Model reduction of singular switched systems in discrete time

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System class and motivation

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$$
$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k)$$

- $x : \mathbb{N} \to \mathbb{R}^n$ state, $u : \mathbb{N} \to \mathbb{R}^m$ input, $y \in \mathbb{N} \to \mathbb{R}^p$ output
- $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{n}\}$ switching signal
- $E_1, E_2, \ldots, E_{\mathtt{n}}, A_1, A_2, \ldots, A_{\mathtt{n}} \in \mathbb{R}^{n\times n}$ with E-matrices possibly singular
- $B_1, B_2, \ldots, B_{\mathtt{n}} \in \mathbb{R}^{n\times m}$, $C_1, C_2, \ldots, C_{\mathtt{n}} \in \mathbb{R}^{p\times n}$

Motivation

- Leontief economic model (Luenberger 1977)
- discretization of continuous-time switched DAEs (e.g. switched electrical circuits)

Goal

Find reduced model with approximately the same input-output behavior

Key challenges

Challenge 1

Surprisingly complex solution theory

Challenge 2

How to incorporate the switching signal in model reduction method?

Solution theory

Simple homogeneous example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{hSSS}$$

Example

Consider (hSSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Nonswitched solution behavior

$$\sigma \equiv 1: \qquad \left.\begin{aligned} x_1(k+1) &= x_1(k) \\ 0 &= x_2(k) \end{aligned}\right\} \quad \rightsquigarrow \quad x(k) = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \ \forall k \in \mathbb{N}$$

$$\sigma \equiv 2: \qquad \left.\begin{aligned} 0 &= x_1(k) \\ x_2(k+1) &= x_2(k) \end{aligned}\right\} \quad \rightsquigarrow \quad x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \ \forall k \in \mathbb{N}$$

Simple homogeneous example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(hSSS)}$$

Example

Consider (hSSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Switched solution behavior $\sigma(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s \end{cases}$

For $k < k_s$ we have $x(k) = \left(\begin{smallmatrix} c_1 \\ 0 \end{smallmatrix}\right)$ and for $k = k_s - 1$ also $x_1(k_s) = x_1(k_s - 1) = c_1$
BUT: For $k = k_s$ also $0 = x_1(k_s)$, hence $c_1 = 0$ necessary!
Furthermore $x_2(k_s)$ not constraint by mode 1 $\rightsquigarrow$ $x_2(k) = c_2$ for all $k \geq k_s$
$\rightsquigarrow$ $x(k) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $k < k_s$ and $x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix}$ for $k \geq k_s$

Simple homogeneous example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{hSSS}$$

Example

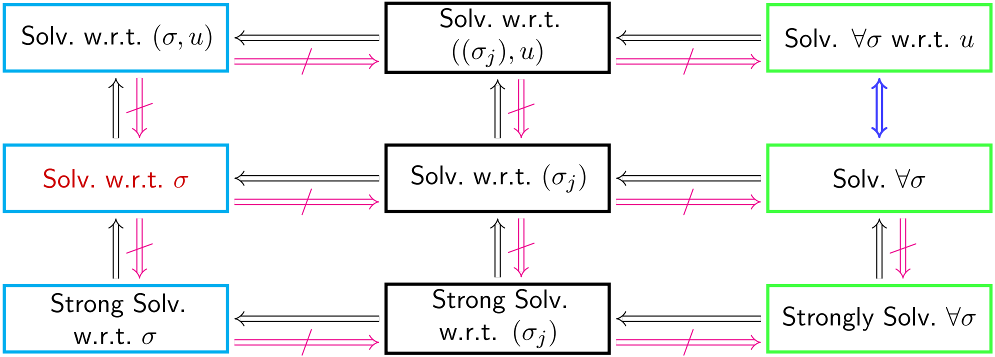
Consider (hSSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

No existence and uniqueness of solutions!

- Non-existence: Not all solutions from the past can be extended to a global solution
- Non-uniqueness: Single initial value leads to multiple solutions in the future
- Non-causality: Loss of causality w.r.t. to switching signal
- Above problems occur despite the individual modes being regular and index-1
- Considering input complicates situation further

Solvability concepts: overview



Sutrisno et al.: "Discrete-time switched descriptor systems: How to solve them?", Math. Control Signals Syst. (2025). https://doi.org/10.1007/s00498-025-00419-7 (open access)

Solvability characterization

Notation for (E_i, A_i, B_i):

$$\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i), \quad \widehat{\mathcal{S}}_i := A_i^{-1}(\operatorname{im}[E_i, B_i]), \quad \widehat{\mathcal{R}}_i := E_i^{-1}(\operatorname{im}[A_i, B_i])$$

Definition

The family $\{(E_i, A_i, B_i)\}_{i\in\{1,\dots,\mathtt{n}\}}$ is called switched index-1 w.r.t. σ $:\Longleftrightarrow$

