Model reduction for switched linear systems

Stephan Trenn

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

Joint work with **Sumon Hossain** (North South University, Dhaka, Bangladesh)

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Problem formulation

System class: Switched linear ODEs

$$
\begin{aligned}
\dot{x} &= A_\sigma x + B_\sigma u \\
y &= C_\sigma x
\end{aligned}
$$

$\sigma : [t_0, t_f) \to \{0, 1, 2, \ldots, \mathtt{m}\}$
$A_0, A_1, \ldots, A_{\mathtt{m}} \in \mathbb{R}^{n\times n}$
$B_0, B_1, \ldots, B_{\mathtt{m}} \in \mathbb{R}^{n\times m}$
$C_0, C_1, \ldots, C_{\mathtt{m}} \in \mathbb{R}^{p\times n}$

Reduced model

$$
\begin{aligned}
\dot{z} &= \widehat{A}_\sigma z + \widehat{B}_\sigma u \\
y &= \widehat{C}_\sigma z
\end{aligned}
$$

$\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n}\times \widehat{n}}$
$\widehat{B}_0, \widehat{B}_1, \ldots, \widehat{B}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n}\times m}$
$\widehat{C}_0, \widehat{C}_1, \ldots, \widehat{C}_{\mathtt{m}} \in \mathbb{R}^{p\times \widehat{n}}$

Related research

- Simultaneous balancing (Monshizadeh et al. 2012)
- Output-depending switching (Papadopoulus & Prandini 2016)
- Enveloping (non-switched) system (Schulze & Unger 2018)
- Gramian-based approaches (Petrezcky, Gosea, ...)

Novel viewpoint

Consider switched linear ODE as special case of time-varying linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}$$

In particular, consider switching signal as given time-varying system parameter

Existing approaches unsuitable

Existing approaches (Imae, Shokoohi, Silverman, Verriest):

- Smoothness of coefficients assumed
- Reduced model is fully time-varying (not piecewise-constant)

Challenge: Mode-wise reduction

Naive mode-wise reduction is not working

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Each mode is input-output equivalent to same scalar system

$$\dot{z} = u, \quad y = z$$

But outputs do not match anymore after switch!

Reducability of modes is effected by other modes

In example:
Second state is unobservable in first mode, but becomes observable in second mode

Challenge: Different reduced state-dimensions

Reduced switched system with non-equal state-dimensions

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Reduced system (with identical input-output behavior):

on $[t_0, t_1)$:

$$\dot{z}^0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} z^0$$

on $[t_1, t_f)$:

$$\dot{z}^1 = 0 \cdot z^1 + u$$
$$y = z^1$$

with concatination condition: $z^1(t_1) = \begin{bmatrix} 1 & 0 \end{bmatrix} z^0(t_1)$

New system class: Switched ODEs with jumps

Model reduction leaves original system class in general.

Challenge: Duration depend reduction

Reducability may depend on mode durations

Example:

on $[t_0, t_1)$:
$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_2)$:
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$
$$y = 0$$

on $[t_2, t_f)$
$$\dot{x} = 0$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

For $t_2 - t_1 = 2k\pi$ reduction possible to

on $[t_0, t_1)$:
$$\dot{z}^0 = 0 \cdot z^0 + u$$
$$y = z^0$$

on $[t_1, t_2)$:
no state
$$y = 0$$

on $[t_2, t_f)$
$$\dot{z}^2 = 0$$
$$y = z^2$$

For almost all other switching durations: First two modes not reducible!

Duration-dependent reduction methods?

Duration dependent methods challenging (numerically expensive, non-robust)

More general system class

Switched linear ODEs with jumps

$$\Sigma_\sigma: \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0, 1, 2, \dots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Key features:

- Known switching signal with $\sigma(t) = k$ on $[t_k, t_{k+1})$
- states $x^k : (t_k, t_{k+1}) \to \mathbb{R}^{n_k}$ may have mode-dependent dimension
- $J_k : \mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$ defines jumps at switch
- Reduced model in same system class
- Certain switched DAEs $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$, $y = C_\sigma x$ fall into this class

Content

System models and challenges

Midpoint balanced truncation method

Academic example and summary

Time-varying Gramians

Key idea: Remove difficult to observe and difficult to reach states

Define suitable reachability and observability Gramians

$$\mathcal{P}_0^\sigma(t) := \int_{t_0}^{t} e^{A_0(\tau - t_0)} B_0 B_0^\top e^{A_0^\top(\tau - t_0)} d\tau, \qquad t \in [t_0, t_1],$$

$$\mathcal{P}_k^\sigma(t) := e^{A_k(t-t_k)} J_k \mathcal{P}_{k-1}^\sigma(t_k) J_k^\top e^{A_k^\top(t-t_k)} + \int_{s_k}^{t} e^{A_k(\tau - t_k)} B_k B_k^\top e^{A_k^\top(\tau - t_k)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

$$\mathcal{Q}_{\mathtt{m}}^\sigma(t) := \int_{t}^{t_f} e^{A_{\mathtt{m}}^\top(t_f - \tau)} C_{\mathtt{m}}^\top C_{\mathtt{m}} e^{A_{\mathtt{m}}(t_f - \tau)} d\tau, \qquad t \in [t_{\mathtt{m}}, t_f],$$

$$\mathcal{Q}_k^\sigma(t) := e^{A_k^\top(s_{k+1}-t)} J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(t_{k+1}) J_{k+1} e^{A_k(t_{k+1}-t)} + \int_{t}^{t_{k+1}} e^{A_k^\top(t_{k+1} - \tau)} C_k^\top C_k e^{A_k(t_{k+1} - \tau)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

