrm{i}\omega) \leq \arg g_2(\mathrm{i}\omega)$
- if $(*)$ is stabilizable, unstable ZD can be stabilized by state feedback, but not by (static) output feedback
- Stable ZD implies stabilizability and detectability, but not the other way around in general

[2] For more on minimum phase see: Ilchmann, A., Wirth, F. (2013). On minimum phase. at–Automatisierungstechnik, 61(12), 805-817.

High gain stabilization for r.d.-one systems



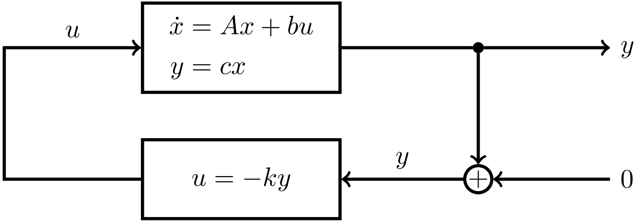
Assumptions:

- Relative degree $r = 1$ $\Leftrightarrow$ $\gamma := cb \neq 0$, in particular:

$$\text{System} \quad \Leftrightarrow \quad \begin{aligned} \dot{y} &= a_{11}y + a_{12}z + \gamma u \\ \dot{z} &= a_{21}y + A_{22}z \end{aligned}$$

- positive high frequency gain $\Leftrightarrow$ $\gamma > 0$
- stable zero-dynamics (minimum phase) $\Leftrightarrow$ A_{22} Hurwitz

High gain stabilization for r.d.-one systems



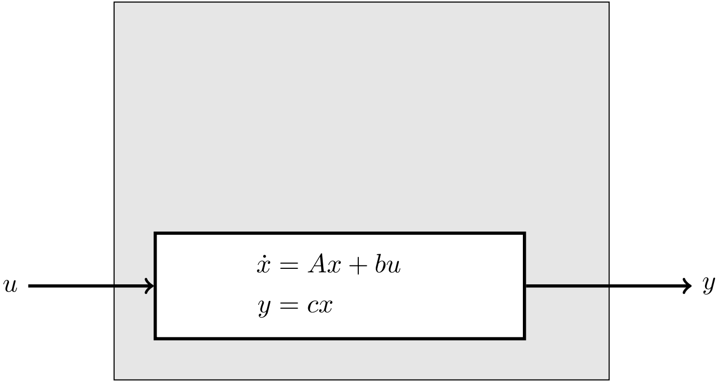
Theorem (High-gain stabilization)

$cb > 0$ and stable zero-dynamics

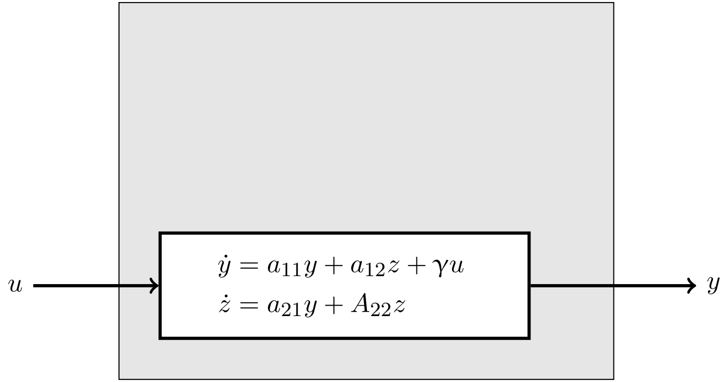
$$\Rightarrow \quad \exists K > 0 \,\forall\, k \geq K : \text{ Closed loop is asymptotically stable}$$

Key idea of proof: Show that $\begin{bmatrix} a_{11} - \gamma k & a_{12} \\ a_{21} & A_{22} \end{bmatrix}$ is Hurwitz for sufficiently large k.