- $\operatorname{im} B_i \subseteq \operatorname{im}[E_i, A_i] \quad \forall i$ and
- $\widehat{\mathcal{R}}_{\sigma(k)} + \widehat{\mathcal{S}}_{\sigma(k+1)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \quad \forall k$

Theorem (Solvability characterization)

(SSS) with given σ is solvable $\Longleftrightarrow$ $\{(E_i, A_i, B_i)\}_i$ is switched index-1 w.r.t. σ

Relationship to index 1

Fact: (E, A) is regular and index 1 $\Longleftrightarrow$ $\ker E \oplus \mathcal{S} = \mathbb{R}^n$
BUT: regularity and index-1 is neither necessary nor sufficient for switched index-1!

Explicite solution formula

Notation:

- $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} + \mathcal{W} \to \mathcal{V}$ denotes any (not necessarily unique) projector such that

$$\Pi_{\mathcal{V}}^{\mathcal{W}} \mathcal{V} = \mathcal{V} \quad \text{and} \quad \Pi_{\mathcal{V}}^{\mathcal{W}} \mathcal{W} = \mathcal{V} \cap \mathcal{W}$$

- M^+ denotes any (not necessarily unique) generalized inverse of M, i.e. $MM^+M = M$

Theorem

(SSS) is solvable w.r.t. σ, then x is a solution on $[k_0, k_1]$ if $x(k_0) \in \mathcal{S}_{\sigma(k_0)} - \{B^a_{\sigma(k_0)} u(k_0)\}$ and

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k) + \Psi^c_{\sigma(k+1),\sigma(k)} u(k) + \Psi^a_{\sigma(k+1),\sigma(k)} u(k+1)$$

where

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} E_j^+ \Pi_{\operatorname{im} E_j}^{\operatorname{im} A_j} A_j, \qquad \Psi^c_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} B^c_j, \qquad \Psi^a_{i,j} := (\Pi_{\mathcal{S}_i}^{\ker E_j} - I) B^a_i,$$

$$B^c_j := E_j^+ \Pi_{\operatorname{im} E_j}^{\operatorname{im} A_j} B_j, \qquad B^a_i := -A_i^+ \Pi_{\operatorname{im} A_i}^{\operatorname{im} E_i} B_i$$

Surrogate system and reduced model

With $\tilde{u}(k) := \begin{pmatrix} u(k) \\ u(k+1) \end{pmatrix}$ and $\tilde{A}_k := \Phi_{\sigma(k+1),\sigma(k)}$, $\tilde{B}_k := [\Psi^c_{\sigma(k+1),\sigma(k)}, \Psi^a_{\sigma(k+1),\sigma(k)}]$, $\tilde{C}_k := C_{\sigma(k)}$, $\tilde{D}_k := [D_\sigma, 0]$ we then have:

Sovable

$$\begin{aligned} E_\sigma x^+ &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

$\Longleftrightarrow$

Surrogate system

$$\begin{aligned} x^+ &= \tilde{A}_k x + \tilde{B}_k \tilde{u} \\ y &= \tilde{C}_k x + \tilde{D}_k \tilde{u} \end{aligned}$$

discrete-time-varying balanced truncation

Reduced system

$$\begin{aligned} \widehat{x}^+ &= \widehat{A}_k \widehat{x} + \widehat{B}_k \tilde{u} \\ \widehat{y} &= \widehat{C}_k \widehat{x} + \tilde{D}_k \tilde{u} \end{aligned}$$

Discrete-time-varying balanced truncation

Time-varying Gramians

$$\begin{aligned} x(k+1) &= \tilde{A}_k x(k) + \tilde{B}_k \tilde{u}(k) \\ y(k) &= \tilde{C}_k x(k) + \tilde{D}_k \tilde{u}(k) \end{aligned} \qquad \text{(Surr)}$$

Definition (Controllability and observability Gramians)

$$P_{k_0} := 0, \qquad P_{k+1} := \tilde{A}_k P_k \tilde{A}_k^\top + \tilde{B}_k \tilde{B}_k^\top, \qquad k = k_0, k_0+1, \ldots, k_f - 1$$

$$Q_{k_f} := \tilde{C}_{k_f}^\top \tilde{C}_{k_f}, \qquad Q_{k-1} := \tilde{A}_{k-1}^\top Q_k \tilde{A}_{k-1} + \tilde{C}_{k-1}^\top \tilde{C}_{k-1}, \qquad k = k_f, k_f - 1, \ldots, k_0 + 1$$

Theorem (Input and output energy)

$\forall x_k \in \operatorname{im} P_k$: $\quad x_k^\top P_k^+ x_k = \min \left\{ \sum_{\ell=k_0}^{k-1} u(\ell)^\top u(\ell) \;\middle|\; \begin{array}{l} u \text{ is s.t. solution } x \text{ of (Surr)} \\ \text{satisfies } x(k_0) = 0 \text{ and } x(k) = x_k \end{array} \right\}$