Midpoint based balanced truncation

$$m_k := (t_k + t_{k+1})/2$$

Use midpoint-Gramians

Use $\mathcal{P}_k^\sigma(m_k)$ and $\mathcal{Q}_k^\sigma(m_k)$ for balance truncation of mode k

Reminder: Balanced Truncation

Given: System matrices (A, B, C), Gramians (P, Q)
Method:

1. Find balancing coordinate transformation T, such that Gramians $\tilde{P}, \tilde{Q}$ of $(\tilde{A}, \tilde{B}, \tilde{C}) = (TAT^{-1}, TB, CT^{-1})$ are equal and diagonal
2. Balanced Gramians $\rightsquigarrow$ how difficult to reach and observe are state directions
3. Remove difficult to observe state directions $\rightsquigarrow$ left- and right-projection matrices Π^l, Π^r
4. Reduced system matrices $(\hat{A}, \hat{B}, \hat{C}) = (\Pi^l A \Pi^r, \Pi^l B, C\Pi^r)$

Overall reduction algorithm for switched systems

$$\Sigma_\sigma: \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0,1,2,\ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Algorithm

Step 1a: Calculate midpoint reachability Gramians $\mathcal{P}_k^\sigma(m_k)$ forward in time
Step 1b: Calculate midpoint observability Gramians $\mathcal{Q}_k^\sigma(m_k)$ backward in time

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Step 2a: Based on singular values of $\mathcal{P}_k^\sigma(m_k)\mathcal{Q}_k^\sigma(m_k)$ decide on reduction order $\widehat{n}_k$
Step 2b: Calculate left/right projectors Π_k^l, Π_k^r via standard balanced truncation

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Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (\Pi_k^l A_k \Pi_k^r, \Pi_k^l B_k, C_k \Pi_k^r)$

Step 4: Calculate reduced jump map $\widehat{J}_k := \Pi_k^l J_k \Pi_{k-1}^r$

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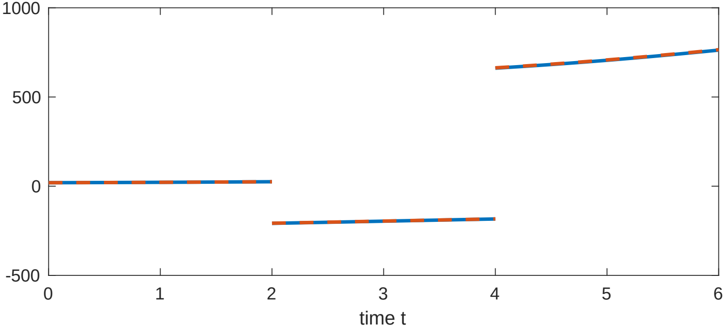
Random medium size example

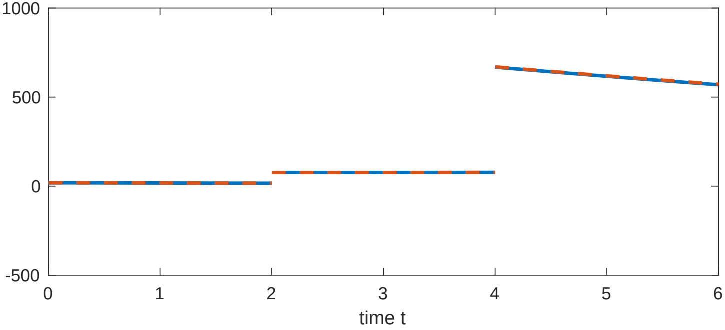
Consider a switched system with 3 random modes of sizes $n_1 = 50$, $n_2 = 60$, $n_3 = 40$



Reduction threshold 10^{-3} $\rightsquigarrow$ reduced sizes $\widehat{n}_1 = 8$, $\widehat{n}_2 = 9$, $\widehat{n}_3 = 5$

Random medium size example

Consider a switched system with 3 random modes of sizes $n_1 = 50$, $n_2 = 60$, $n_3 = 40$



Reduction threshold 10^{-3} ⇝ reduced sizes $\widehat{n}_1 = 8$, $\widehat{n}_2 = 9$, $\widehat{n}_3 = 5$

Rerun simulation with a different input and different initial values (same reduced model!)

Summary

- Midpoint-based balanced truncation method, suitable for switched linear systems with
  - known switching signal
  - mode-dependent state dimension
  - state-jumps
  - arbitrary non-zero initial values
  - arbitrary input signals
- Remaining challenges
  - resolve numerical challenges with (almost) singular Gramians
  - relax dependence on exact knowledge of switching signal
  - error bounds
  - extension to nonlinear case