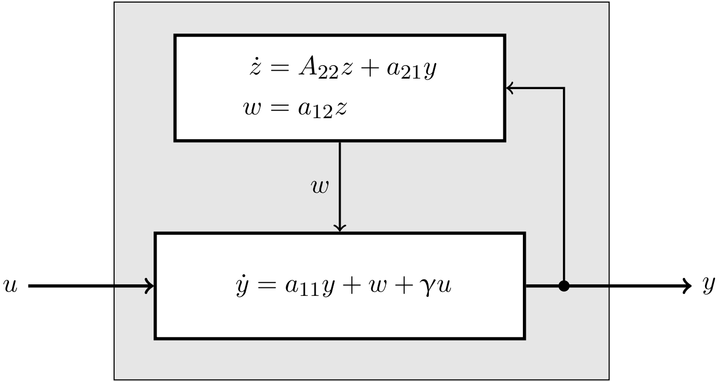
From linear to nonlinear systems



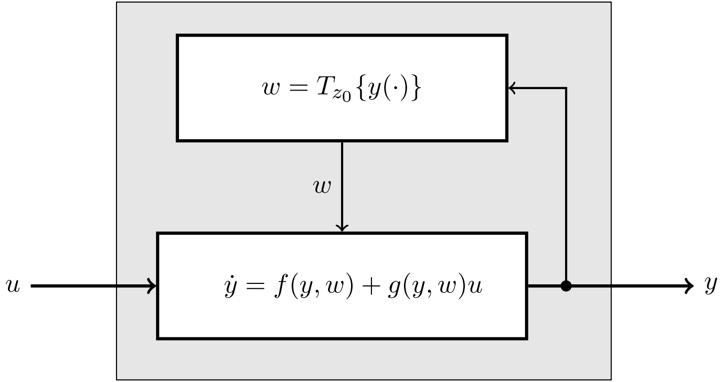
From linear to nonlinear systems



From linear to nonlinear systems



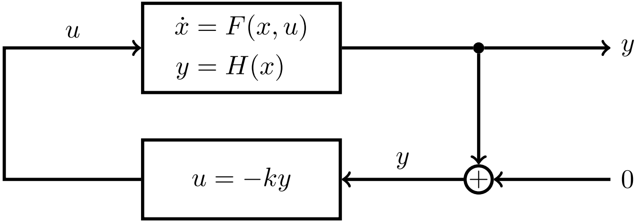
From linear to nonlinear systems



Assumptions:

- T_{z_0} is causal BIBO operator, i.e. $\exists \kappa(\cdot): \quad \|w\| \leq \kappa(\|y\|)$
- f and g continuous and $g > 0$

High gain stabilization for nonlinear systems



Theorem

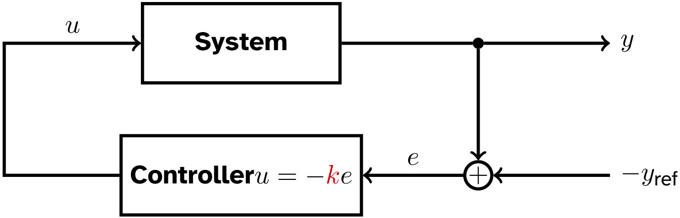
Assume there exists (nonlinear) coordinate transformation such that system is equivalent to

$$\dot{y} = f(y,w) + g(y,w)u, \quad w = T_{z_0}\{y(\cdot)\}$$

with f, g continuous, T_{z_0} causal BIBO operator and $g > 0$, then

$$\forall y_0 \, \forall \eta_0 \, \forall \varepsilon > 0 \, \exists K > 0 \, \forall k \geq K \, \exists T > 0 : \quad |e(t)| < \varepsilon \quad \forall t \geq T$$

Summary high gain feedback



Goal: Output tracking

Challenge: Unknown system parameters

Structural assumptions

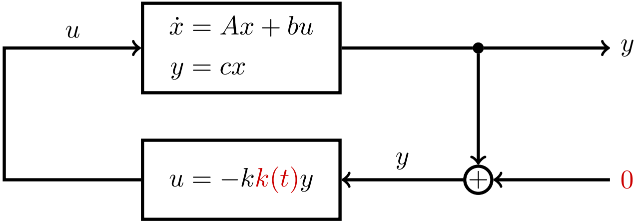
› Relative degree one with known sign of "high frequency gain"

› Stable zero dynamics

High gain feedback: $u = -ke$ "works" for sufficiently large gain $k > 0$

Remaining challenge: When is k sufficiently large?

Choosing gain adaptively, linear case



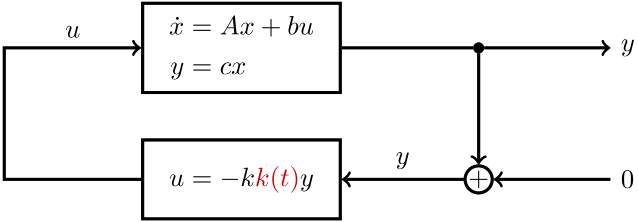
Theorem (High-gain stabilization)

$cb > 0$ *and stable zero-dynamics* $\quad \Rightarrow \quad \exists K > 0 \; \forall \, k \geq K : y(t) \to 0$

Key idea

Why not make k time-varying with $\dot{k}(t) > 0$ as long as $y(t) > 0$?

Choosing gain adaptively, linear case



Theorem (Adaptive High-Gain Feedback, Byrnes & Willems 1984)

$cb > 0$ and stable zero-dynamics $\Rightarrow$

$$\dot{k}(t) = y(t)^2 \text{ makes closed loop asymptotically stable}$$

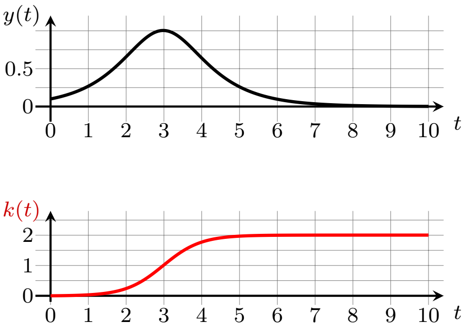
and $k(\cdot)$ remains bounded

Boundedness of $k(t) = \int_0^t y(s)^2 \, \mathrm{d}s$ follows from final exponential decay of y.

