Discrete-time switched descriptor systems: How to solve them?

S. Sutrisno[1], Duc Thuan Do[2], Phi Ha[2], M. Munadi[3], Stephan Trenn[4,*]

[1]Dept. of Mathematics, Universitas Diponegoro, Jalan Jacub Rais, Semarang, 50275, Indonesia.
[2]Faculty of Mathematics and Informatics (FaMI), Hanoi University of Science and Technology, 01 Dai Co Viet Str., Hai Ba Trung, Hanoi, Vietnam.
[3]Dept. of Mechanical Engineering, Universitas Diponegoro, Jalan Prof. Soedarto SH, Semarang, 50275, Indonesia.
[4*]Bernoulli Institute for Mathematics, CS, and AI, University of Groningen, Nijenborgh 9, Groningen, 9747 AG, The Netherlands.

Contributing authors: s.sutrisno@live.undip.ac.id; thuan.doduc@hust.edu.vn; phi.ha@hust.edu.vn; munadi@ft.undip.ac.id; s.trenn@rug.nl;

Abstract

We study the solution theory of singular linear switched systems with inputs (also known as switched descriptor systems). These systems are highly relevant in many applications; in particular, in economics the well known dynamic Leontief model with changing coefficient matrices falls into this class. Theorem 5.1 in the paper by Anh et al. (2019) stated that if a singular linear switched system is jointly index-1 then there exists an explicit surrogate switched system having identical solution behavior for all switching signals. However, it was not clear yet whether the jointly index-1 condition is a necessary and sufficient condition for the existence and uniqueness of a solution. Furthermore, it was also not clear what conditions are actually required to guarantee existence and uniqueness of solutions for particular switching signals only. In this article, we provide necessary and sufficient conditions for existence and uniqueness of solutions for singular linear switched systems with respect to fixed switching signals (both mode sequences and switching times are fixed), fixed mode sequences (switching times are arbitrary), and arbitrary switching signals (both mode sequences and switching times are arbitrary). In all three cases we provide an explicit surrogate system with the same solution set; our approach improves the results presented in Anh et al. (2019) as the coefficient matrices describing the transition from $\boldsymbol{x}(\boldsymbol{k})$ to $\boldsymbol{x}(\boldsymbol{k}+\mathbf{1})$ only depend on original system matrices at time $\boldsymbol{k}$ and $\boldsymbol{k}+\mathbf{1}$ and not on $\boldsymbol{k}-\mathbf{1}$ as in Anh et al. (2019). We illustrate the theoreticals findings with the dynamic Leontief model and investigate the solvability properties of discretizations of continuous-time singular systems.

Keywords: descriptor systems, difference-algebraic systems, Leontief economic model, switched systems

1 Introduction

The act of transitioning among diverse system structures is a fundamental element in various systems, including power systems [1] and electronics [2]. Additionally, switched systems naturally emerge in sampled-data systems [3–5]. In this study, we consider inhomogeneous switched linear singular systems of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1}$$

where $k \in \mathbb{N}$ represents the time instant or time step, $x(k) \in \mathbb{R}^n$, $n \in \mathbb{N}$ denotes the vector of states, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ stands for the vector of inputs, the map $\sigma : \mathbb{N} \to \mathtt{M}$ expresses the switching signal which determines which mode from the (finite or infinite) index set $\mathtt{M}$ is active at a time instant k, and $E_i, A_i \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times m}$ are constant matrices for every $i \in \mathtt{M}$.

Applications of the systems of the form (1), which are also known as descriptor or implicit systems, can be found in numerous fields such as electrical circuits [6], industrial processes [7], power systems [8], economic systems [9], constrained mechanical systems [10, 11], robotics [12–14], and neural networks [15], among others. Furthermore, the dynamic Leontief economic model, or input-output analysis, has the form of system (1) (without switching), see e.g. [16, 17]; the switched case occurs when the parameters change in time. This Leontief model is crucial for analyzing interdependencies among economic parties and helps policymakers and businesses optimize production, assess supply chain impacts, and predict economic shifts due to policy changes or external shocks. In civil engineering, it can be used to evaluate the risk in complex interconnected infrastructures [18]. In the economic analysis of a country, it enables the estimation of the resource and value-added of inter-sectoral relations [19].

The matrices E_i are in general singular, but we do allow that some of the matrices E_i in (1) are invertible; however, if all matrices E_i are invertible, then (1) can easily be rewritten as an explicit switched linear systems for which the solution theory is trivial. In fact, if all E_i are invertible, then for arbitrary initial value $x_0 \in \mathbb{R}^n$, for arbitrary switching signal, and for arbitrary input sequence $(u(0), u(1), \ldots)$, the system with the initial condition $x(0) = x_0$ has a unique solution at any time instant $k \in \mathbb{N}$. In this case, the solution for the state at any time instant can be calculated by simply propagating the equation forward in time, see e.g. [20], and no particular solvability notions are necessary to be defined for its well-posedness. Additionally, an explicit switched system is strictly causal in the sense that the current state only depends on the previous state, previous switching signal, and previous input. Those features may not be possessed by a singular switched system of the form (1). For an example of a singular switched system that is not well-posed (although each mode is well-posed), see [21, Example 1.1], whereas for an example of a singular system which is not strictly causal w.r.t. the input, consider the singular (non-switched) system

$$0 = x(k) + u(k). \tag{2}$$

Furthermore, for switched singular systems, solvability in general depends on the switching signal as well as on the input. To illustrate this, consider the switched system (1) with $(E_0, A_0, B_0) = (1, 0, 1)$ and $(E_1, A_1, B_1) = (0, 1, 1)$. Note that both modes correspond to well-posed non-switched systems. For a switching signal of the form $\sigma(k) = 0$ for $k < k^s$ and $\sigma(k) = 1$ for $k \geqslant k^s$, the switched system (1) takes the form

$$\begin{array}{r|l} k < k^s : & k \geqslant k^s : \\ x(k+1) = u(k) & 0 = x(k) + u(k). \end{array}$$

We can easily see that a unique solution exists for any given initial value $x(0)$ if, and only if $u(k^s - 1) = -u(k_s)$. On the other hand, if we reverse the order of modes, we see that the value for $x_2(k^s)$ is not restricted, i.e. we do not have uniqueness of solutions for this reversed switching signal.

The fact that the well-posedness of the individual modes is in general not sufficient for the well-posedness of the switched singular system has often been overlooked in the literature on switched singular systems and only recently a complete solution theory for the homogeneous case (i.e. $u(k) = 0$ for all $k \in \mathbb{N}$) has been presented [22]. In there, the three different notions *jointly index-1*, *sequentiel index-1* and *switched index-1* have been introduced for families $\{(E_i, A_i)\}_{i \in \mathtt{M}}$ of matrix pairs and these notions have been shown to be equivalent to certain solvability concepts. However, the presence of an input complicates the analysis significantly. For example, the existence of at least one solution (namely $x(k) = 0$ for all $k \in \mathbb{N}$) for the homogeneous switched system is always guaranteed, whereas this is not the case anymore for inhomogeneous systems. Furthermore, the initial value can in general not be chosen independently from the input.

To the best of our knowledge the only available rigorous solution theory for (discrete time) inhomogeneous switched system is contained in the last part of the conference contribution [21]. The well-posedness of the switched singular system is shown for the jointly index-1 case, however, the solution formula for $x(k+1)$ does not only depend on coefficient matrices at time $k+1$ and k (which one would intuitively expect) but also on $k-1$. Furthermore, necessary assumptions for solvability for given switching signals and/or inputs have not been studied yet. The general case of time-varying inhomogeneous descriptor systems in discrete time with $B_k = I$ has however been studied in [23] and is based on global transformations and the strangeness index; the delicate interplay between the input space (given by B_k) and the switching signal is however not discussed therein, neither is the important concept of causality with respect to the switching signal considered.

Our goal with this contribution is to close this gap in the literature and provide a comprehensive solution theory for switched singular systems (1). Therefore, we define novel solvability notions that differ w.r.t. the role of the switching signal and the inputs. In total this leads to nine different solvability notions which we are able to fully characterize; these results are summarized in Figure 1.

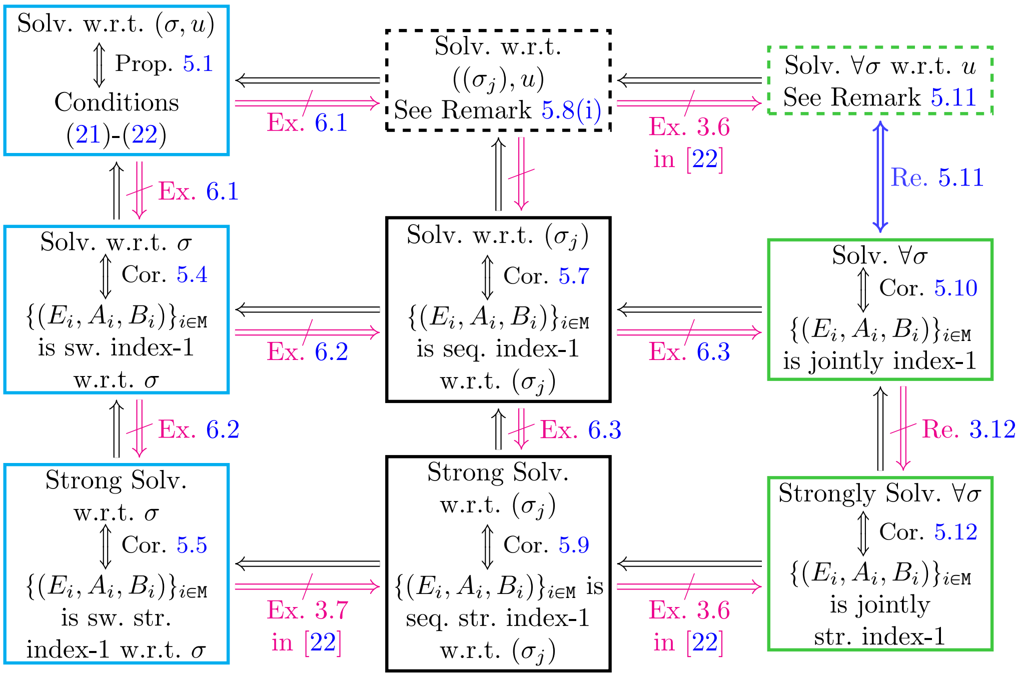

Fig. 1 Summary of the solvability characterizations; all implications and nonimplications are discussed in Section 6.

Furthermore, we propose surrogate systems (explicit systems that have identical solution behaviors), which can be used to further analyze the system behavior (e.g. reachability and stabilizability) in future work.

This article is structured as follows. In Section 2, we introduce the solvability notions studied in this paper for system (1). These solvability notions are motivated by the solvability issues discussed in Section 1. In Section 3, some concepts from algebra are revisited. Some lemmas are also presented, which are used later in most parts of the study. In Section 4, a key lemma presenting a necessary and sufficient condition for a generic system of linear equations of system (1) is also presented here. This later is used as the foundation to study the solvability of the switched system (1).

The main results for the solvability characterizations are presented in Section 5. The next three sections present some (counter) examples, alternative approaches for the solvability characterizations, and applications of the results for Leontief economic models and discretized switched differential-algebraic equations.

Throughout the manuscript, we use the following standard notation. $\mathbb{R}$ and $\mathbb{N}$ denote the real and natural numbers (including zero). For two subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, $\mathcal{V} \oplus \mathcal{W}$ means the direct sum of $\mathcal{V}$ and $\mathcal{W}$, in particular, $\mathcal{V} \cap \mathcal{W} = \{0\}$ is implicitly required. For a (possible singular or rectangular) matrix $A \in \mathbb{R}^{\ell \times n}$, let $\operatorname{im} A$ be the image (or range or column space) of A, $\ker A$ be the kernel (null space) of A, and $A^{-1}(\mathcal{V}) = \{ \xi \in \mathbb{R}^n \mid A\xi \in \mathcal{V} \}$ be the preimage of A over a set $\mathcal{V} \subseteq \mathbb{R}^n$. For two integers $k_1 < k_2$ we define the "closed" (discrete-time) interval $[k_1, k_2] := \{k_1, k_1 + 1, \ldots, k_2 - 1, k_2\}$ and "half-open" interval $[k_1, k_2) := \{k_1, k_1 + 1, \ldots, k_2 - 1\}$.

