Model reduction of switched systems via midpoint Gramians

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Problem formulation

Switched linear ODEs with state jumps

$$\dot{x} = A_\sigma x + B_\sigma u$$
$$x(t_k^+) = J_k^x x(t_k^-) + J_k^v v_k$$
$$y = C_\sigma x$$

$\sigma : [t_0, t_f) \to \{0, 1, 2, \dots, \mathtt{m}\}$
$A_0, A_1, \dots, A_\mathtt{m}, J_0^x, J_1^x, \dots, J_\mathtt{m}^x \in \mathbb{R}^{n\times n}$
$B_0, \dots, B_\mathtt{m} \in \mathbb{R}^{n\times m_u},\ J_0^v, \dots, J_\mathtt{m}^v \in \mathbb{R}^{n\times m_v}$
$C_0, C_1, \dots, C_\mathtt{m} \in \mathbb{R}^{p\times n}$

Reduced model

$$\dot{\widehat{x}} = \widehat{A}_\sigma \widehat{x} + \widehat{B}_\sigma u$$
$$\widehat{x}(t_k^+) = \widehat{J}_k^x \widehat{x}(t_k^-) + \widehat{J}_k^v v_k$$
$$\widehat{y} = \widehat{C}_\sigma z$$

$\widehat{A}_k \in \mathbb{R}^{r_k\times r_k}, \widehat{J}_k^x \in \mathbb{R}^{r_k\times r_{k-1}}$
$\widehat{B}_k \in \mathbb{R}^{r_k\times m_u},\ \widehat{J}_k^v \in \mathbb{R}^{r_k\times m_v}$
$\widehat{C}_k \in \mathbb{R}^{p\times r_k}$

Motivation

- Switched differential-algebraic equations
- Most general linear system class (covers switched and impulsive systems)

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$B_0, \ldots, B_\mathtt{m} \in \mathbb{R}^{n\times m_u}, \; J_0^v, \ldots, J_\mathtt{m}^v \in \mathbb{R}^{n\times m_v}$
$C_0, C_1, \ldots, C_\mathtt{m} \in \mathbb{R}^{p\times n}$

Reduced model

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Related research

- Simultaneous balancing (Monshizadeh et al. 2012)
- Output-depending switching (Papadopoulus & Prandini 2016)
- Enveloping (non-switched) system (Schulze & Unger 2018)
- Gramian-based approaches (Petrezcky, Gosea, ...)

Novel viewpoint

Consider switched linear ODE (without jumps) as special case of time-varying linear system

$$\begin{aligned}\dot{x} &= A(t)x + B(t)u \\ y &= C(t)x\end{aligned}$$

In particular, consider switching signal as given time-varying system parameter

Existing approaches unsuitable

Existing approaches (Imae, Shokoohi, Silverman, Verriest):

- Smoothness of coefficients assumed
- Reduced model is fully time-varying (not piecewise-constant)
- State jumps not considered

Challenge 1: Naive mode-wise reduction unsuitable

Example: Naive mode-wise reduction is not working

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Each mode is input-output equivalent to same scalar system

$$\dot{z} = u, \quad y = z$$

But outputs do not match anymore after switch!

Reducability of modes is effected by other modes

In example:
Second state is unobservable in first mode, but becomes observable in second mode

Challenge 2: Different reduced state-dimensions

Example: Reduced switched system with non-equal state-dimensions

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Reduced system (with identical input-output behavior):

on $[t_0, t_1)$:

$$\dot{z}^0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} z^0$$

on $[t_1, t_f)$:

$$\dot{z}^1 = 0 \cdot z^1 + u$$

$$y = z^1$$

with concatination condition: $z^1(t_1) = \begin{bmatrix} 1 & 0 \end{bmatrix} z^0(t_1)$