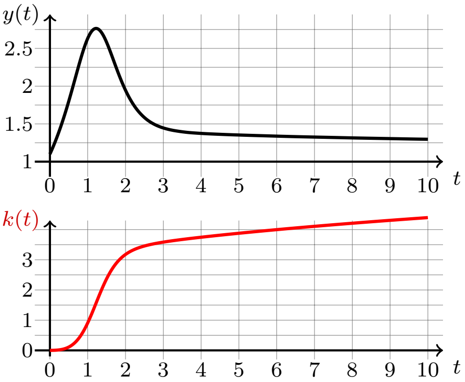
Simulations

$$\dot{y} = y + u, \quad u(t) = -k(t)(y(t) - y_{\text{ref}}(t)), \quad \dot{k} = (y - y_{\text{ref}})^2$$

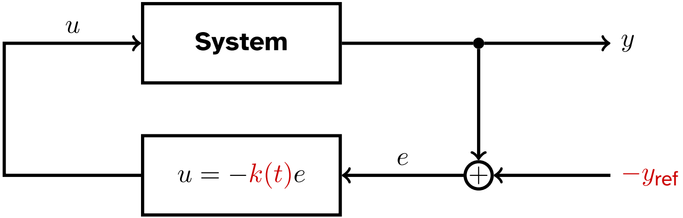
output and gain for $y_{\text{ref}} = 0$



output and gain for $y_{\text{ref}} = 1$



High gain adaptive control and tracking?

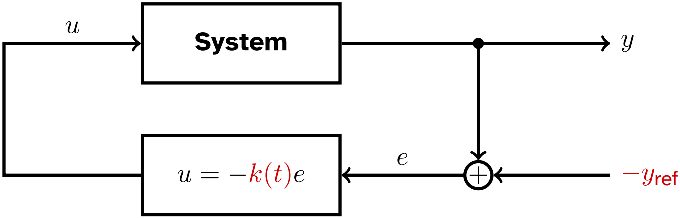


Unbounded gain

For $y_{\text{ref}} \neq 0$ the adaptation rule $\dot{k} = e^2$ leads to unbounded gain.

Recall: Constant gain for $y_{\text{ref}} \neq 0$ only leads to practical tracking, i.e. $e(t) \not\to 0$

High gain adaptive control and tracking?

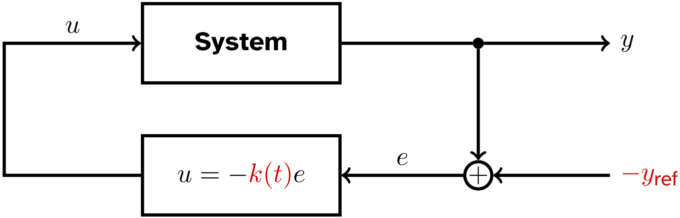


How to prevent unbounded growth?

Stop increasing gain when error is sufficiently small, e.g. via

$$\dot{k}(t) = \begin{cases} 0 & |e(t)| \leq \lambda \\ |e(t)|(|e(t)| - \lambda) & |e(t)| > \lambda \end{cases}$$

High gain adaptive control and tracking?



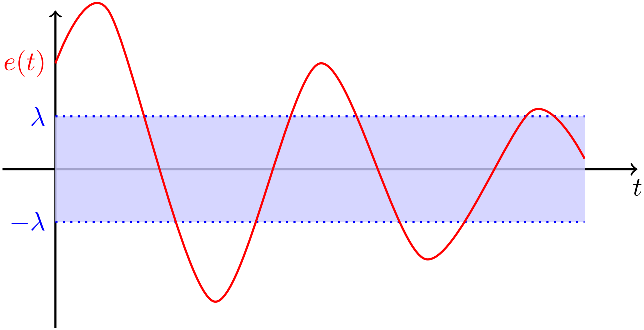
Theorem (λ-tracking, Ilchmann & Ryan 1994)

Assume r.d.-one with "$\gamma > 0$", stable zero-dynamics and $y_{\text{ref}}, \dot{y}_{\text{ref}}$ bounded. For $\lambda > 0$ consider

$$\dot{k}(t) = \begin{cases} 0, & |e(t)| \leq \lambda, \\ |e(t)|\big(|e(t)| - \lambda\big), & |e(t)| > \lambda. \end{cases}$$

Then the closed loop is practically stable, i.e. $\limsup_{t\to\infty} |e(t)| \leq \lambda$.

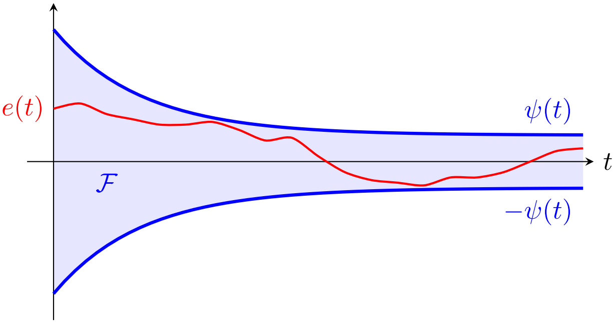
Remaining problems of λ-tracker



Problems:

- No guarantees when $|e(t)| \leq \lambda$
- No bounds on transient behaviour
- Monotonically growing $k(\cdot)$ $\Rightarrow$ Measurement noise unnecessarily amplified