2 Solvability notions

2.1 Classes of switching signals

In this study, we assume that the switching signal σ has the form

$$\sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s), k_{j+1}^s > k_j^s,\ j = 0, 1, 2, \ldots \tag{3}$$

where $k_j^s \in \mathbb{N}$, $j = 0, 1, \ldots$ are the switching times with the initial (switching) time $k_0^s = 0$ and $\sigma_j \in \mathtt{M}$. Note that the switching signal σ is triggered only by the time and not triggered by states or inputs, and furthermore, it can be seen as a piecewise constant function (see Fig. 1 in [22] for an illustration). For every $i \in \mathtt{M}$, the corresponding (nonswitched) system $E_i x(k+1) = A_i x(k) + B_i u(k)$ is called the i-th mode or subsystem.

Apart from unrestricted switching signals, there are two types of restricted switching signals: fixed mode sequences and fixed switching signals. Those three classes of switching signals are described precisely as follows:

2.1.1 Arbitrary Switching Signals

The term "arbitrary switching signals" means only the set of modes $\mathtt{M}$ (and the corresponding family of coefficient matrices $\{E_i, A_i, B_i\}_{i \in \mathtt{M}}$ is known), and both mode sequences and switching times are unknown. Thus, study results under arbitrary switching signals are also valid for specific or constrained switching signals. However, the characterization of the solvability is in general not necessary when restricted switching signals are being considered, and thus studies under restricted switching signals are also crucial for switched systems.

2.1.2 Fixed Mode Sequences

A fixed mode sequence, denoted by $(\sigma_0, \sigma_1, \ldots) =: (\sigma_j)_{j=0,1,\ldots}$ (for short just (σ_j)), has the information of the initial mode which actives at the initial time $k = 0$ and its subsequent modes in the future, however, the switching times are unknown. If a finite time interval $[0, K]$, $K \in \mathbb{N}$ is under consideration, then a fixed (finite) mode sequence on this time interval refers to $(\sigma_0, \sigma_1, ..., \sigma_J) =: (\sigma_j)_{j=0,1,\ldots,J}$ (the short notation (σ_j) can also be used together with the information of a finite time interval being considered). Therefore, with respect to a fixed mode sequence, investigations are done under a known mode sequence but with arbitrary mode durations. This implies that results are valid for all switching signals with the same mode sequence. However, in general, solvability for other switching signals with a different mode sequence cannot be concluded.

2.1.3 Fixed Switching Signals

A fixed switching signal σ is uniquely determined by its mode sequence $(\sigma_0, \sigma_1, \ldots)$ and the sequence of mode durations $(k^s_{j+1} - k^s_j)_{j=0,1,\ldots}$ defined as in (3). Mode σ_0 is referred to as the initial mode.

2.2 Causal solvability notions

In most practical applications and also from a theoretical standpoint causality is a very important property. We therefore restrict our attention to descriptor systems with certain causality properties. The key idea is that in order to determine the value $x(k)$ from the past, only information up to time k should be utilized (which includes the actual equations but also the input). This viewpoint may seem a bit artificial for the unswitched case (because the equations in the future are the same as in the past), however, this viewpoint is quite natural when considering the switched case (in which we are ultimately interested in), because there the future equations may be different to the current or past ones. Furthermore, causality also means that a local solution can always be extended into the future.

Definition 2.1 (Solvability notions w.r.t. fixed switching signal) (a) We call (1) with given σ *locally uniquely causally solvable* (short: *solvable*) w.r.t. a given input $u : \mathbb{N} \to \mathbb{R}^m$ if for all $k_0 \leqslant k_1$ there exists a state trajectory $x : [k_0, k_1] \to \mathbb{R}^n$ and some arbitrary $x(k_1 + 1) \in \mathbb{R}^n$ such that (1) is satisfied for $k \in [k_0, k_1]$; furthermore, the solution has to be causal in the sense that for every $k_1' \in [k_0, k_1]$ any solution on $[k_0, k_1']$ can be uniquely extended to a solution on $[k_0, k_1]$.

(b) We call (1) with given σ (locally uniquely causally) solvable if it is solvable w.r.t. all inputs.

(c) We call (1) with given σ *strongly locally uniquely causally solvable* (short: *strongly solvable*) if for all $k_0 < k_1$, all $x_0 \in \widehat{\mathcal{S}}_{\sigma(k_0)} := A^{-1}_{\sigma(k_0)}(\operatorname{im}[E_{\sigma(k_0)}, B_{\sigma(k_0)}])$ and all $u : [k_0, k_1] \to \mathbb{R}^m$ there exists a unique $x : [k_0, k_1] \to \mathbb{R}^n$ with $x(k_0) = x_0$ and some $x(k_1 + 1) \in \mathbb{R}^n$ such that (1) is satisfied for $k \in [k_0, k_1]$.

Before further studying the necessary and sufficient conditions for solvability, we would like to highlight some important aspects of the solvability notions.

Remarks 2.2 (Discussion of solvability notions) (i) The case $k_0 = k_1$ in the first solvability definition is an important special case. In general, the system (1) with a given input may not have a solution at all or may have a solution only for particular initial values. As a trivial example consider the (non-switched) singular system $Ex(k+1) = Ax(k)+Bu(k)$ with $(E, A, B) = (0, 0, 1)$ which is not locally solvable on $[k_0, k_0]$ if $u(k_0) \neq 0$. Existence of a solution on the interval $[k_0, k_1]$ with $k_1 = k_0$ simply requires that there is at least one consistent initial value (which will in general depend on $u(k_0)$). The required unique extendibility then implies that the initial value problem for a given input has a unique solution, provided the initial value is consistent with the initial input.

(ii) Causality is embedded in all three solvability definitions, which is deduced from requiring existence of a unique solution on any interval $[k_0, k_1]$ by only considering the equations in (1) for $k \in [k_0, k_1]$ and without restricting the future value $x(k_1 + 1)$ appearing in (1) for $k = k_1$. In particular, for $k_1 = k_0 + 1$ this means that for a solvable system, the value $x(k_0 + 1)$ has to be uniquely determined by $x(k_0)$, $u(k_0)$ and $u(k_0 + 1)$.

(iii) The solvability notions differ w.r.t. the relationship of the initial value to the input signal. For solvable systems w.r.t. a specific input, it is only required that for this specifically given u an initial value exists which is consistent with (1) (and for this initial value there may be no solution for some other input signal); while for strong solvability it is required that the initial value and the input can be chosen independently from each others. Here the set $\widehat{S}_i := A_i^{-1}(\operatorname{im}[E_i, B_i])$ denotes the *augmented consistency space* of mode i, i.e. it is the set of all possible x_0 for which some x_1 and some u_0 exists, such that $E_i x_1 = A_i x_0 + B_i u_0$.

(iv) It is also possible to define *weak solvability* by requiring that for all initial values in the (augmented) consistency space there exists an input such that the system is solvable w.r.t. that input and that initial condition. This solvability notion is highly relevant when considering controllability and reachability, where usually the initial value is first given and then one needs to find a suitable input. However, it turns out that in all situations (non-switched, switched with or without fixed switching signal) weak solvability is equivalent to solvability. We therefore, do not further consider this solvability notion here.

3 Preliminaries

3.1 Linear algebra preliminaries

Definition 3.1 (Projector) Let $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ be two subspaces. Then

$$\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} + \mathcal{W} \to \mathcal{V}$$

denotes any (not necessarily unique) projector such that $\Pi_{\mathcal{V}}^{\mathcal{W}} \mathcal{V} = \mathcal{V}$ and $\Pi_{\mathcal{V}}^{\mathcal{W}} \mathcal{W} = \mathcal{V} \cap \mathcal{W}$. In case $\mathcal{V} \cap \mathcal{W} = \{0\}$ then $\Pi_{\mathcal{V}}^{\mathcal{W}}$ is unique, furthermore, if $\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n$ then $I - \Pi_{\mathcal{V}}^{\mathcal{W}} = \Pi_{\mathcal{W}}^{\mathcal{V}}$.

Definition 3.2 (Generalized inverse) For a matrix $M \in \mathbb{R}^{m \times n}$, a generalized inverse of M is defined as a matrix $M^+ \in \mathbb{R}^{n \times m}$ that satisfies $MM^+M = M$.

A generalized matrix inverse always exists but is not necessarily unique; one possible choice is the well-known Moore-Penrose pseudoinverse [24]. Furthermore, for two generalized inverses M_1 and M_2 of M, we have that $(M_1 - M_2)y \in \ker M$ for all $y \in \operatorname{im} M$. In particular, for calculations, the well-known Moore-Penrose inverse can be used, for which efficient algorithms are available in the literature, e.g. by using a singular value decomposition [25]. Furthermore, it is easily seen that MM^+ restricted to $\operatorname{im} M$ is the identity map, in particular, we have

$$MM^+m = m \quad \forall m \in \operatorname{im} M \tag{4}$$

and, for any matrix $\widehat{M}$ with $\operatorname{im}\widehat{M} \subseteq \operatorname{im} M$,

$$MM^+\widehat{M} = \widehat{M}.$$

Lemma 3.3 (Preimage property) *For any matrix $M \in \mathbb{R}^{n\times n}$ let M^+ be some generalized inverse of M; furthermore, let $y \in \mathbb{R}^n$ and let $\mathcal{V} \subseteq \mathbb{R}^n$ be a subspace.*

(a) If $y \in \operatorname{im} M$, then

$$M^{-1}\{y\} = \{M^+y\} + \ker M.$$

(b) If $(\{y\} + \mathcal{V}) \cap \operatorname{im} M \neq \varnothing$ then

$$M^{-1}(\{y\} + \mathcal{V}) = \{M^+\Pi^{\mathcal{V}}_{\operatorname{im} M}y\} + M^{-1}\mathcal{V}.$$

Proof (a) This property is well known, for proof see e.g. [22, Lem. 2.2].

(b) First observe that $(\{y\} + \mathcal{V}) \cap \operatorname{im} M \neq \varnothing$ implies that $y \in \mathcal{V} + \operatorname{im} M$ and hence $\hat{y} := \Pi^{\mathcal{V}}_{\operatorname{im} M}y \in \operatorname{im} M$ is indeed well defined. Now the following equivalences hold:

$$\begin{aligned}
x \in M^{-1}(\{y\} + \mathcal{V}) &\iff \exists v \in \mathcal{V} : Mx = y + v = \hat{y} + y - \hat{y} + v \\
&\iff \exists \hat{v} \in \mathcal{V} \cap \operatorname{im} M : Mx = \hat{y} + \hat{v} \\
&\iff \exists \hat{v} \in \mathcal{V} \cap \operatorname{im} M : x \in M^{-1}(\{\hat{y} + \hat{v}\}) \\
&\quad \overset{(a)}{=} \{M^+\hat{y} + M^+\hat{v}\} + \ker M \\
&\quad \overset{(a)}{=} M^+\hat{y} + M^{-1}\{\hat{v}\} \\
&\iff x \in \{M^+\hat{y}\} + M^{-1}(\mathcal{V} \cap \operatorname{im} M) \\
&\iff x \in \{M^+\hat{y}\} + M^{-1}\mathcal{V}. \ \square
\end{aligned}$$

The following lemma provides a property of an intersection of two affine sets and the representation of the intersection via a projector.