$\forall$ *solutions* x *of (Surr) on* $[k, k_f]$: $\quad x(k)^\top Q_k x(k) = \sum_{\ell=k}^{k_f} y(\ell)^\top y(\ell)$

Time-varying balancing

$$\begin{aligned} x(k+1) &= \tilde{A}_k x(k) + \tilde{B}_k \tilde{u}(k) \\ y(k) &= \tilde{C}_k x(k) + \tilde{D}_k \tilde{u}(k) \end{aligned} \qquad \text{(Surr)}$$

Definition (Balanced system)

(Surr) is called balanced $:\Longleftrightarrow$ $\exists$ positive definite diagonal matrix Σ_k, Σ_k^r, Σ_k^o:

$$P_k = \operatorname{diag}(\Sigma_k, \Sigma_k^r, 0, 0) \quad \text{and} \quad Q_k = \operatorname{diag}(\Sigma_k, 0, \Sigma_k^o, 0)$$

Theorem (cf. Thm. 7.5 in Zhou & Doyle 1999)

There always exists a (time-varying) coordinate transformation resulting in a balanced system.

Note: For $x(k) = T_k z(k)$ the transformed system is
$$\begin{aligned} z(k+1) &= \overline{A}_k z(k) + \overline{B}_k \tilde{u}(k) \\ y(k) &= \overline{C}_k z(k) + \tilde{D}_k \tilde{u}(k) \end{aligned}$$
with

$$\overline{A}_k := T_{k+1}^{-1} \tilde{A}_k T_k, \quad \overline{B}_k := T_{k+1}^{-1} B_k, \quad \overline{C}_k := C_k T_k, \quad \text{and} \quad \overline{P}_k = T_k^{-1} P_k T_k^{-\top}, \quad \overline{Q}_k = T_k^\top Q_k T_k$$

Time-varying balanced truncation

For each k choose reduction size r_k (e.g. by defining threshold for diagonal entries in Σ_k) and let

$$\Pi_k^l := [I_{r_k}\ 0]T_k^{-1} \quad \text{and} \quad \Pi_k^r := T_k \begin{bmatrix} I_{r_k} \\ 0 \end{bmatrix}$$

The reduced model is then

$$\begin{aligned} \widehat{x}(k+1) &= \widehat{A}_k \widehat{x}(k) + \widehat{B}_k \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix} \\ y(k) &= \widehat{C}_k \widehat{x}(k) + D_k u(k) \end{aligned}$$

with

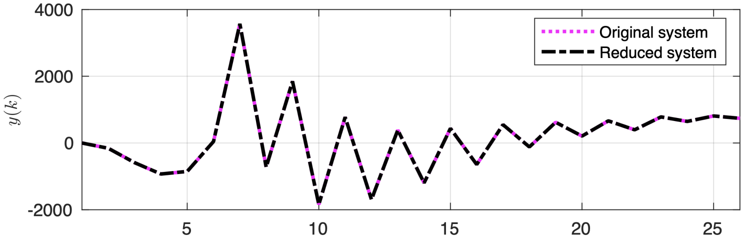
$$\widehat{A}_k := \Pi_{k+1}^l \tilde{A}_k \Pi_k^r, \quad \widehat{B}_k := \Pi_{k+1}^l \tilde{B}_k, \quad \widehat{C}_k := \tilde{C}_k \Pi_k^r$$

Example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$$
$$y(k) = C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k)$$

- $n = 100$, $m = p = 1$, rank $E_i = 50$, otherwise random matrices
- $[k_0, k_f] = [1, 2, \dots, 26]$, switching sequence $(1, 2, 1, 2, 1)$ with switching times $(6, 11, 16, 21)$
- With reduction threshold 0.1 the reduced model sizes are

k	1	2	3	4	5	6	7	$\cdots$	19	20	21	22	23	24	25	26
r_k	0	2	4	6	6	6	7	$\cdots$	7	6	6	5	4	3	2	1



Summary

Sovable

$$E_\sigma x^+ = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x + D_\sigma u$$

Surrogate system

$$x^+ = \tilde{A}_k x + \tilde{B}_k \tilde{u}$$
$$y = \tilde{C}_k x + \tilde{D}_k \tilde{u}$$

discrete-time-varying balanced truncation

Reduced system

$$\widehat{x}^+ = \widehat{A}_k \widehat{x} + \widehat{B}_k \tilde{u}$$
$$\widehat{y} = \widehat{C}_k \widehat{x} + \tilde{D}_k \tilde{u}$$

Remaining challenges

- Error bounds
- Robust and efficient numerical implementations
- Uncertain switching signal