The funnel as time-varying error bound

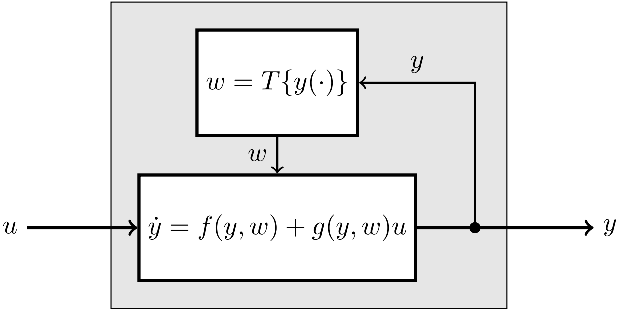


$$\mathcal{F} = \mathcal{F}(\psi) := \{(t,e) \mid |e| < \psi(t)\}$$

Idea: $k(t)$ large $\iff$ Distance of $e(t)$ to funnel boundary small

$\rightsquigarrow$ **Funnel gain:** $$k(t) = \frac{1}{\psi(t) - |e(t)|}$$

Funnel controller works



System class

Equivalent to structure left:

- T is causal and BIBO
- f, g continuous
- $g > 0$

Theorem (Ilchmann, Ryan, Sangwin 2002)

Assume $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded*, $\liminf_{t\to\infty} \psi(t) > 0$ *and* $|e(0)| < \psi(0)$ *where* $e := y - y_{\text{ref}}$. *Then*

$$u(t) = -k(t)e(t) \quad \text{with} \quad k(t) = \frac{1}{\psi(t) - |e(t)|}$$

ensures that $e(t)$ *remains within funnel* $\mathcal{F}(\psi)$ *while* $k(t)$ *remains bounded.*

Proof

Step 1: Existence of solution

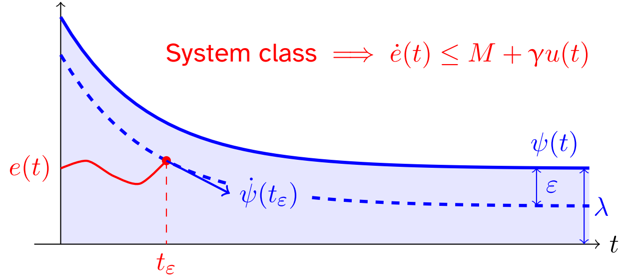
- Standard ODE theory: solution of closed loop exists on $[0,\omega)$ for $\omega \in (0,\infty]$
- Choose $\omega > 0$ maximal
- If $\omega < \infty$ then "$|e(\omega)| = \psi(\omega)$"

Step 2: We show that $\omega < \infty$ implies $|e(t)| - \psi(t) > \varepsilon$ for some $\varepsilon > 0$
Error dynamics are given by

$$\dot{e} = f(y,w) - \dot{y}_{\text{ref}} + g(y,w)u$$

Step 2a: Boundedness of e, y, and w
$e(t)$ within funnel for $t \in [0,\omega)$ *(domain of ODE)*
$\Rightarrow e$ bounded on $[0,\omega)$ *(because ψ is bounded)*
$\Rightarrow y$ bounded on $[0,\omega)$ *(because y_{ref} is bounded)*
$\Rightarrow w$ bounded on $[0,\omega)$ *(because T is BIBO)*
$\Rightarrow f(y,w)$ bounded and $g(y,w)$ bounded away from zero on $[0,\omega)$ *(continuity)*
$\Rightarrow \dot{e}(t) \leq M + \gamma u(t)$ if $u(t) < 0$ and $\dot{e}(t) \geq -M + \gamma u(t)$ if $u(t) > 0$

Step 2b: Funnel invariant (case $e(t) > 0$)



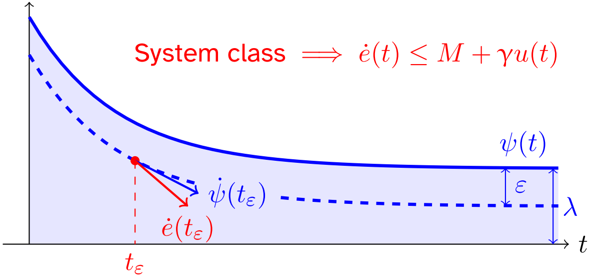
Assumptions: $\varepsilon < \psi(0) - e(0)$ $\quad \varepsilon < \lambda/2 \quad \psi(t) \geq \lambda$

$$e(t_\varepsilon) = \psi(t_\varepsilon) - \varepsilon \implies k(t_\varepsilon) = \frac{1}{\psi(t_\varepsilon) - |e(t_\varepsilon)|} = \frac{1}{\varepsilon}$$

$$\implies u(t_\varepsilon) = -k(t_\varepsilon)e(t_\varepsilon) \leq -\frac{1}{\varepsilon}\frac{\lambda}{2}$$

$$\implies \dot{e}(t_\varepsilon) \leq M - \frac{\gamma\lambda}{2\varepsilon}$$

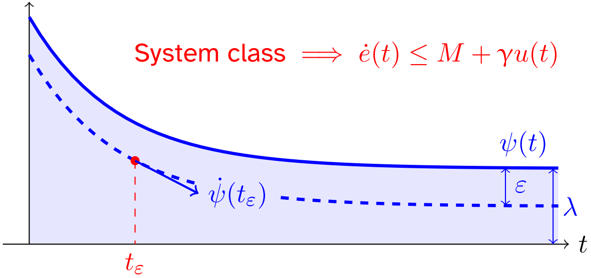
Step 2b: Funnel invariant (case $e(t) > 0$)