Lemma 3.4 (Intersection of affine spaces) *Consider sets $\mathbf{Z}, \mathbf{U} \subseteq \mathbb{R}^n$ and subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$. Then, for all pairs $(z,u) \in \mathbf{Z} \times \mathbf{U}$, $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is a singleton if, and only if,*

$$\mathbf{U} - \mathbf{Z} \subseteq \mathcal{V} \oplus \mathcal{W},$$

where $\mathbf{U} - \mathbf{Z} = \{\ u - z \ \mid \ z \in \mathbf{Z}, u \in \mathbf{U}\ \}$. In that case,

$$(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W}) = \left\{\Pi^{\mathcal{W}}_{\mathcal{V}}(u - z) + z\right\} = \left\{\Pi^{\mathcal{V}}_{\mathcal{W}}(z - u) + u\right\}. \tag{5}$$

Furthermore, if $\mathcal{V} + \mathcal{W} = \mathbb{R}^n$, then $(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W})$ is always non-empty and is a singleton if, and only if, $\mathcal{V} \cap \mathcal{W} = \{0\}$; in that case

$$(\{z\} + \mathcal{V}) \cap (\{u\} + \mathcal{W}) = \left\{\Pi^{\mathcal{W}}_{\mathcal{V}}u + \Pi^{\mathcal{V}}_{\mathcal{W}}z\right\}. \tag{6}$$

Proof *Step 1:* We show that the intersection $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is nonempty for all pairs $(z,u)\in\mathbf{Z}\times\mathbf{U}$ if, and only if, $\mathbf{U}-\mathbf{Z}\subseteq\mathcal{V}+\mathcal{W}$.
Step 1a: Necessity.
Seeking a contradiction, assume $\mathbf{U}-\mathbf{Z}\nsubseteq\mathcal{V}+\mathcal{W}$, i.e. there exists $(z,u)\in\mathbf{Z}\times\mathbf{U}$ with $u-z\in\mathbf{U}-\mathbf{Z}$ which is not in $\mathcal{V}+\mathcal{W}$. By assumption there exists $x\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$, hence there are $v\in\mathcal{V}$ and $w\in\mathcal{W}$ with $x=z+v=u+w$. But this implies $u-z=v-w\in\mathcal{V}+\mathcal{W}$, which contradicts the choice of u and z.
Step 1b: Sufficiency.
Pick an arbitrary pair $(z,u)\in\mathbf{Z}\times\mathbf{U}$, then by assumption $u-z\in\mathbf{U}-\mathbf{Z}\subseteq\mathcal{V}+\mathcal{W}$. Choose $v\in\mathcal{V}$ and $w\in\mathcal{W}$ such that $u-z=v+w$. Then $z+v=u-w\in\{u\}+\mathcal{W}$. Hence, $z+v\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$, i.e., the latter intersection is not empty.

Step 2: We will prove that if $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is non-empty for at least one pair $(z,u)\in\mathbf{Z}\times\mathbf{U}$ then $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is at most a singleton for all pairs $(z,u)\in\mathbf{Z}\times\mathbf{U}$, if, and only if, $\mathcal{V}\cap\mathcal{W}=\{0\}$.
Step 2a: Necessity.
Seeking a contradiction, assume that $\mathcal{V}\cap\mathcal{W}\neq\{0\}$ and choose $0\neq p\in\mathcal{V}\cap\mathcal{W}$. Choose some $z\in\mathbf{Z}$ and $u\in\mathbf{U}$ for which $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is non-empty and choose $x\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$. Then there are $v\in\mathcal{V}$ and $w\in\mathcal{W}$ with $x=z+v=u+w$. Since $z+v+p=u+w+p$ and $v+p\in\mathcal{V}$ as well as $w+p\in\mathcal{W}$ we arrive at $z+v+p\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$, and since $z+v+p\neq z+v$, the set $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is not a singleton (and also not empty).
Step 2b: Sufficiency.
For some $z\in\mathbf{Z}$ and $u\in\mathbf{U}$ for which $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is non-empty, let $x_1,x_2\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$. Then, there exists $v_1,v_2\in\mathcal{V}$ and $w_1,w_2\in\mathcal{W}$ with $x_1=z+v_1=u+w_1$ and $x_2=z+v_2=u+w_2$. Consequently, $x_1-x_2=z+v_1-z-v_2=u+w_1-u-w_2=v_1-v_2=w_1-w_2$. Consequently $v_1-v_2=w_1-w_2\in\mathcal{V}\cap\mathcal{W}=\{0\}$, which implies $x_1=x_2$, i.e., $(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$ is a singleton.

Step 3: We show (5).
Let $u-z\in\mathbf{U}-\mathbf{Z}\subseteq\mathcal{V}\oplus\mathcal{W}$ and choose (unique) $v\in\mathcal{V}$ and $w\in\mathcal{W}$ such that $u-z=v+w$. Then $x:=v+z=u-w\in(\{z\}+\mathcal{V})\cap(\{u\}+\mathcal{W})$. Furthermore, from $\Pi_{\mathcal{V}}^{\mathcal{W}}(u-z)=v$ and $\Pi_{\mathcal{W}}^{\mathcal{V}}(z-u)=-w$ we can conclude that $\Pi_{\mathcal{V}}^{\mathcal{W}}(u-z)+z=v+z=x=u-w=\Pi_{\mathcal{W}}^{\mathcal{V}}(z-u)+u$ as desired. □

Note that we will later on rewrite the right-hand sides of (5) as

$$\Pi_{\mathcal{V}}^{\mathcal{W}}(u-z)+z=\Pi_{\mathcal{V}}^{\mathcal{W}}u+(I-\Pi_{\mathcal{V}}^{\mathcal{W}})z,$$

which is only well defined if we extend the projector $\Pi_{\mathcal{V}}^{\mathcal{W}}$ uniquely defined on $\mathcal{V}\oplus\mathcal{W}$ to a projector defined on the whole space $\mathbb{R}^n$. This extension is in general non-unique, and if $\mathcal{V}\oplus\mathcal{W}$ is not the whole space it is *not possible* to preserve *all* of the following properties of a projector

$$\operatorname{im}\Pi_{\mathcal{V}}^{\mathcal{W}}=\mathcal{V},\quad \ker\Pi_{\mathcal{V}}^{\mathcal{W}}=\mathcal{W},\quad (I-\Pi_{\mathcal{V}}^{\mathcal{W}})=\Pi_{\mathcal{W}}^{\mathcal{V}}.$$

In particular, (6) does not hold for a general extension of the projector $\Pi_{\mathcal{V}}^{\mathcal{W}}$ to $\mathbb{R}^n$.

We conclude this subsection by highlighting a property of projectors which we will utilize later.

Lemma 3.5 *Let $\mathcal{V}_1, \mathcal{V}_2, \mathcal{W} \subseteq \mathbb{R}^n$ be subspaces such that $\mathcal{V}_i \oplus \mathcal{W} = \mathbb{R}^n$ for $i = 1, 2$. Then for all $x \in \mathcal{V}_1$ we have for the corresponding projectors $\Pi_{\mathcal{V}_i}^{\mathcal{W}}$ onto $\mathcal{V}_i$ along $\mathcal{W}$ that*

$$\Pi_{\mathcal{V}_1}^{\mathcal{W}} \Pi_{\mathcal{V}_2}^{\mathcal{W}} x = x.$$

Proof From $\Pi_{\mathcal{V}_2}^{\mathcal{W}} x - x \in \mathcal{W}$ together with $\Pi_{\mathcal{V}_1}^{\mathcal{W}} x = x$ we have

$$\Pi_{\mathcal{V}_1}^{\mathcal{W}} \Pi_{\mathcal{V}_2}^{\mathcal{W}} x - x = \Pi_{\mathcal{V}_1}^{\mathcal{W}} (\Pi_{\mathcal{V}_2}^{\mathcal{W}} x - x) = 0.$$

□

3.2 Index-1 notions

3.2.1 Matrix pairs (E, A) and matrix triplets (E, A, B)

A matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is called *regular*, if the polynomial $\det(sE - A) \in \mathbb{R}[s]$ is not identically zero. It is well known that this is equivalent to the existence of two invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{7}$$

where $J \in \mathbb{R}^{n_J \times n_J}$ is some matrix and $N \in \mathbb{R}^{n_N \times n_N}$ is a nilpotent matrix (i.e. there exists $\nu \leqslant n_N$ with $N^\nu = 0$). Following [26] we call (7) the quasi-Weierstrass form (QWF) of the matrix pair (E, A). Furthermore, the *index* of a regular matrix pair (E, A) is defined as the nilpotency index of N in the corresponding QWF (7); in particular, (E, A) is *index-1* if, and only if[1], $N = 0$. In that case it can be easily seen that $T = [V, W]$ and $S = [EV, AW]^{-1}$, where V and W are full column matrices such that

$$\operatorname{im} V = \mathcal{S} := A^{-1}(\operatorname{im} E) \quad \text{and} \quad \operatorname{im} W = \ker E.$$

The property of index-1 will play a central role in the remainder of this work and we therefore present several equivalent characterizations.

Lemma 3.6 (Index-1 characterizations) *Consider a matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ and let $\mathcal{S} := A^{-1}(\operatorname{im} E)$. Then the following statements are equivalent to (E, A) being regular and index-1.*

(IC1) $\mathcal{S} \cap \ker E = \{0\}$.

(IC2) $\mathcal{S} \oplus \ker E = \mathbb{R}^n$.

(IC3) $\deg \det(sE - A) = \operatorname{rank} E$.

[1] This characterization is formally not correct, because if N is a 0×0 matrix (which is the case if E is invertible), then $N^0 = I_{0 \times 0} = 0_{0 \times 0}$. However, since all of our results concerning index-1 remain valid if E is invertible, we choose to include the index-0 case in our index-1 definition.

(IC4) *A_4 is invertible in $\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} := PAQ$ where P, Q are invertible matrices such that $PEQ = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ (with matching block sizes in PEQ and PAQ).*

(IC5) *$\begin{bmatrix} S_1 E \\ S_2 A \end{bmatrix}$ is invertible, where $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix}$ is invertible such that $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} E = \begin{bmatrix} S_1 E \\ 0 \end{bmatrix}$ and $S_1 E$ has full rank (i.e. $\operatorname{im} S_2^\top = \ker E^\top$ or $\ker S_2 = \operatorname{im} E$).*

Proof The equivalent characterizations (IC1) and (IC2) are well known, see e.g. [27, Lem. 2.9]. The characterization (IC3) follows from the QWF and that $\det(sE - A) = c\det(sI - J)$; consequently $\deg\det(sE - A)$ is the size of the J-block in the QWF and $\operatorname{rank} E$ is equal to that size if, and only if, $N = 0$. Utilizing the Schur-complement, it is easily seen that (E, A) has a QWF with $N = 0$ if, and only if, A_4 in (IC4) is invertible. For the last equivalence, we first observe that by construction $S_1 E$ has full row rank, hence $\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \operatorname{im} E = \operatorname{im} \begin{bmatrix} S_1 E \\ 0 \end{bmatrix} = \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix}$. Consequently,

$$\mathcal{S} = \left(\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} A \right)^{-1} \left(\begin{bmatrix} S_1 \\ S_2 \end{bmatrix} \operatorname{im} E \right) = \begin{bmatrix} S_1 A \\ S_2 A \end{bmatrix}^{-1} \operatorname{im} \begin{bmatrix} I \\ 0 \end{bmatrix} = \ker SA_2.$$

Therefore, we have $\mathcal{S} \cap \ker E = \ker S_2 A \cap \ker SE_1$, from which equivalence between (IC1) and (IC5) can be concluded. □

For a matrix triplet (E, A, B), the index-1 notion remains the same, but we also introduce the notion of strictly index-1 as follows.

Definition 3.7 A matrix triplet (E, A, B) is called

(i) index-1 if (E, A) is (regular and) index-1,

(ii) strictly index-1 if (E, A) is index-1 and $\operatorname{im} B \subseteq \operatorname{im} E$.

Remark 3.8 With $\widehat{\mathcal{S}} := A^{-1}(\operatorname{im}[E, B])$ it is possible to equivalently express strictly index-1 for (E, A, B) as

$$\ker E \oplus \widehat{\mathcal{S}} = \mathbb{R}^n. \tag{8}$$

Necessity is clear because $\operatorname{im} B \subseteq \operatorname{im} E$ implies that $\mathcal{S} = \widehat{\mathcal{S}}$ and sufficiency follows from first observing that (8) together with $\mathcal{S} \subseteq \widehat{\mathcal{S}}$ implies $\ker E \cap \mathcal{S} = \{0\}$, i.e. (8) implies regularity and index-1 of the matrix pair (E, A). Now by utilizing S and T in the QWF (7) with $SB =: \begin{bmatrix} B_J \\ B_N \end{bmatrix}$ we can see that (8) implies $B_N = 0$ which is equivalent to $\operatorname{im} B \subseteq \operatorname{im} E$.

3.2.2 Family of matrix triplets $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$

In the solvability characterizations of system (1), we also utilize the so-called index-1 notions for a family of matrix triplets $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ (already indicated in Fig. 1). Those index-1 notions are defined according to the switching signal under consideration.