Reduced system necessarily contains jumps

Jumps occur in reduced model even if original system does not contain jumps

Challenge 3: Duration depend reduction

Reducability may depend on mode durations

Example:

on $[t_0, t_1)$:
$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_2)$:
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$
$$y = 0$$

on $[t_2, t_f)$
$$\dot{x} = 0$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

For $t_2 - t_1 = 2k\pi$ reduction possible to

on $[t_0, t_1)$:
$$\dot{z}^0 = 0 \cdot z^0 + u$$
$$y = z^0$$

on $[t_1, t_2)$:
no state
$$y = 0$$

on $[t_2, t_f)$
$$\dot{z}^2 = 0$$
$$y = z^2$$

But for almost all other switching durations: First two modes not reducible!

Duration-dependent reduction methods?

Effective reduction method necessarily duration dependent, but poses numerical challenges.

Challenge 4: Two types of input

$$\begin{aligned}\dot{x} &= A_\sigma x + B_\sigma u\\ x(t_k^+) &= J_k^x x(t_k^-) + J_k^v v_k\\ y &= C_\sigma x\end{aligned}$$

Decoupling of inputs possible?

What does “difficult to control” mean for this system class?

Content

Decoupling of inputs and overall reduction method

$$\boxed{\begin{aligned}\dot{x} &= A_\sigma x + B_\sigma u\\ x(t_k^+) &= J_k^x x(t_k^-) + J_k^v v_k\end{aligned}}$$

$$\boxed{\begin{aligned}\dot{x}_u &= A_\sigma x_u + B_\sigma u\\ x_u(t_k^+) &= J_k^x x_u(t_k^-)\end{aligned}} \qquad \boxed{\begin{aligned}\dot{x}_v &= A_\sigma x_v\\ x_v(t_k^+) &= J_k^x x_v(t_k^-) + J_k^v v_k\end{aligned}}$$

$$x = x_u + x_v$$

Key idea for model reduction

Identify difficult to reach and difficult to observe states via suitable Gramians:

- Calculated two types of reachability Gramians (one for u and one for v_k)
- Calculate observability Gramian
- Apply balanced truncation via midpoint Gramians

Time-varying reachability Gramian

Consider system with input u only (i.e. $v_k = 0$)

Definition (Time-varying reachability Gramian)

$$t \in (t_0, t_1]: \quad \mathcal{P}^\sigma(t) := \int_{t_0}^{t} e^{A_0(\tau - t_0)} B_0 B_0^\top e^{A_0^\top(\tau - t_0)} d\tau$$

$$t \in (t_k, t_{k+1}]: \quad \mathcal{P}^\sigma(t) := e^{A_k(t-t_k)} J_k^x \mathcal{P}^\sigma(t_k) J_k^{x\top} e^{A_k^\top(t-t_k)} + \int_{t_k}^{t} e^{A_k(\tau - t_k)} B_k B_k^\top e^{A_k^\top(\tau - t_k)} d\tau$$

Theorem (Required input energy to reach x_t on $[t_0, t]$)

$\operatorname{im} P^\sigma(t^-)$ *is the reachability space of the switched system with jumps and for all* $x_t \in \operatorname{im} P_k(t^-)$*:*

$$\min_{0 \xrightarrow{u} x_t} \int_{t_0}^{t} u(t)^\top u(t) \mathrm{d}t = x_t^\top P^\sigma(t^-)^\dagger x_t.$$

Time-varying observability Gramian

Consider homogeneous switched system (i.e. $u = 0$ and $v_k = 0$).