Assume $\dot{\psi}(t) > -\Psi$ and $\varepsilon \leq \dfrac{\gamma\lambda}{2(\Psi + M)}$ we have

$$\dot{e}(t_\varepsilon) \leq M - \frac{\gamma\lambda}{2\varepsilon} \leq -\Psi < \dot{\psi}(t_\varepsilon)$$

Step 2b: Funnel invariant (case $e(t) > 0$)



Consequence: For sufficiently small $\varepsilon > 0$,

$$\mathcal{F}_\varepsilon := \{(t,e) \mid |e(t)| < \psi(t) - \varepsilon\}$$

is positively invariant, i.e.

$$(0, e(0)) \in \mathcal{F}_\varepsilon \quad \Rightarrow \quad (t, e(t)) \in \mathcal{F}_\varepsilon \ \forall t \geq 0$$

and $\omega < \infty$ impossible!

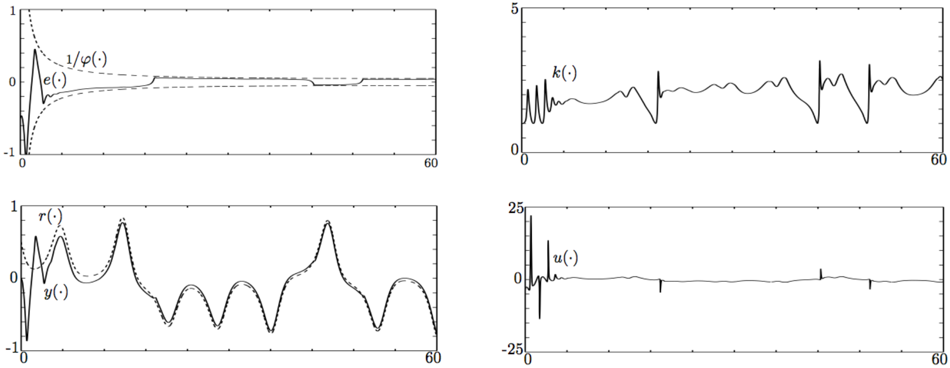
Extensions of funnel controller

- Asymptotic tracking (Lee & Trenn 2019)
- Multi-Input Multi-Output (MIMO) (already in Ilchmann et al. 2002)
- Higher relative degree (Ilchmann et al. 2007, Berger et al. 2018)
- Input saturation (Ilchmann et al. 2004, Hopfe et al. 2010)
- Bang-Bang funnel control (Liberzon & Trenn 2013)
- Funnel synchronization for multi-agent systems (Shim & Trenn 2015)
- For DAE-systems (Berger 2016)
- For impulsive systems (Karimi Pour & Trenn 2025)

Relative degree two via backstepping

For rel. deg. two systems, Funnel Controller is given by (Ilchmann et al. 2007):

$$u(t) = -k(t)e(t) - (\|e(t)\|^2 + k(t)^2)k(t)^4(1 + \|\xi(t)\|^2)(\xi(t) + k(t)e(t))$$
$$k(t) = 1/(1 - \varphi(t)^2\|e(t)\|^2)$$
$$\dot{\xi}(t) = -\xi(t) + u(t)$$



Taken from: Ilchmann, Ryan, Townsend 2007, SICON

Alternative Approach for relative degree two

Use two funnels, one for error and one for derivative of error

Simple Control Law

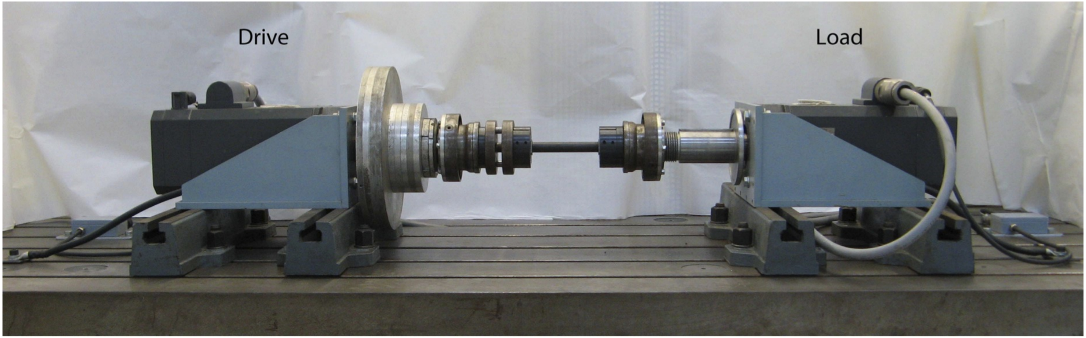
$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$$
$$k_i(t) = \frac{1}{\psi_i(t) - |e^{(i)}(t)|}, \quad i = 0, 1$$

System class: $\ddot{y}(t) = f(p_f(t), T_f\{y, \dot{y}\}(t)) + g(p_g(t), T_g\{y, \dot{y}\}(t))u(t)$

Theorem (Hackl et al. 2012)

The above Funnel Controller for relative-degree-two-systems works (under mild assumptions on ψ_0 and ψ_1).