We first define index-1 notions for a fixed and given switching signal σ of the form (3) as follows:

Definition 3.9 A family of matrix triplets $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ is called

(i) switched index-1 w.r.t. σ if

$$1)\ \operatorname{im} B_i \subseteq \operatorname{im}[E_i, A_i]\ \forall i \in \mathtt{M},$$
$$2)\ \widehat{\mathcal{R}}_{\sigma(k)} + \widehat{\mathcal{S}}_{\sigma(k+1)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \quad \forall k \in \mathbb{N},$$

where

$$\widehat{\mathcal{R}}_i := E_i^{-1}(\operatorname{im}[A_i, B_i]), \quad \widehat{\mathcal{S}}_i := A_i^{-1}(\operatorname{im}[E_i, B_i])$$

(ii) switched strictly index-1 w.r.t. σ if

$$1)\ \operatorname{im} B_i \subseteq \operatorname{im} E_i\ \forall i \in \mathtt{M},$$
$$2)\ \widehat{\mathcal{R}}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \widehat{\mathcal{S}}_{\sigma(k+1)}\ \forall k \in \mathbb{N}.$$

By imposing the switched index-1 conditions to all switching signals from a particular mode sequence, we define the index-1 notions for a mode sequence (σ_j) as follows:

Definition 3.10 A family of matrix triplets $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ is called

(i) sequentially index-1 w.r.t. (σ_j) if

$$1)\ (E_i, A_i)\ \text{index-1}\ \forall i \in \mathtt{M},$$
$$2)\ \widehat{\mathcal{R}}_{\sigma_j} + \widehat{\mathcal{S}}_{\sigma_{j+1}} \subseteq \ker E_{\sigma_j} \oplus \mathcal{S}_{\sigma_{j+1}} \quad \forall j \in \mathbb{N}.$$

(ii) sequentially strictly index-1 w.r.t. (σ_j) if

$$1)\ (E_i, A_i, B_i)\ \text{strictly index-1}\ \forall i \in \mathtt{M},$$
$$2)\ \widehat{\mathcal{R}}_{\sigma_j} \subseteq \ker E_{\sigma_j} \oplus \widehat{\mathcal{S}}_{\sigma_{j+1}} \quad \forall j \in \mathbb{N}.$$

Finally, we define the index-1 notions for $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ w.r.t. arbitrary switching signals as follows:

Definition 3.11 A family of matrix triplets $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ is called

(i) jointly index-1 if $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n \quad \forall i, j \in \mathtt{M}$,

(ii) jointly strictly index-1 if $\ker E_i \oplus \widehat{\mathcal{S}}_j = \mathbb{R}^n \quad \forall i, j \in \mathtt{M}$.

Remark 3.12 Note that in all three strict index-1 notions defined above, we have that $\widehat{\mathcal{S}}_j = \mathcal{S}_j$ because $\operatorname{im} B_j \subseteq \operatorname{im} E_j$ (for the jointly strictly index-1 case cf. the discussion after (8)). Since the latter is explicitly required for the switched and sequential index-1 property, we could also replace $\widehat{\mathcal{S}}_j$ by $\mathcal{S}_j$ in Definitions 3.9(ii) and 3.10(ii). Nevertheless, this replacement is not possible for the jointly strictly index-1 definition, unless we explicitly add the condition $\operatorname{im} B_j \subseteq \operatorname{im} E_j$. Furthermore, every non-strict index-1 notion becomes strict, if, and only if, $\operatorname{im} B_j \subseteq \operatorname{im} E_j$.

4 A key lemma

The causal solvability notions lead to the consideration of the following set of two equations:

$$E_0x_1 = A_0x_0 + B_0u_0, \tag{9a}$$
$$E_1x_2 = A_1x_1 + B_1u_1, \tag{9b}$$

where $E_0, A_0, E_1, A_1 \in \mathbb{R}^{n\times n}$, $B_0, B_1 \in \mathbb{R}^{n\times m}$. The key question now is whether for a given x_0, u_0, u_1 there exists x_1 and x_2 such that (9) is satisfied and that x_1 is uniquely determined by x_0, u_0, u_1.

Inspired by Definition 2.1, we call (9) solvable w.r.t. $u_0, u_1 \in \mathbb{R}^m$ if for all x_0 consistent with (9a), i.e. for all $x_0 \in \mathcal{S}_0^{u_0} := A_0^{-1}(\operatorname{im} E_0 - \{B_0u_0\})$ (assumed to be non-empty), there exists a unique $x_1 \in \mathbb{R}^n$ and some $x_2 \in \mathbb{R}^m$ such that (9) holds; we call (9) solvable if it is solvable for all $u_0, u_1 \in \mathbb{R}^m$. Finally, we call (9) strongly solvable if for all $x_0 \in \widehat{\mathcal{S}}_0 = A_0^{-1}(\operatorname{im}[E_0, B_0])$ and all $u_0, u_1 \in \mathbb{R}^m$ there exists a unique $x_1 \in \mathbb{R}^n$ and some $x_2 \in \mathbb{R}^n$ such that (9) holds.

Lemma 4.1 *The linear system of equations* (9) *is solvable w.r.t.* u_0, u_1 *if, and only if*

$$B_0u_0 \in \operatorname{im}[E_0, A_0], \quad B_1u_1 \in \operatorname{im}[E_1, A_1], \tag{10}$$
$$\mathcal{R}_0^{u_0} - \mathcal{S}_1^{u_1} \subseteq \ker E_0 \oplus \mathcal{S}_1, \tag{11}$$

where

$$\begin{aligned}\mathcal{R}_0^{u_0} &:= E_0^{-1}(\operatorname{im} A_0 + \{B_0u_0\}),\\ \mathcal{S}_1^{u_1} &:= A_1^{-1}(\operatorname{im} E_1 - \{B_1u_1\}),\\ \mathcal{S}_1 &:= \mathcal{S}_1^0 = A_1^{-1}(\operatorname{im} E_1).\end{aligned}$$

In that case, for any $x_0 \in \mathcal{S}_0^{u_0} := A_0^{-1}(\operatorname{im} E_0 - \{B_0u_0\})$ *the unique solution is given by*

$$x_1 = \Phi_{1,0}x_0 + \Psi_{1,0}^c u_0 + \Psi_{1,0}^a u_1, \tag{12}$$

where

$$\begin{aligned}\Phi_{1,0} &:= \Pi_{\mathcal{S}_1}^{\ker E_0} E_0^+ \Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0} A_0,\\ \Psi_{1,0}^c &:= \Pi_{\mathcal{S}_1}^{\ker E_0} E_0^+ \Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0} B_0,\\ \Psi_{1,0}^a &:= (\Pi_{\mathcal{S}_1}^{\ker E_0} - I) A_1^+ \Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1.\end{aligned}$$

Proof Clearly, (10) is necessary for the existence of x_0, x_1, x_2 such that (9) is satisfied. Under this assumption, we have that the sets $\mathcal{S}_0^{u_0}$ and $\mathcal{S}_1^{u_1}$ are both nonempty. Furthermore, for any $x_0 \in \mathcal{S}_0^{u_0}$ we can conclude that $Ax_0 + Bu_0 \in \operatorname{im} E_0$ and hence the set

$$\mathcal{R}_0^{x_0,u_0} := E_0^{-1}\{Ax_0 + Bu_0\}$$

is also nonempty. Now, x_1 satisfies (9a) if, and only if, $x_1 \in E_0^{-1}\{Ax_0 + Bu_0\} = \mathcal{R}_0^{x_0,u_0}$ and x_1 satisfies (9b) for some x_2 if, and only if, $x_1 \in A_1^{-1}(\operatorname{im} E_1 - \{B_1u_1\}) = \mathcal{S}_1^{u_1}$. Consequently, solvability of (9) w.r.t. u_0, u_1 is now equivalent to

$$\mathcal{R}_0^{x_0,u_0} \cap \mathcal{S}_1^{u_1} \tag{13}$$

being nonempty and a singleton for all $x_0 \in \mathcal{S}_0^{u_0}$. Using Lemma 3.3 (part (a) for $\mathcal{R}_0^{x_0,u_0}$ and part (b) for $\mathcal{S}_1^{u_1}$), we can rewrite (13) as

$$(\ker E_0 + \{E_0^+(A_0x_0 + B_0u_0)\}) \cap (A_1^{-1}(\operatorname{im} E_1) - \{A_1^+ \Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1\}). \tag{14}$$

According to Lemma 3.4 with $z = E_0^+(A_0x_0 + B_0u_0)$, $\mathbf{Z} = \{z\}$, $\mathcal{V} = \ker E_0$, $u = -A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1$, $\mathbf{U} = \{u\}$ and $\mathcal{W} = A_1^{-1}(\operatorname{im} E_1)$, the set (14) is nonempty and a singleton if, and only if

$$\{E_0^+(A_0x_0 + B_0u_0)\} + \{A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1\} \subseteq \ker E_0 \oplus \mathcal{S}_1. \tag{15}$$

Adding $\ker E_0$ and $\mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1)$ on the left side of (15) results in an equivalent set relation ship and using Lemma 3.3 backwards (part (a) for the first and part (b) for the second) we obtain

$$\begin{aligned}\{E_0^+(A_0x_0 + B_0u_0)\} + \ker E_0 &= E_0^{-1}\{A_0x_0 + B_0u_0\},\\ \{A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1\} + A_1^{-1}(\operatorname{im} E_1) &= A_1^{-1}(\operatorname{im} E_1 + \{B_1u_1\}) = -\mathcal{S}_1^{u_1},\end{aligned}$$

consequently, (15) is satisfied for all $x_0 \in \mathcal{S}_0^{u_0}$ if, and only if

$$E_0^{-1}(A_0\mathcal{S}_0^{u_0} + \{B_0u_0\}) - \mathcal{S}_1^{u_1} \subseteq \ker E_0 \oplus A_1^{-1}(\operatorname{im} E_1), \tag{16}$$

where we also used that $\bigcup_{x_0\in\mathcal{S}_0^{u_0}} E_0^{-1}\{A_0x_0+B_0u_0\} = E_0^{-1}(A_0\mathcal{S}_0^{u_0}+\{B_0u_0\})$. The condition (16) is in fact identical to (11), because

$$\begin{aligned}\mathcal{R}_0^{u_0} &= E_0^{-1}(\operatorname{im} A_0 + \{B_0u_0\})\\ &= E_0^{-1}\left((\operatorname{im} A_0 + \{B_0u_0\}) \cap \operatorname{im} E_0\right)\\ &\overset{(*)}{=} E_0^{-1}\left(\operatorname{im} A_0 \cap (\operatorname{im} E_0 - \{B_0u_0\}) + \{B_0u_0\}\right)\\ &= E_0^{-1}\left(A_0\mathcal{S}_0^{u_0} + \{B_0u_0\}\right),\end{aligned}$$

where $(*)$ follows from the general property $M \cap N = ((M - \{p\}) \cap (N - \{p\})) + \{p\}$ for any sets $M, N \subseteq \mathbb{R}^n$ and any point $p \in \mathbb{R}^n$. If the intersection (14) is indeed nonempty and a singleton, then Lemma 3.4 (with $\mathcal{V}$, $\mathcal{W}$, u, z as above) implies that the unique element x_1 in that intersection is given by

$$x_1 = \Pi_{\mathcal{S}_1}^{\ker E_0}\left(E_0^+(A_0x_0 + B_0u_0) + A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1\right) - A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1,$$

which is the claimed solution formula after taking into account that $A_0x_0 + B_0u_0 \in \operatorname{im} E_0$ and hence $A_0x_0 + B_0u_0 = \Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0}(A_0x_0 + B_0u_0)$. □

Remark 4.2 From the proof of Lemma 4.1, it becomes clear that condition (11) is equivalent to

$$\mathcal{R}_0 + \{E_0^+\Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0} B_0u_0 + A_1^+\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} B_1u_1\} \subseteq \ker E_0 \oplus \mathcal{S}_1,$$

where $\mathcal{R}_0 := \mathcal{R}_0^0 = E_0^{-1}(\operatorname{im} A_0)$, which may be more practical because the involved subspaces and matrices can be calculated independently of u_0 and u_1. In particular, we can immediately conclude the condition $\mathcal{R}_0 \subseteq \ker E_0 \oplus \mathcal{S}_1$ for solvability of the homogeneous case of (9).

As a direct consequence of Lemma 4.1, we have the following characterization for the solvability (for all u_0 and all u_1) of (9):

Corollary 4.3 *The linear system of equations* (9) *is solvable if, and only if*

$$\operatorname{im} B_0 \subseteq \operatorname{im}[E_0, A_0], \quad \operatorname{im} B_1 \subseteq \operatorname{im}[E_1, A_1], \tag{17}$$

$$\widehat{\mathcal{R}}_0 + \widehat{\mathcal{S}}_1 \subseteq \ker E_0 \oplus \mathcal{S}_1, \tag{18}$$

where

$$\widehat{\mathcal{R}}_0 := E_0^{-1}(\operatorname{im}[A_0, B_0]), \quad \widehat{\mathcal{S}}_1 := A_1^{-1}(\operatorname{im}[E_1, B_1]).$$

Utilizing the characterization for the solvability above, we can now derive the characterization for the strong solvability as follows.