Definition (Time-varying observability Gramian)

$$t \in [t_{\mathtt{m}}, t_f): \quad Q^\sigma(t) := \int_t^{t_f} e^{A_{\mathtt{m}}^\top (t_f - \tau)} C_{\mathtt{m}}^\top C_{\mathtt{m}} e^{A_{\mathtt{m}}(t_f-\tau)} d\tau$$

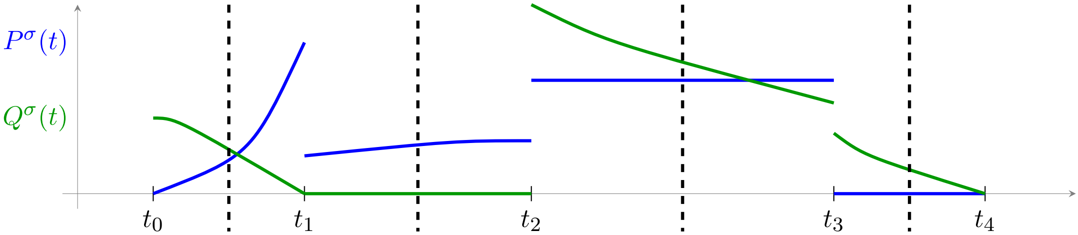
$$t \in [t_k, t_{k+1}): \quad Q^\sigma(t) := e^{A_k^\top (t_{k+1}-t)} {J_{k+1}^x}^\top Q^\sigma(t_{k+1}) J_{k+1}^x e^{A_k(t_{k+1}-t)} + \int_t^{t_{k+1}} e^{A_k^\top (t_{k+1}-\tau)} C_k^\top C_k e^{A_k(t_{k+1}-\tau)} d\tau$$

Theorem (Observable output energy from x_t on $[t, t_f]$)

$\ker Q^\sigma(t^+)$ *is the unobservable space of the switched system with jumps and for all* $x_t \in \mathbb{R}^n$*:*

$$\int_t^{t_f} y(t)^\top y(t) \mathrm{d}t = x_t^\top Q^\sigma(t^+) x_t.$$

Midpoint Gramians



Fully time-varying Gramians

- need one reachability and one observability Gramian per mode
- $\rightsquigarrow$ consider midpoint Gramians: $P^\sigma(\frac{t_k+t_{k+1}}{2})$ and $Q^\sigma(\frac{t_k+t_{k+1}}{2})$

Effect of discrete input v_k on reachability Gramian?

$P^\sigma(t)$ only considers reachability w.r.t. continuous input u, effect of v_k not yet considered
$\rightsquigarrow$ utilize decoupling $x = x_u + x_v$

Discrete reachability Gramian

Consider switched system with $u = 0$:
$$\begin{aligned}\dot{x} &= A_\sigma x\\ x(t_k^+) &= J_k^x x(t_k^-) + J_k^v v_k\end{aligned}$$

Solution $x(\frac{t_k+t_{k+1}}{2})$ at midpoints is given by solution of discrete time system

$$x_{k+1}^m = A_k^m x_k^m + B_k^m v_k$$

with

$$A_k^m := \mathrm{e}^{A_k \frac{t_{k+1}-t_k}{2}} J_k^x \mathrm{e}^{A_{k-1}\frac{t_k-t_{k-1}}{2}} \quad \text{and} \quad B_k^m := \mathrm{e}^{A_k \frac{t_{k+1}-t_k}{2}} J_k^v$$

Definition (Discrete reachability Gramian)

$$P_k^v := A_k^m P_{k-1}^v {A_k^m}^\top + B_k^v {B_k^v}^\top$$

with either $P_{-1}^v := 0$ for $x^m(0) = 0$ or $P_{-1}^v := \gamma X_0 X_0^\top$ for $x(0) \in \operatorname{im} X_0$

Midpoint based balanced truncation

Proposed reduction method

Mode-wise balanced truncation with w.r.t. combined midpoint reachability Gramian $P_k^m := P^\sigma(\frac{t_k+t_{k+1}}{2}) + \lambda P_k^v$ and midpoint observability Gramian $Q_k^m := P^\sigma(\frac{t_k+t_{k+1}}{2})$

Reminder: Balanced Truncation

Given: Midpoint Gramians (P_k^m, Q_k^m)

Method:

1. Find balancing transformation T_k, such that Gramians P_k^m, Q_k^m are equal and diagonal
2. Balanced Gramians $\rightsquigarrow$ how difficult to reach and observe are state directions
3. Remove simultaniously difficult to reach and observe state directions
   $\rightsquigarrow$ left- and right-projection matrices W_k, V_k as corresponding rows/columns of T_k^{-1}/T_k

Overall reduction algorithm for switched impulsive systems

$$\Sigma_\sigma: \quad \begin{aligned} \dot{x} &= A_k x + B_k u, && \text{on } (t_k, t_{k+1}) \\ x(t_k^+) &= J_k x(t_k^-) + J_k^v v_k, && k = 0,1,2,\ldots, \qquad x(t_0^-) \in \operatorname{im} X_0 \\ y &= C_k x \end{aligned}$$

Algorithm

Step 1a: Calculate midpoint reachability Gramians P_k^m forward in time w.r.t. u and v_k
Step 1b: Calculate midpoint observability Gramians Q_k^m backward in time

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Step 2a: Based on singular values of $P_k^m Q_k^m$ decide on reduction order r_k
Step 2b: Calculate left/right projectors W_k, V_k via standard balanced truncation

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Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (W_k A_k V_k, \quad W_k B_k, \quad C_k V_k)$

Step 4: Calculate reduced jump maps $\widehat{J}_k^x := W_k J_k^x V_{k-1}$ and $\widehat{J}_k^v := W_k J_k^v$

Step 5: Calculate $\widehat{X}_0 = I_{r_{-1}}$ with $r_{-1} := \dim \operatorname{im} X_0 - \dim(\operatorname{im} X_0 \cap \ker W_0 J_0^x)$

Content

System models and challenges

Reduction proceedure

Academic example and summary

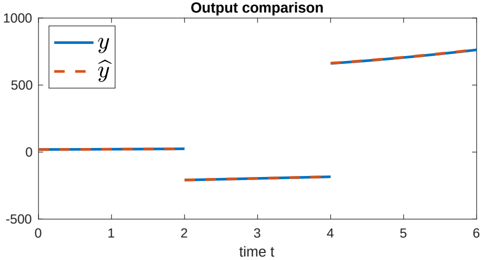
Random medium size example

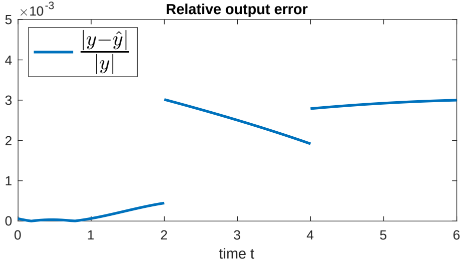
Consider a switched system with 3 random modes of sizes $n_1 = 50$, $n_2 = 60$, $n_3 = 40$ and with non-zero initial conditions in $\operatorname{im} X_0$ with dimension 5

Reduction parameters:

- $\lambda = 1$ (weight of discrete reachability Gramian)
- $\gamma = 0.1$ (weight of initial conditions)
- $\varepsilon_k = 10^{-3}$ (Hankel singular values threshold)

$\rightsquigarrow$ reduced switched system with reduced sizes $r_0 = 8$, $r_1 = 10$, $r_2 = 6$





Summary

- Midpoint-based balanced truncation method, suitable for switched linear systems with
  - known switching signal
  - mode-dependent state dimension
  - state-jumps
  - arbitrary non-zero initial values
- Remaining challenges
  - resolve numerical challenges with (almost) singular Gramians
  - relax dependence on exact knowledge of switching signal
  - error bounds
  - extension to nonlinear case

Hossain & T.: Midpoint based balanced truncation for switched linear systems with known switching signal. IEEE TAC 2024.

Hossain & T.: Model reduction for switched differential-algebraic equations with known switching signal. DAE Panel 2025 (Matlab implementation available on Zenodo).

T., Sutrisno, Thuan & Ha: Model reduction of singular switched systems in discrete time. ECC 2025.