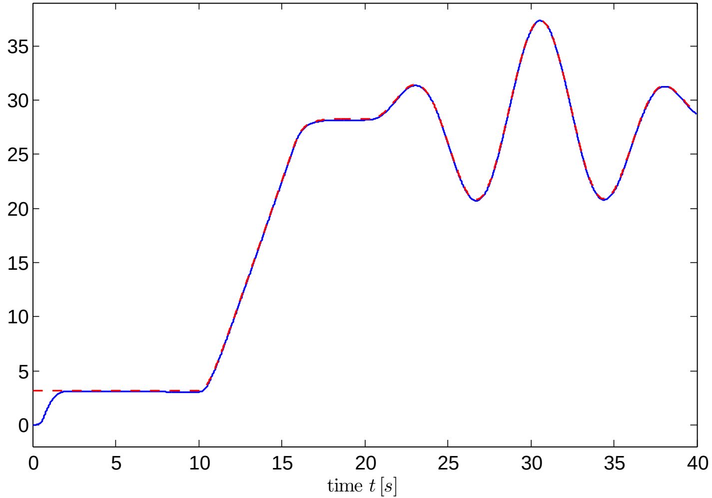
Experimental verification



$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \big(u(t) + u_L(t) - (Tx_2)(t)\big),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

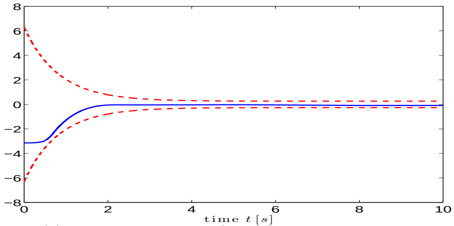
x_1: angle of rotating machine, $x_2 = \dot{x}_1$: angular velocity
u_L: unknown (bounded) load
$T : \mathbb{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathbb{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ friction operator

Tracking control in experiment

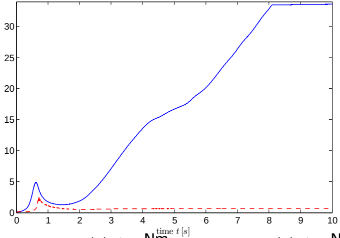


Measured angle $y(t)$ in rad, reference angle $y_{\text{ref}}(t)$ in rad

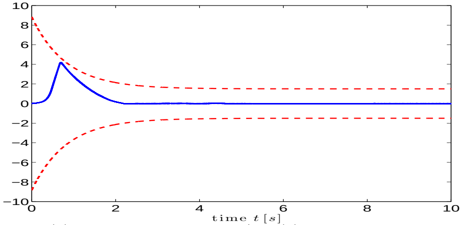
Experiment: Error, gains, input



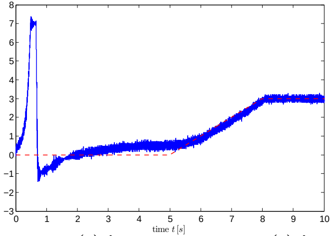
$e(t)$ in rad, $1/\varphi_0(t)$



$k_0(t)$ in $\frac{\text{Nm}}{\text{rad}}$, $k_1(t)$ in $\frac{\text{Nms}}{\text{rad}}$