Lemma 4.4 *The linear system of equations* (9) *is strongly solvable if, and only if*

$$\operatorname{im} B_0 \subseteq \operatorname{im} E_0, \quad \operatorname{im} B_1 \subseteq \operatorname{im}[E_1, A_1], \tag{19}$$

$$\widehat{\mathcal{R}}_0 + \widehat{\mathcal{S}}_1 \subseteq \ker E_0 \oplus \mathcal{S}_1 \tag{20}$$

Furthermore, if in addition to $\operatorname{im} B_0 \subseteq \operatorname{im} E_0$ *also* $\operatorname{im} B_1 \subseteq \operatorname{im} E_1$*, then (strong) solvability is equivalent to*

$$\widehat{\mathcal{R}}_0 \subseteq \ker E_0 \oplus \widehat{\mathcal{S}}_1$$

and the solution formula simplifies to

$$x_1 = \Phi_{1,0} x_0 + \Psi^c_{1,0} u_0$$

Proof For strong solvability of (9), the first equation (9a) must be solvable for $x_0 = 0 \in \widehat{\mathcal{S}}_0$ and all $u_0 \in \mathbb{R}^m$, which immediately implies that $\operatorname{im} B_0 \subseteq \operatorname{im} E_0$ is necessary for solvability. With this restriction, the characterization for strong solvability is equal to the characterization of solvability. If additionally $\operatorname{im} B_1 \subseteq \operatorname{im} E_1$ then $\mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1) = A_1^{-1}(\operatorname{im}[E_1, B_1]) = \widehat{\mathcal{S}}_1$ from which the simplified characterization immediately follows. Finally, $\Psi^a_{1,0} = 0$ follows from $A_1^+ \Pi^{\operatorname{im} E_1}_{\operatorname{im} A_1} \operatorname{im} B_1 \subseteq \mathcal{S}_1$, because $\operatorname{im} B_1 \subseteq E_1$ implies, by definition, $\Pi^{\operatorname{im} E_1}_{\operatorname{im} A_1} \operatorname{im} B_1 \subseteq \operatorname{im} E_1 \cap \operatorname{im} A_1$ and because $\mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1 \cap \operatorname{im} A_1) = A_1^+(\operatorname{im} E_1 \cap \operatorname{im} A_1) + \ker A_1$. □

Example 4.5 Consider (9) with

$$E_0 = \begin{bmatrix} 0&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0&1&1&0 \\ 0&0&0&1 \\ 0&0&0&1 \\ 0&0&0&1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 1&0&0&0 \\ 1&0&0&0 \\ 1&0&0&0 \\ 0&1&1&0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

We first see that neither (E_0, A_0) nor (E_1, A_1) are regular (because each pair has a common zero-column), but $\operatorname{im}[E_0, A_0] = \mathbb{R}^n = \operatorname{im}[E_1, A_1]$, hence condition (10) is satisfied for any choice of $u_0, u_1 \in \mathbb{R}$ (and in fact, for any choice of B_0, B_1). In order to check the solvability condition (11) we first calculate

$$\ker E_0 = \operatorname{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1) = \operatorname{im} \begin{bmatrix} 1&0 \\ 1&0 \\ 1&0 \\ 0&1 \end{bmatrix} \text{ and } \ker E_0 \oplus \mathcal{S}_1 = \operatorname{im} \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix}.$$

Furthermore, utilizing Lemma 3.3(b) we have

$$\mathcal{R}_0^{u_0} = E_0^{-1}(\operatorname{im} A_0 + \{B_0 u_0\}) = E_0^{-1} \operatorname{im} A_0 + \{E_0^+ \Pi^{\operatorname{im} A_0}_{\operatorname{im} E_0} B_0 u_0\} = \operatorname{im} \begin{bmatrix} 1&0 \\ 0&1 \\ 0&1 \\ 0&1 \end{bmatrix} + \left\{ \begin{bmatrix} 0 \\ u_0 \\ 0 \\ 0 \end{bmatrix} \right\},$$

$$\mathcal{S}_1^{u_1} = A_1^{-1}(\operatorname{im} E_1 - \{B_1 u_1\}) = A_1^{-1}(\operatorname{im} E_1) - A_1^+ \Pi^{\operatorname{im} E_1}_{\operatorname{im} A_1} B_1 u_1 = \operatorname{im} \begin{bmatrix} 1&0 \\ 1&0 \\ 1&0 \\ 0&1 \end{bmatrix} - \left\{ \begin{bmatrix} 0 \\ 0 \\ u_1 \\ 0 \end{bmatrix} \right\},$$

where we have chosen $E_0^+ = \Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0} = E_0$ and $A_1^+ = \Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} = A_1$. The condition (11) now reads as

$$\operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left\{ \begin{bmatrix} 0 \\ u_0 \\ u_1 \\ 0 \end{bmatrix} \right\} \subseteq \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which is satisfied if, and only if, $u_0 = u_1$. Now assume $u_0 = u_1 = 1$ and we want to find a specific solution for a given consistent x_0. We first calculate the set of consistent initial values as

$$\mathcal{S}_0^{u_0} = A_0^{-1}(\operatorname{im} E_0 - \{B_0 u_0\}) = A_0^{-1}(\operatorname{im} E_0) - A_0^+ \Pi_{\operatorname{im} A_0}^{\operatorname{im} E_0} B_0 u_0 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{3} \end{bmatrix} \right\},$$

where we have chosen $A_0^+ = \frac{1}{6} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 0 & 2 & 2 & 2 \end{bmatrix}$ and $A_0 A_0^+ = \Pi_{\operatorname{im} A_0}^{\operatorname{im} E_0} = \frac{1}{3} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Note that $\mathcal{S}_0^{u_0} = \mathcal{S}_0$, hence any consistent initial value x_0 is given by $x_0 = \begin{bmatrix} x_0^1 \\ x_0^2 \\ x_0^3 \\ x_0^4 \end{bmatrix}$ with $x_0^3 = -x_0^2$. Consider now the consistent $x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, then the unique x_1 satisfying (9) for $u_0 = u_1 = 1$ is given by (12), where

$$\Pi_{\mathcal{S}_1}^{\ker E_0} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and hence } \Phi_{1,0} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Psi_{1,0}^c = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Psi_{1,0}^a = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

resulting in $x_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$. It can easily be verified that $E_0 x_1 = A_0 x_0 + B_0 u_0$ and that $A_1 x_1 + B_1 u_1 \in \operatorname{im} E_1$, which shows that x_1 is indeed a solution of (9). Note that $\widehat{\mathcal{R}}_0 + \widehat{\mathcal{S}}_1 = \mathbb{R}^4 \nsubseteq \ker E_0 \oplus \mathcal{S}_1$, which means that according to Corollary 4.3 (Lemma 4.4) the system (9) is not (strongly) solvable.

5 Solvability characterizations

Recall the switched system (1) together with the solvability notions in Definition 2.1. The proof of Lemma 4.1 provides the key argument for the solvability characterizations of (1). By extending the arguments of this lemma to the switching signals under consideration, the characterizations can then be derived straightforwardly.

5.1 Solvability characterizations for given switching signals

We first are interested in obtaining the weakest possible condition which guarantees solvability for a *given* switching signal and *given* input. Surprisingly, for a fixed switching signal and fixed input, neither index-1 nor regularity for each individual mode is necessary anymore. In fact, based on Lemma 4.1 and Remark 4.2 we arrive at the following solvability characterization, where we use the following notation:

$$\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i) \quad \text{and} \quad \mathcal{R}_i := E_i^{-1}(\operatorname{im} A_i), \quad \text{for } i \in \mathtt{M}.$$

Proposition 5.1 (Solvability w.r.t. σ and w.r.t. u) *System (1) with given switching signal σ and with given input u is (locally uniquely causally) solvable if, and only if, for $k = 0, 1, \ldots$*

$$B_{\sigma(k)} u(k) \in \operatorname{im}[E_{\sigma(k)}, A_{\sigma(k)}] \text{ and} \tag{21}$$

$$\mathcal{R}_{\sigma(k)} + \left\{B^c_{\sigma(k)}u(k) + B^a_{\sigma(k+1)}u(k+1)\right\} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \tag{22}$$

where

$$B^c_j := E_j^+ \Pi^{\operatorname{im} A_j}_{\operatorname{im} E_j} B_j \quad \text{and} \quad B^a_i := A_i^+ \Pi^{\operatorname{im} E_i}_{\operatorname{im} A_i} B_i.$$

In that case, x is a solution on $[k_0, k_1]$ if, and only if, the corresponding initial value $x(k_0)$ satisfies $x(k_0) \in \mathcal{S}_{\sigma(k_0)} - \left\{B^a_{\sigma(k_0)}u(k_0)\right\}$, and x satisfies (the surrogate system)

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k) + \Psi^c_{\sigma(k+1),\sigma(k)}u(k) + \Psi^a_{\sigma(k+1),\sigma(k)}u(k+1) \tag{23}$$

where, for $i, j \in \mathtt{M}$,

$$\Phi_{i,j} := \Pi^{\ker E_j}_{\mathcal{S}_i} E_j^+ \Pi^{\operatorname{im} A_j}_{\operatorname{im} E_j} A_j,$$
$$\Psi^c_{i,j} := \Pi^{\ker E_j}_{\mathcal{S}_i} B^c_j, \qquad \Psi^a_{i,j} := (\Pi^{\ker E_j}_{\mathcal{S}_i} - I)B^a_i.$$

In particular, for all k, $x(k) \in \mathcal{S}_{\sigma(k)} - \left\{B^a_{\sigma(k)}u(k)\right\}$.

Proof We first observe that (1) being solvable w.r.t. σ and u implies that $A^{-1}_{\sigma(k_0)}(\operatorname{im} E_{\sigma(k_0)} - \{B_{\sigma(k_0)}u(k_0)\}) = \mathcal{S}^{u(k_0)}_{\sigma(k_0)}$ is non-empty for all $k_0 \in \mathbb{N}$ and that $x(k_0) = x_0$ is a solution of (1) on $[k_0, k_0]$ for all $x_0 \in \mathcal{S}^{u(k_0)}_{\sigma(k_0)}$. Solvability, in particular unique extendability, implies that for all $x_0 \in \mathcal{S}^{u(k_0)}_{\sigma(k_0)}$ there exists a unique $x_1 \in \mathbb{R}^n$ such that $x(k_0) = x_0$ and $x(k_1) = x_1$ is a solution of (1). In other words, for all $x_0 \in \mathcal{S}^{u(k_0)}_{\sigma(k_0)}$ there exist a unique $x_1 \in \mathbb{R}^n$ and some $x_2 \in \mathbb{R}^n$ such that

$$E_{\sigma(k_0)}x_1 = A_{\sigma(k_0)}x_0 + B_{\sigma(k_0)}u(k_0),$$
$$E_{\sigma(k_0+1)}x_2 = A_{\sigma(k_0+1)}x_1 + B_{\sigma(k_0+1)}u(k_0+1).$$

Now Lemma 4.1 together with Remark 4.2 implies that (21) and (22) are necessary for solvability w.r.t. to σ and u.

Sufficiency is clear by simply recursively extending a solution found on $[k_0, k_1]$ to $[k_0, k_1 + 1]$ by solving

$$E_{\sigma(k_1)}x_1 = A_{\sigma(k_1)}x(k_1) + B_{\sigma(k_1)}u(k_1),$$
$$E_{\sigma(k_1+1)}x_2 = A_{\sigma(k_1+1)}x_1 + B_{\sigma(k_1+1)}u(k_1+1).$$

for x_1 and setting $x(k_1 + 1) = x_1$; Lemma 4.1 together with Remark 4.2 ensures the unique existence of such an x_1. □

Remark 5.2 (Homogeneous case) When considering the homogeneous case, i.e. (1) with $u = 0$ (or $B_i = 0$), we immediately see that the inhomogeneous solvability characterization from Theorem 5.1 reduces to

$$\mathcal{R}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)},$$

or equivalently (by Lemma 3.3(a))

$$E^+_{\sigma(k)}(\operatorname{im} E_{\sigma(k)} \cap \operatorname{im} A_{\sigma(k)}) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)},$$

which is the *switched index-1* condition for homogeneous systems already reported in [22].

