Determining a piecewise Lyapunov function for a biological piecewise affine system

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1 Introduction

Motivated by the olfactory system (i.e. the biological pathway of identifying smells), we would like to be able to determine meaningful interactions between neurons, with potential application to electronic noses. For our purposes, we make use of the model proposed in [1], which encompasses the interactions between the excitatory and inhibitory neurons within a subsection of the olfactory system in a threshold-linear network frame. The scalar version of this ODE system is given by

$$\begin{aligned}\dot r(t) &= -\alpha r(t) - \beta[u(t)-\theta_u]_+ + i(t),\\ \dot u(t) &= -\gamma u(t) + \delta[r(t)-\theta_r]_+,\end{aligned} \tag{1}$$

where $r(t), u(t) \in \mathbb{R}$ are the firing rates of the excitatory/inhibitory neuron; $\theta_r, \theta_u > 0$ are firing thresholds; $\alpha, \beta, \gamma, \delta > 0$ are biological parameters; $i(t)$ is an external input; $[.]_+$ denotes the half-way rectification function (i.e. $[p]_+ = \max\{0, p\}$).
We begin our work by analysing the stability of the homogeneous scalar version of this model, aiming for proving global asymptotic stability (GAS), which represents a key aspect in further working within input-to-state stability (ISS).

2 The Wanner Friedrich model - stability analysis

Due to the presence of the half-way rectification functions, the model (1) (with $i(t) = 0$) can be rewritten as a piecewise affine system of the form ($x = (r, u)$)

$$\dot x = A_k x + b_k, \quad x \in X_k, \quad \bigcup_k X_k = \mathbb{R}^2,$$

with four regions determined by the thresholds θ_r and θ_u. For the stability analysis and in view of the different qualitative behavior of the dynamics within these regions we propose a further refinement of these regions in a total of nine regions as indicated in Figure 1. As an example, for region X_0, which represents the below threshold space for both neuron types, the piecewise affine system will have the following form

$$\dot x = A_0 x + b_0 = \begin{pmatrix} -\alpha & 0 \\ 0 & -\gamma \end{pmatrix} x.$$

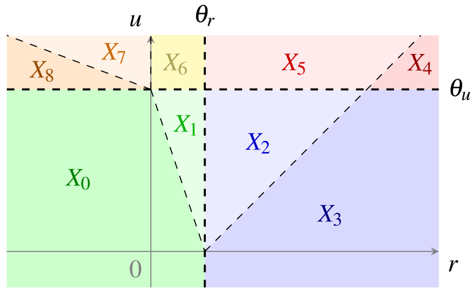

Figure 1: PAS proposed regions

In order to proceed in proving GAS, we would require certain ingredients for a piecewise Lyapunov candidate, as indicated in [2]. The main idea revolves around formulating regions that would be used in a polyhedral partition. Then, we generate the corresponding ray matrix of the cone, which will serve as a component in linear matrix inequalities (LMI). Solving these LMI‘s would result in the determination of local Lyapunov candidates, hence concluding the proof of GAS in the scalar case.

3 Conclusion

We are approaching the Wanner Friedrich homogeneous scalar model from a system theory perspective, since we are interested in studying ISS. As a key component, GAS, requires a more complex proof for our system in the form of a piecewise Lyapunov function suitable for this biological piecewise affine system. Our results and further work in the direction of ISS would aid in the extraction of key aspects related to interactions and stability of neural networks, that can be further applied to electronic noses.

References

[1] A. Wanner and R.W. Friedrich, ”Whitening of odor representations by the wiring diagram of the olfactory bulb,” Nat Neurosci. 23(3), 433-442, 2020.

[2] R. Iervolino, S. Trenn, F. Vasca, ”Asymptotic stability of piecewise affine systems with Filippov solutions via discontinuous piecewise Lyapunov functions”, IEEE Transactions on Automatic Control 66(4), 1513-1528, 2021.