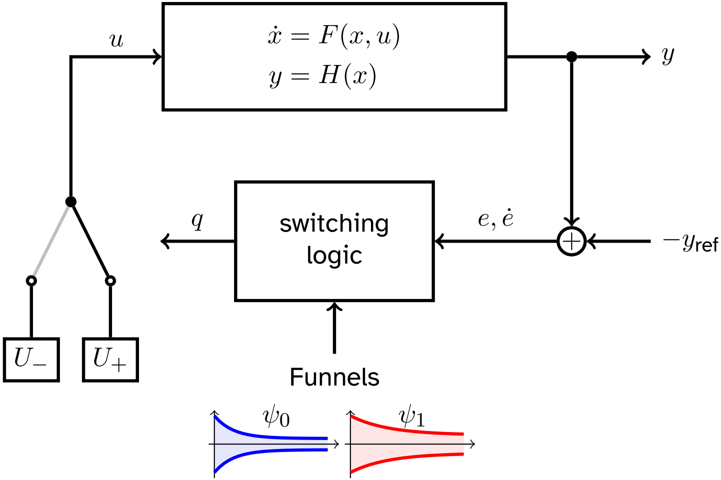


$\dot{e}(t)$ in rad/s, $1/\varphi_1(t)$

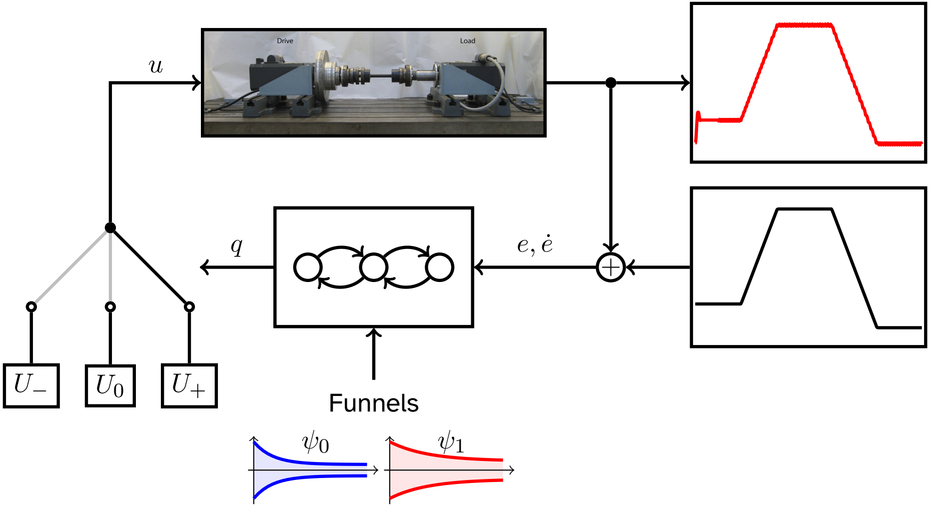


$u(t)$ in Nm, $u_L(t)$ in Nm

Bang-Bang Funnel Control



Bang-Bang Funnel Control



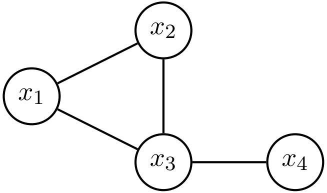
Funnel synchronization - setup

Given

- N agents with individual n-dimensional dynamics:

$$\dot{x}_i = f_i(t, x_i) + u_i$$

- undirected connected coupling-graph $G = (V, E)$
- local feedback $u_i = \gamma_i(x_i, x_{\mathcal{N}_i})$



$$u_1 = \gamma_1(x_1, x_2, x_3)$$
$$u_2 = \gamma_2(x_2, x_1, x_3)$$
$$u_3 = \gamma_3(x_3, x_1, x_2, x_4)$$
$$u_4 = \gamma_4(x_4, x_3)$$

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$

A „high-gain“ result

Let $\mathcal{N}_i := \{j \in V \mid (j,i) \in E\}$ and $d_i := |\mathcal{N}_i|$ and $\mathcal{L}$ be the Laplacian of G.

Diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j) \quad \text{or, equivalently,} \quad u = -k\,\mathcal{L}\,x$$

Theorem (Practical synchronization, Kim et al. 2013)

Assumptions: G connected, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K$: Diffusive coupling results in

$$\boxed{\limsup_{t \to \infty} |x_i(t) - x_j(t)| < \varepsilon \quad \forall i,j \in V}$$

Remarks on high-gain result

Common trajectory

It even holds that

$$\limsup_{t\to\infty} |x_i(t) - s(t)| < \varepsilon/2,$$

where $s(\cdot)$ solves $\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \quad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i.$

Independent of coupling structure and amplification k.

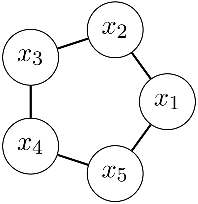
Error feedback

With $e_i := x_i - \overline{x}_i$ and $\overline{x}_i := \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j$ diffusive coupling has the form

$$u_i(t) = -k e_i(t)$$

Attention: $e_i \neq x_i - s$, in particular, agents do not know „limit trajectory“ $s(\cdot)$

Example (taken from Kim et al. 2015)



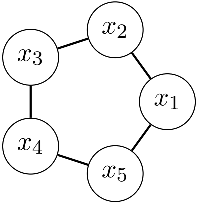
Simulations in the following for $N = 5$ agents with dynamics

$$f_i(t, x_i) = (-1 + \delta_i)x_i + 10 \sin t + 10 m_i^1 \sin(0.1t + \theta_i^1) + 10 m_i^2 \sin(10t + \theta_i^2),$$

with randomly chosen parameters $\delta_i, m_i^1, m_i^2 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

Example (taken from Kim et al. 2015)

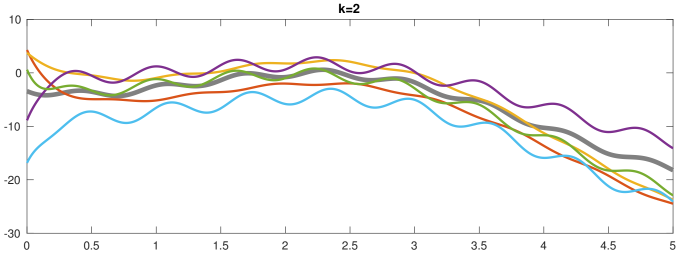


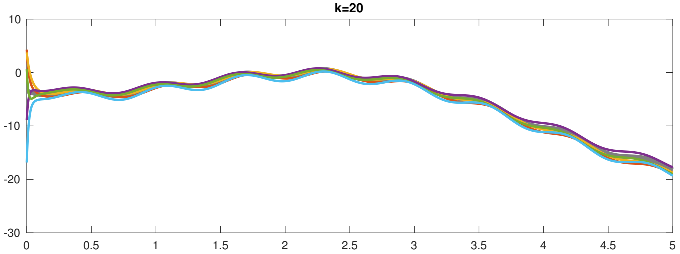
$$u = -k\,\mathcal{L}\,x$$

gray curve:

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t))$$

$$s(0) = \tfrac{1}{N}\sum_{i=1}^{N} x_i(0)$$





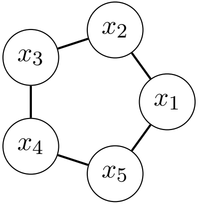
Funnel synchronization: Initial idea

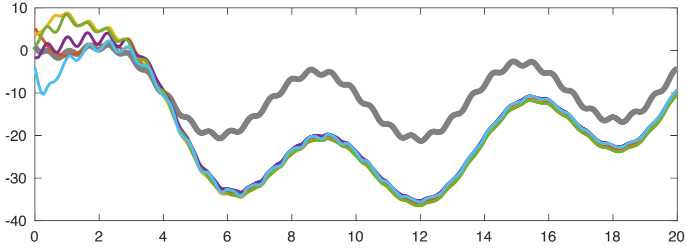
Reminder diffusive coupling: $u_i = -k_i e_i$ with $e_i = x_i - \overline{x}_i$.

Combine diffusive coupling with Funnel Controller

$$u_i(t) = -k_i(t)\, e_i(t) \quad \text{with} \quad k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

First simulations



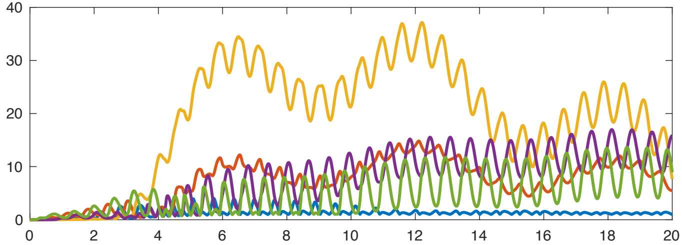


$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

$$\psi(t) = \underline{\psi} + (\overline{\psi} - \underline{\psi})e^{-\lambda t}$$

$$\overline{\psi} = 20,\ \underline{\psi} = 1,\ \lambda = 1$$



Observations from simulations

Funnel synchronization seems to work

- errors remain within funnel
- practical synchronization is achieved
- limit trajectory does not coincide with solution $s(\cdot)$ of

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \tfrac{1}{N}\sum_{i=1}^{N} x_i(0).$$

What determines the new limiting trajectory?

- Coupling graph?
- Funnel shape?
- Gain function?

Diffusive coupling revisited

Diffusive coupling for weighted graph

$$u_i = -k \sum_{i}^{N} \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_{i}^{N} k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j)$$

where $\alpha_{ij} = \alpha_{ji} \in \{0, 1\}$ is the weight of edge (i, j)

Conjecture

If $k_{ij} = k_{ji}$ are all sufficiently large, then practical synchronization occurs with desired limit trajectory s of average dynamics.

Proof technique from Kim et al. 2013 should still work in this setup.

Edgewise Funnel synchronization

Diffusive coupling → edgewise Funnel synchronization

$$u_i = -\sum_i^N k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_i^N k_{ij}(t) \cdot \alpha_{ij} \cdot (x_i - x_j)$$

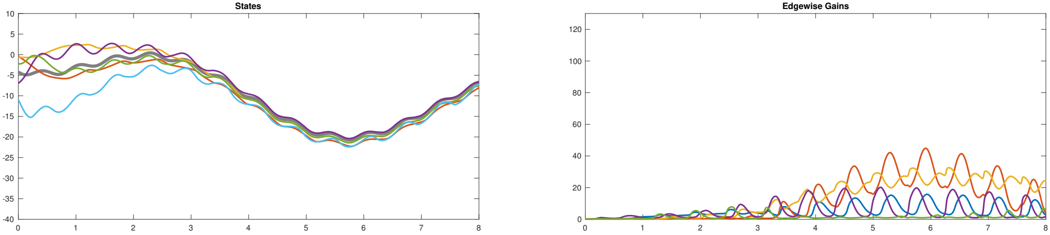
Edgewise error feedback

$$k_{ij}(t) = \frac{1}{\psi(t) - |e_{ij}|}, \quad \text{with} \quad e_{ij} := x_i - x_j$$

Properties:

- Decentralized, i.e. u_i only depends on state of neighbors
- Symmetry, $k_{ij} = k_{ji}$
- Laplacian feedback, $u = -\mathcal{L}_K(t,x)x$

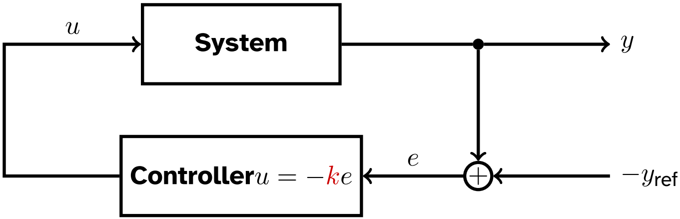
Simulation (from Trenn 2017)



Properties

+ Synchronization occurs
+ Predictable limit trajectory (given by average dynamics)
+ Local feedback law
+ Convergence recently proved (Lee et al. 2023)

Summary high gain feedback and funnel control



Goal: Output tracking

Challenge: Unknown system parameters

Structural assumptions

- Relative degree one with known sign of "high frequency gain"
- Stable zero dynamics

High gain feedback: $u = -ke$ "works" for sufficiently large gain $k > 0$

Funnel gain: $k(t) = \frac{1}{\psi(t) - |e(t)|}$ achieves tracking with prescribed perfomance