Corollary 5.3 (Augmented consistency space) *Consider the switched system* (1) *with a fixed switching signal and assume it is solvable for all inputs. The space of all consistent initial values* $x(k_0)$ *at* k_0 *is given by* $\operatorname{im} B^a_{\sigma(k_0)} + \mathcal{S}_{\sigma(k_0)}$ *which is equal to*

$$\widehat{\mathcal{S}}_{\sigma(k_0)} := A^{-1}_{\sigma(k_0)}(\operatorname{im}[E_{\sigma(k_0)}, B_{\sigma(k_0)}].$$

Proof We have that

$$\widehat{\mathcal{S}}_{\sigma(k_0)} = \bigcup_{u \in \mathbb{R}^m} A^{-1}_{\sigma(k_0)}(\{B_{\sigma(k_0)}u\} + \operatorname{im} E_{\sigma(k_0)}).$$

From (21) it follows that $(\{B_{\sigma(k_0)}u\} + \operatorname{im} E_{\sigma(k_0)}) \cap \operatorname{im} A_{\sigma(k_0)} \neq \emptyset$ and hence by Lemma 3.3(b) we have

$$\widehat{\mathcal{S}}_{\sigma(k_0)} = \bigcup_{u \in \mathbb{R}^m} (\{B^a_{\sigma(k_0)}u\} + \mathcal{S}_{\sigma(k_0)}) = \operatorname{im} B^a_{\sigma(k_0)} + \mathcal{S}_{\sigma(k_0)}.$$ □

Now, from Corollary 4.3 and the switched index-1 notion in Definition 3.9, it becomes clear that switched index-1 is the necessary and sufficient condition for the solvability of (1) for a given switching signal (and arbitrary inputs).

Corollary 5.4 (Solvability w.r.t. σ) *The switched system* (1) *with given switching signal* σ *is (locally uniquely causally) solvable for all inputs if, and only if,* $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ *is switched index-1 w.r.t. the given* σ*. If solvable, the surrogate system* (23) *is also valid.*

Proof This follows straightforwardly from Proposition 5.1 by observing that validity of the conditions (10) and (11) for all $u_0, u_1 \in \mathbb{R}^m$ is equivalent to the condition for switched index-1 w.r.t. σ. □

Similarly, from Lemma 4.4, it also becomes clear that switched strictly index-1 is the necessary and sufficient condition for the strong solvability of (1) for a given switching signal.

Corollary 5.5 (Strong solvability w.r.t. σ) *System* (1) *is strongly solvable w.r.t. a given switching signal* σ *if, and only if,* $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ *is switched strictly index-1 w.r.t. the given* σ*. If solvable, the surrogate system* (23) *is also valid with* $\Psi^a_{i,j} = 0$.

Proof Clearly, (1) is strongly solvable if, and only if, it is strongly solvable on each interval $[k, k+1]$. Using Lemma (4.4), we can therefore first conclude $\operatorname{im} B_{\sigma(k)} \subseteq \operatorname{im} E_{\sigma(k)}$ for all $k \in \mathbb{N}$. Consequently, we can use again Lemma 4.4 to conclude that solvability is equivalent to $\widehat{\mathcal{R}}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}$, i.e. the familiy $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ is switched strictly index-1 w.r.t. σ. □

Remark 5.6 (Augmented consistency space) We have established in Corollary 5.3 that the set of initial values at time k_0 for which a solution exists is given by $\widehat{\mathcal{S}}_{\sigma(k_0)}$. However, for solvable systems, it is not possible to choose $x_0 \in \widehat{\mathcal{S}}_{\sigma(k_0)}$ independently from the input $u(k_0)$. This motivated the definition of strong solvability. However, as it turns out, strong solvability implies that $\operatorname{im} B_{\sigma(k_0)} \subseteq E_{\sigma(k_0)}$ so that the augmented consistency space $\widehat{\mathcal{S}}_{\sigma(k_0)}$ is equal to the (homogeneous) consistency space $\mathcal{S}_{\sigma(k_0)}$. Consequently, the possibility to choose an initial value outside the homogeneous consistency space comes with the limitation that then this initial value cannot be chosen independently from the input.

5.2 Solvability characterizations for given mode sequence

We now consider the case that only the mode sequence $(\sigma_j)_{j\in\mathbb{N}}$ of the switching signal is known (and given by (3)), but the mode durations are unknown. This situation is quite common in practice when considering single faults, whose occurrence is known (or predicted), but it is not known when the fault occurs.

We can now utilize the already obtained result for solvability for fixed switching signals to conclude the solvability characterization for fixed mode sequences (with arbitrary switching times), however, in order to obtain a simple characterization we have to consider the case of arbitrary inputs. The reason is, that for a specific input u the solvability condition (22) relates the input value $u(k)$ (and $u(k+1)$) at the current time k with the switching signal $\sigma(k)$ (and $\sigma(k+1)$) which is not directly possible anymore when the switching times are not known; however, see the forthcoming Remark 5.8(i).

Corollary 5.7 (Solvability w.r.t. (σ_j)) *The switched system* (1) *with a given (surjective) mode sequence* (σ_j) *is solvable (for all inputs* u*) if, and only if,* $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ *is sequentially index-1 w.r.t. the given mode sequence. In that case,* x *is a solution on* $[k_0, k_1]$ *if, and only if, the corresponding initial value* $x(k_0)$ *satisfies* $x(k_0) \in \mathcal{S}_{\sigma(k_0)} - \left\{B^a_{\sigma(k_0)} u(k_0)\right\}$*, and* x *satisfies the surrogate system* (23).

Proof It is clear, that property 2) of sequentially index-1 w.r.t. switching sequence (σ_j) implies property 2) of switched index-1 for all switching signals σ with switching sequence (σ_j). Furthermore, (regularity and) index-1 of each (E_i, A_i) implies that $\operatorname{im}[E_i, A_i] = \mathbb{R}^n$ (this is a simple consequence of the QWF (7)) which shows the sufficiency part of the proof.

To show necessity, we first consider property 1) of sequentially index-1, i.e. the condition that (E_i, A_i) is index-1. By assumption the mode sequence (σ_j) is surjective, i.e. we can choose a switching signal σ with sequence (σ_j) such that there exists k_0 with $\sigma(k_0) = i = \sigma(k_0 + 1)$. Since, the switched system is solvable for that specific switching signal, property 2) of the switched index condition has to hold, in particular $\{0\} = \ker E_{\sigma(k_0)} \cap \mathcal{S}_{\sigma(k_0+1)} = \ker E_i \cap \mathcal{S}_i$. Then Lemma 3.6 implies that indeed (E_i, A_i) has to be regular and index-1. Necessity of property 2) of the sequential index-1 property follows by considering (the necessary) condition 2) of the switched index-1 property for k_0 such that $\sigma(k_0) = \sigma_j$ and $\sigma(k_0 + 1) = \sigma_{j+1}$. □

Remarks 5.8 (i) It is possible to formulate a characterization of solvability w.r.t. a fixed mode sequence and a *fixed input* by defining first

$$J(k) := \left\{ j \in \mathtt{M} \;\middle|\; \begin{array}{l} \exists\sigma \text{ with mode sequence} \\ (\sigma_j) \text{ s.t. } \sigma_j = \sigma(k) \end{array} \right\}.$$

This set describes which modes can be active at time k. For example, for a periodic mode sequence $(0, 1, 2, 3, 0, 1, 2, 3, ...)$ we have that $J(0) = \{0\}$, $J(1) = \{0, 1\}$, $J(2) = \{0, 1, 2\}$

and $J(k) = \{0, 1, 2, 3\}$ for all $k \geqslant 3$. Then (22) has to be replaced by

$$\mathcal{R}_{\sigma_j} + \left\{B^c_{\sigma_j} u(k) + B^a_{\sigma_{j+1}} u(k+1)\right\} \subseteq \ker E_{\sigma_j} \oplus \mathcal{S}_{\sigma_{j+1}} \quad \forall k \geqslant 0 \ \forall j \in J(k).$$

Additionally, regularity and index-1 for each mode need to be assumed, which then implies that $B^c_{\sigma_j} u(k) \in \mathcal{R}_{\sigma_j}$, so that the dependence on $u(k)$ can be removed.

(ii) The characterization for solvability in Corollary 5.7 can be generalized to the situation where the allowed mode sequences are given in a more complicated way, e.g. by a directed graph with nodes $\{0, 1, \ldots\}$ and where the edges describe which mode transitions are possible. The pairs (σ_j, σ_{j+1}) then need to be replaced by all possible mode pairs (v, w) which are edges in the graph. For a full graph, the jointly index-1 condition is then recovered.

(iii) When considering the homogeneous case (i.e. $B_i = 0$), the definition of sequential index-1 from [22] is recovered.

Corollary 5.9 (Strong solvability w.r.t. (σ_j)) *System* (1) *is* strongly *solvable for a given mode sequence* (σ_j) *if, and only if,* $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ *is sequentially strictly index-1 w.r.t.* (σ_j)*. If solvable, the surrogate system* (23) *is also valid in this case.*

Proof Strong solvability w.r.t. (σ_j) implies solvability w.r.t. (σ_j) as well as strong solvability w.r.t. any switching signal σ with mode sequence (σ_j). Hence, combining Corollary 5.7 with Corollary 5.5 shows that both sequential index-1 w.r.t. (σ_j) as well as switched strictly index-1 w.r.t. any σ with mode sequence (σ_j) are necessary. However, this implies that (E_i, A_i, B_i) needs to be strictly index-1 for all $i \in \mathtt{M}$ and that $\widehat{\mathcal{S}}_{\sigma_{j+1}} = \mathcal{S}_{\sigma_{j+1}}$, so that condition 2) of sequential index-1 becomes condition 2) of sequentially strictly index-1; concluding the necessity part of the proof.

For sufficiency, we observe that sequentially strictly index-1 w.r.t. (σ_j) implies switched strictly index-1 w.r.t. all σ with mode sequence (σ_j), which in turn, using Corollary 5.5, implies strong solvability w.r.t. all σ with mode sequencence (σ_j), i.e. strong solvability w.r.t. (σ_j). □

5.3 Solvability for arbitrary switching signals

From the previous characterizations, intuitively, the jointly (strictly) index-1 notion is the necessary and sufficient condition for the (strong) solvability of (1) for *arbitrary* switching signals, and this is indeed true.

Corollary 5.10 (Solvability for arbitrary switching signals) *The switched system* (1) *is solvable for all switching signals if, and only if,* $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}}$ *is* jointly index-1.

Proof Sufficiency follows from the fact that jointly index-1 implies property 2) of switched index-1 trivially for all switching signals. Furthermore, regularity of (E_i, A_i) (implied by jointly index-1) implies that $\operatorname{im}[E_i, A_i] = \mathbb{R}^n$ and hence property 1) of switched index-1 is also trivially satisfied. Consequently, the switched system is solvable for arbitrary switching signals.

For necessity, we can conclude similar as in Corollary 5.7 that indeed (E_i, A_i) must be (regular and) index-1, which implies that $\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^n$. Furthermore, solvability implies that $\ker E_i \cap \mathcal{S}_j = \{0\}$ for all $i, j \in \mathtt{M}$. Now, a simple dimensional argument (see [21, Lem. 3.3]) shows that indeed $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n$ as required. □

Remark 5.11 (Solvability for arbitrary switching signals and w.r.t. u) It follows that (1) is solvable for all switching signals and w.r.t. a given u if and only if $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ is jointly index-1. This is due to the regularity and index-1 requirement for each mode, i.e., $\ker E_i \oplus \mathcal{S}_i$ must be equal to $\mathbb{R}^n$. This means that the solvability conditions (21)-(22) hold for a particular input if and only if it holds for arbitrary inputs.

Corollary 5.12 (Strong solvability for arbitrary switching) *System* (1) *is* strongly *solvable for all switching signals if and only if $\{(E_i, A_i, B_i)\}_{i\in\mathtt{M}}$ is jointly strictly index-1. If solvable, the surrogate system* (23) *is also valid with $\widetilde{\Psi}^a_{i,j} = 0$.*

Proof By Remark 3.12 we see that jointly strictly index-1 implies that $\operatorname{im} B_i \subseteq \operatorname{im} E_i$ for all $i \in \mathtt{M}$ and hence jointly strictly index-1 implies switched strictly index-1 for any σ. By Corollary 5.5, (1) is therefore strongly solvable for any σ.

Conversely, strong solvability for arbitrary switching signals implies strong solvability for any mode sequence, which, by Corollary 5.9 implies sequentially strictly index-1 for arbitrary mode sequences. This implies $\operatorname{im} B_i \subseteq \operatorname{im} E_i$ and hence $\mathcal{S}_i = \widehat{\mathcal{S}}_i$. Furthermore, as already observed in the proof of Corollary 5.10, $\ker E_i \cap \mathcal{S}_j = \{0\}$ for all $i, j \in \mathtt{M}$ implies $\mathbb{R}^n = \ker E_i \oplus \mathcal{S}_j = \ker E_i \oplus \widehat{\mathcal{S}}_j$, which is jointly strictly index-1. □

5.4 Explicit solution formula

From the establishment of the surrogate system (23) for solvable systems (with $\Psi^a_{i,j} = 0$ for strongly solvable systems), we can define the transition matrix of the switched system as follows:

$$\Phi_\sigma(k_1, k_0) := \Phi_{\sigma(k_1),\sigma(k_1-1)}\Phi_{\sigma(k_1-1),\sigma(k_1-2)}\cdots\Phi_{\sigma(k_0+1),\sigma(k_0)}$$

for $k_1 > k_0$ and $\Phi(k_0, k_0) := I$. Clearly, for homogeneous systems, we have that the solution is given by $x(k) = \Phi_\sigma(k, k_0)x_0$ for every consistent $x_0 \in \mathcal{S}_{\sigma(k_0)}$.

Furthermore, for $k > \ell > 0$ let

$$\begin{aligned}
\Psi_\sigma(0,0) &:= 0,\\
\Psi_\sigma(k,0) &:= \Phi_\sigma(k,1)\Psi^c_{1,0},\\
\Psi_\sigma(k,\ell) &:= \Phi_\sigma(k,\ell+1)\Psi^c_{\sigma(\ell+1),\sigma(\ell)} + \Phi_\sigma(k,\ell)\Psi^a_{\sigma(\ell),\sigma(\ell-1)},\\
\Psi_\sigma(k,k) &:= \Psi^a_{\sigma(k),\sigma(k-1)}.
\end{aligned}$$

Then the explicit solution formula of a general solvable system (1) is given by

$$x(k) = \Phi_\sigma(k,0)x_0 + \sum_{j=0}^{k} \Psi_\sigma(k,j)u(j) \tag{24}$$

where $x_0 \in \mathcal{S}_{\sigma(0)}^{u(0)}$ for solvable systems or with $x_0 \in \widehat{\mathcal{S}}_{\sigma(0)} = \mathcal{S}_{\sigma(0)}$ for strongly solvable systems.

6 (Counter) examples

All implications depicted in Fig. 1 are rather obvious and can be seen directly from the solvability definitions, i.e., with respect to switching signals, solvability for all switching signals implies solvability w.r.t. all mode sequences and the latter implies solvability of a specific switching signal with a given switching sequence. Similar observations also apply to solvability with respect to inputs.

Meanwhile, some of the nonimplications can be deduced directly from the homogeneous case (with $u = 0$ or with $B_i = 0$, see Examples 3.6 and 3.7 in [22]). Now, we provide counter-examples for the rest of the nonimplications.

Example 6.1 (Solvability w.r.t. σ and u) Consider the switched system (1) with

$$(E_0, A_0, B_0) = \left(\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right] \right),$$
$$(E_1, A_1, B_1) = \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \right).$$

Note that after some simple row and column operations this example is a combination of the second (scalar) example discussed in the introduction (showing that solvability depends on the input) and the scalar singular system $0 = 0$ (which if active only for one time step, can still be part of a solvable switched system). Using Corollary 5.1 we will now show that this switched system is solvable w.r.t. the switching signal $\sigma(0) = 0$, $\sigma(k) = 1$, $k \geqslant 1$ and w.r.t. the input $u(k) = (-1)^k$, $k \geqslant 0$, but is neither uniquely solvable for arbitrary switching signals with mode sequence $(0, 1)$ (and the same fixed input), nor is it solvable for arbitrary inputs (and the same fixed switching signal). First, observe that

$$\ker E_0 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right], \quad \mathcal{S}_0 = \mathbb{R}^2, \quad \mathcal{R}_0 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right],$$
$$\ker E_1 = \mathbb{R}^2, \qquad \mathcal{S}_1 = \{0\}, \quad \mathcal{R}_1 = \mathbb{R}^2.$$

Clearly, the first solvability condition (10) of Corollary 5.1 is satisfied; in order to check the second solvability condition (22), we first calculate $E_0^+ = \frac{1}{4} \left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$, $A_1^+ = \left[\begin{smallmatrix} 1 & 0 \\ -1 & 1 \end{smallmatrix}\right]$, $\Pi_{\operatorname{im} E_0}^{\operatorname{im} A_0} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$, $\Pi_{\operatorname{im} A_1}^{\operatorname{im} E_1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then (22) for $k = 0$ reads as

$$\operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] + \left\{ \left[\begin{smallmatrix} u(0)/2 + u(1) \\ u(0)/2 \end{smallmatrix}\right] \right\} \subseteq \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right] \oplus \{0\}$$

Plugging in $u(0) = 1$ and $u(1) = -1$, we see that this condition is indeed satisfied. Since $\ker E_1 \oplus \mathcal{S}_1 = \mathbb{R}^n$, condition (22) is also trivially satisfied for $k \geqslant 1$, hence we can conclude that this switched singular system is solvable w.r.t. the given switching signal and given input. However, if we choose another input signal, e.g. $u(k) = 1$, $k \geqslant 0$, then we see that the solvability condition (22) for $k = 0$ is not satisfied. On the other hand, considering a switching signal where the switching happens later, i.e. $\sigma(0) = \sigma(1) = 0$, we see that (22) for $k = 0$ is also not satisfied because, $\ker E_0 \cap \mathcal{S}_0 \neq \{0\}$.

Example 6.2 (Solvability w.r.t. σ) Consider the scalar switched system (1) with $(E_0, A_0, B_0) = (0, 0, 0)$ and $(E_1, A_1, B_1) = (0, 1, 1)$ and with switching signal $\sigma(0) = 0$, $\sigma(k) = 1$, $k \geqslant 1$ and arbitrary input u. For an arbitrary initial value $x(0)$, the unique solution is given by $x(k) = -u(k)$. It can also be verified easily that $\{(E_0, A_0, B_0), (E_1, A_1, B_1)\}$ is switched index-1 w.r.t. the given σ, hence Corollary 5.4 shows that the switched system is solvable w.r.t. σ. However, $\operatorname{im} B_1 \nsubseteq \operatorname{im} E_1$, hence the switched system cannot be (locally) strongly solvable. Note however, that this example is actually *globally* strongly solvable, because when only considered on the whole time interval $[0, \infty)$, there exists a unique solution for all initial values x_0 and all inputs u. Furthermore, (unique) solvability is lost, when staying longer than one time step in the initial mode because $\ker E_0 \cap \mathcal{S}_0 \neq \{0\}$, i.e. this example is not solvable w.r.t. the mode sequence $(0, 1)$.

Example 6.3 (Solvability w.r.t. (σ_j)) Consider the switched system (1) given by

$$(E_0, A_0, B_0) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]\right), \quad (E_1, A_1, B_1) = \left(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right).$$

We will show in the following that this switched system is solvable w.r.t. switching signals with a mode sequence $(0, 1)$, but it is not strongly solvable w.r.t. this mode sequence and it is also not solvable for arbitrary switching signals. It is easily verified that (E_i, A_i) is regular and index-1 for $i = 0, 1$ and that

$$\ker E_0 = \operatorname{im} \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \quad \mathcal{S}_0 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad \ker E_1 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad \mathcal{S}_1 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right].$$

Furthermore, we have that

$$\ker E_0 \oplus \mathcal{S}_0 = \ker E_0 \oplus \mathcal{S}_1 = \ker E_1 \oplus \mathcal{S}_1 = \mathbb{R}^2.$$

Consequently, $\{(E_0, A_0, B_0), (E_1, A_1, B_1)\}$ is sequentially index-1 w.r.t. the mode sequence $(0, 1)$ and Corollary 5.7 yields that the switched system is solvable for all switching signals with the mode sequence $(0, 1)$ (and all inputs). The above system is clearly not strongly solvable (w.r.t. the mode sequence $(0, 1)$) because $\operatorname{im} B_i \nsubseteq \operatorname{im} E_i$, $i = 0, 1$. Since $\ker E_1 \cap \mathcal{S}_0 \neq \{0\}$, the family $\{(E_0, A_0, B_0), (E_1, A_1, B_1)\}$ is not jointly index-1 and hence the switched system cannot be solvable for arbitrary switching signals.

7 Alternative approaches for jointly index-1 systems

The simplicity of the jointly index-1 condition $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n \ \forall i, j \in \mathtt{M}$ makes it possible to establish some other approaches to check solvability (w.r.t. all switching signals) of the switched system (1).

7.1 Decoupling approach

We recall the following definitions for a regular matrix pair (E, A) with QWF (7) from the continuous time case (see e.g. [28]):

$$\begin{aligned}
\Pi &:= T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}, \\
\Pi^{\text{diff}} &:= T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S, \qquad A^{\text{diff}} := \Pi^{\text{diff}} A, \qquad B^{\text{diff}} := \Pi^{\text{diff}} B, \\
\Pi^{\text{imp}} &:= T \left[\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right] S, \qquad E^{\text{imp}} := \Pi^{\text{imp}} E, \qquad B^{\text{imp}} := \Pi^{\text{imp}} B.
\end{aligned}$$

Note that although S and T for obtaining a QWF are not unique, the above matrices are uniquely determined by the matrix pair (E, A) Furthermore, if (E, A) is index-1 then it is easy to see that

$$\mathcal{S} = A^{-1}(\operatorname{im} E) = \operatorname{im}\Pi = \operatorname{im}\Pi^{\text{diff}} \text{ and } \ker E = \ker\Pi = \operatorname{im}(I-\Pi) = \operatorname{im}\Pi^{\text{imp}}.$$

Similar to the continuous time case (cf. [29, Lem. 1]) we have that x solves the unswitched singular system

$$Ex(k+1) = Ax(k) + Bu(k), \quad x(0) = x_0$$

if, and only if, $x = x^c \oplus x^a$, where

$$\begin{aligned} x^c(k+1) &= A^{\text{diff}}x^c(k) + B^{\text{diff}}u(k), \; x^c(0) = \Pi x_0, \\ E^{\text{imp}}x^a(k+1) &= x^a(k+1) + B^{\text{imp}}u(k), \; x^a(0) = (I-\Pi)x_0. \end{aligned}$$

We now want to utilize this decoupling also for the switched case. Towards this goal let us define for each regular matrix triplet (E_i, A_i, B_i) the matrices Π_i, Π_i^{diff}, Π_i^{imp}, A_i^{diff}, B_i^{diff}, E_i^{imp}, B_i^{imp} as above.

By defining

$$\begin{aligned} x^c(k) &:= \Pi_k x(k), & x^a(k) &:= (I-\Pi_k)x(k), \\ \bar{x}^c(k+1) &:= \Pi_k x(k+1), & \bar{x}^a(k+1) &:= (I-\Pi_k)x(k+1), \end{aligned}$$

we have $x(k) = x^c(k) + x^a(k) = \bar{x}^c(k) + \bar{x}^a(k)$ for $k \geqslant 1$. Furthermore, by multiplying (1) either with $\Pi^{\text{diff}}_{\sigma(k)}$ or $\Pi^{\text{imp}}_{\sigma(k)}$ from the left, it follows that every solution of (1) also satisfies

$$\bar{x}^c(k+1) = A^{\text{diff}}_{\sigma(k)}x^c(k) + B^{\text{diff}}_{\sigma(k)}u(k), \tag{25a}$$
$$x^c(0) = x_0^c := \Pi_{\sigma(k)}x_0,$$
$$E^{\text{imp}}_{\sigma(k)}\bar{x}^a(k+1) = x^a(k) + B^{\text{imp}}_{\sigma(k)}u(k), \tag{25b}$$
$$x^a(0) = x_0^a := (I-\Pi_{\sigma(k)})x_0.$$

However, the equations cannot be solved directly because there is no explicit relationship between $\bar{x}^c$ and x^c as well as between $\bar{x}^a$ and x^a. Note however that the jointly index-1 property implies that $E_i^{\text{imp}} = 0$ for all modes i, which leads to the immediate solution of (25b):

$$x^a(k) = -B^{\text{imp}}_{\sigma(k)}u(k), \quad k \geqslant 0.$$

Furthermore, $\bar{x}^c(k+1)$ (but not $x^c(k+1)$) is uniquely given by $x^c(k)$ and $B^{\text{diff}}_{\sigma(k)}u(k)$. By construction, we know that $x^c(k+1) \in \operatorname{im}\Pi_{\sigma(k+1)} = \mathcal{S}_{\sigma(k+1)}$ and $\bar{x}^a(k+1) \in$

$\operatorname{im}(I - \Pi_{\sigma(k)}) = \ker E_{\sigma(k)}$, consequently

$$\begin{aligned} x(k+1) &= x^a(k+1) + x^c(k+1) = \bar{x}^c(k+1) + \bar{x}^a(k+1) \\ &\in \big(\{x^a(k+1)\} + \mathcal{S}_{\sigma(k+1)}\big) \cap \big(\{\bar{x}^c(k+1)\} + \ker E_{\sigma(k)}\big) \end{aligned}$$

Jointly index-1 together with Lemma 3.4 implies that the above intersection is a singleton and hence

$$x(k+1) = \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}} \big(\bar{x}^c(k+1) - x^a(k+1)\big) + x^a(k+1).$$

We can now formulate the solvability characterization as well as the surrogate system in terms of the decoupled states x^c and x^a.

Proposition 7.1 *Consider the jointly index-1 switched system (1) with the corresponding matrices Π_i, Π_i^{diff}, Π_i^{imp}, A_i^{diff}, E_i^{imp}, B_i^{diff} and B_i^{imp} as above. Then x is a solution of (1) on $[k_0, k_1]$ for some given input u if, and only if, $x(k_0) \in \mathcal{S}_{\sigma(k_0)} - \{B^{\text{imp}}_{\sigma(k_0)} u(k_0)\}$ and x is a solution of the surrogate system*

$$x(k+1) = \bar{\Phi}_{\sigma(k+1),\sigma(k)} x(k) + \bar{\Psi}^c_{\sigma(k+1),\sigma(k)} u(k) + \bar{\Psi}^a_{\sigma(k+1),\sigma(k)} u(k+1), \tag{26}$$

where

$$\begin{aligned} \bar{\Phi}_{i,j} &:= \Pi^{\ker E_j}_{\mathcal{S}_i} A_j^{\text{diff}}, \\ \bar{\Psi}^c_{i,j} &:= \Pi^{\ker E_j}_{\mathcal{S}_i} B_j^{\text{diff}}, \qquad \bar{\Psi}^a_{i,j} := (\Pi^{\ker E_j}_{\mathcal{S}_i} - I) B_i^{\text{imp}}. \end{aligned}$$

Proof "⇒" If x is a solution of (1) then by the above arguments we have

$$x(k+1) = \Pi^{\ker E_{\sigma(k)}}_{\mathcal{S}_{\sigma(k+1)}} (\bar{x}^c(k+1) - x^a(k+1)) + x^a(k+1),$$

and together with $\bar{x}^c(k+1) = A^{\text{diff}}_{\sigma(k)} x^c(k) + B^{\text{diff}}_{\sigma(k)} u(k)$ and $x^a(k+1) = -B^{\text{imp}}_{\sigma(k)} u(k+1)$, we arrive at the claimed surrogate system. Furthermore, we can conclude that $x(k_0) = x^c(k_0) + x^a(k_0) \in \operatorname{im} \Pi_{\sigma(k_0)} - \{B^{\text{imp}}_{\sigma(k_0)} u(k_0)\}$ which shows the claim concerning the initial value because $\operatorname{im} \Pi_{\sigma(0)} = \mathcal{S}_{\sigma(0)}$
"⇐" We first observe that $\Pi_i^{\text{diff}} + \Pi_i^{\text{imp}} = T_i S_i$ and hence is invertible. Therefore x is a solution of (1) if, and only if, x solves

$$\begin{aligned} &(\Pi^{\text{diff}}_{\sigma(k)} + \Pi^{\text{imp}}_{\sigma(k)}) E_{\sigma(k)} x(k+1) \\ &\qquad = (\Pi^{\text{diff}}_{\sigma(k)} + \Pi^{\text{imp}}_{\sigma(k)}) A_{\sigma(k)} x(k) + (\Pi^{\text{diff}}_{\sigma(k)} + \Pi^{\text{imp}}_{\sigma(k)}) B_{\sigma(k)} u(k). \end{aligned}$$

Since $\operatorname{im} \Pi_i^{\text{diff}} \oplus \operatorname{im} \Pi_i^{\text{imp}} = \mathbb{R}^n$ the latter equation holds, if and only if, the following two equations hold

$$\Pi^{\text{diff}}_{\sigma(k)} E_{\sigma(k)} x(k+1) = A^{\text{diff}}_{\sigma(k)} x(k) + B^{\text{diff}}_{\sigma(k)} u(k) \quad \text{and} \tag{27a}$$

$$E^{\text{imp}}_{\sigma(k)} x(k+1) = \Pi^{\text{imp}}_{\sigma(k)} A_{\sigma(k)} x(k) + B^{\text{imp}}_{\sigma(k)} u(k). \tag{27b}$$

We will now show that x given by the surrogate system satisfies these two equations for all $k \in \mathbb{N}$. Towards this goal we first observe that $\Pi_i^{\text{diff}} E_i = \Pi_i$, $\Pi_i A_i^{\text{diff}} = A_i^{\text{diff}}$ and $\Pi_i B_i^{\text{diff}} = B_i^{\text{diff}}$

which means that (27a) can equivalently be written as (28a) below. Furthermore, $\Pi_i^{\text{imp}} A_i = (I - \Pi_i)$ and (jointly) index-1 implies that $E_i^{\text{imp}} = 0$, hence (27b) takes the form (28b) below.

$$\Pi_{\sigma(k)}(x(k+1) - A^{\text{diff}}_{\sigma(k)} x(k) - B^{\text{diff}}_{\sigma(k)} u(k)) = 0 \tag{28a}$$

$$0 = (I - \Pi_{\sigma(k)})x(k) + B^{\text{imp}}_{\sigma(k)} u(k) \tag{28b}$$

Step 1: We show that (28a) holds.

Using the proposed surrogate system, we can replace $x(k+1)$ in (28a) by the right hand side of (26); by observing that $\operatorname{im} \bar{\Psi}^a_{i,j} \subseteq \ker E_j = \ker \Pi_j$ for all mode pairs i, j and recalling that $A_j^{\text{diff}} = \Pi_j^{\text{diff}} A_j$ and $\Pi_j \Pi_j^{\text{diff}} = \Pi_j^{\text{diff}}$ we see that (28a) is satisfied if

$$\Pi_j \Pi_{\mathcal{S}_i}^{\ker E_j} \Pi_j^{\text{diff}} = \Pi_j^{\text{diff}}.$$

The latter is a simple consequence from Lemma 3.5 and utilizing that $\Pi_j = \Pi_{\mathcal{S}_j}^{\ker E_j}$.

Step 2: We show that (28b) holds.

For $k = 0$, (28b) clearly holds because $x(0) = x_0 \in \mathcal{S}_{\sigma(0)} - \{B^{\text{imp}}_{\sigma(0)} u(0)\}$, $\mathcal{S}_{\sigma(0)} \subseteq \ker(I - \Pi_{\sigma(k)})$ and $B^{\text{imp}}_{\sigma(0)} u(0) \subseteq \operatorname{im} \Pi^{\text{imp}}_{\sigma(0)} = \operatorname{im}(I - \Pi_{\sigma(0)})$. For $k \geqslant 1$, we can replace $x(k)$ in (28b) by the (time-shifted) right-hand side of (26); by observing that both $\operatorname{im} \bar{\Phi}_{i,j}$ and $\operatorname{im} \bar{\Psi}^c_{i,j}$ are subsets of $\operatorname{im} \Pi_j = \ker(I - \Pi_i)$ for all mode pairs i, j, it remains to be shown that

$$(I - \Pi_i)(\Pi_{\mathcal{S}_i}^{\ker E_j} - I)\Pi_i^{\text{imp}} = -\Pi_i^{\text{imp}}.$$

But this follows again from Lemma 3.5 by observing that $I - \Pi_i = \Pi^{\mathcal{S}_i}_{\ker E_i}$ and $(\Pi_{\mathcal{S}_i}^{\ker E_j} - I) = -\Pi^{\mathcal{S}_i}_{\ker E_j}$. □

Remark 7.2 In general the matrices in the surrogate systems (23) and (26) are different (but result in the same solution trajectories). While (23) doesn't rely on the regularity and index-1 assumption for each mode, there is some freedom in choosing the matrices (because the pseudo-inverse is not unique and the projector $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} + \mathcal{W} \to \mathcal{V}$ is non-unique if $\mathcal{V} \cap \mathcal{W} \neq \{0\}$). On the other hand, (26) is only valid if each mode is regular and index-1, but then all involved matrices are uniquely defined.

7.2 Row reduced approach

Corollary 5.1 and Proposition 7.1 present a nice geometric and coordinate-free solution formula for switched descriptor systems (because all involved matrices can be seen as linear maps defined in the originally considered vector spaces). However, in real applications a specific coordinate system needs to be chosen and all calculations are done with specific matrix representations of the underlying linear maps. In that case, Corollary 5.1 and Proposition 7.1 involve quite many matrix operations and may not always be very efficient; therefore we present another, more practicable, solvability characterization and solution formula as follows (inspired by [23]): Choose a family of invertible matrices S_j (in fact, each S_j can be chosen as a product of simple Gauss eliminations and row permutations) such that, for $j \in \mathtt{M}$,

$$S_j E_j = \begin{bmatrix} E_j^1 \\ 0 \end{bmatrix}, \quad E_j^1 \text{ is full row rank.} \tag{29}$$

Furthermore, let

$$\begin{bmatrix} A_j^1 \\ A_j^2 \end{bmatrix} := S_j A_j, \quad \begin{bmatrix} B_j^1 \\ B_j^2 \end{bmatrix} := S_j B_j, \tag{30}$$

where A_j^1 and B_j^1 have the same number of rows as E_j^1.

Proposition 7.3 (Numerical solvability characterization) *The switched system* (1) *is (locally uniquely causally) solvable for all input u and all switching signals σ if, and only if,* $\begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix}$ *is square and invertible for all $i, j \in \mathtt{M}$, where the notation of* (29) *and* (30) *is used. In that case, x is a solution of* (1) *if, and only if, the initial value $x_0 = x(0)$ satisfies $A_{\sigma(0)}x_0 + B_{\sigma(0)}u(0) \in \operatorname{im} E_{\sigma(0)}$ and*

$$x(k+1) = \widetilde{\Phi}_{\sigma(k+1),\sigma(k)}x(k) + \widetilde{\Psi}^c_{\sigma(k+1),\sigma(k)}u(k) + \widetilde{\Psi}^a_{\sigma(k+1),\sigma(k)}u(k+1), \tag{31}$$

where

$$\widetilde{\Phi}_{i,j} := \begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix}^{-1} \begin{bmatrix} A_j^1 \\ 0 \end{bmatrix},$$

$$\widetilde{\Psi}^c_{i,j} := \begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix}^{-1} \begin{bmatrix} B_j^1 \\ 0 \end{bmatrix}, \qquad \widetilde{\Psi}^a_{i,j} := \begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -B_i^2 \end{bmatrix}.$$

Proof With the same arguments as in the proof of Lemma 3.6, it follows that $\mathcal{S}_i = A_i^{-1}(\operatorname{im} E_i) = \ker A_i^2$ and $\ker E_j = \ker E_j^1$. Consequently, we have that $\mathcal{S}_i \cap \ker E_j = \{0\}$ if, and only if, $\ker \begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix} = \{0\}$. Taking into account that jointly index-1 implies that $\operatorname{rank} E_i = \operatorname{rank} E_j$ (see [21, Lem. 3.3]), we can immediatly conclude that jointly index-1 is equivalent to $\begin{bmatrix} E_j^1 \\ A_i^2 \end{bmatrix}$ being square and invertible for all pairs i, j. Now multiplying the switched system (1) from the left with $S_{\sigma(k)}$ we obtain

$$\begin{bmatrix} E^1_{\sigma(k)} \\ 0 \end{bmatrix} x(k+1) = \begin{bmatrix} A^1_{\sigma(k)} \\ A^2_{\sigma(k)} \end{bmatrix} x(k) + \begin{bmatrix} B^1_{\sigma(k)} \\ B^2_{\sigma(k)} \end{bmatrix} u(k),$$

$$\begin{bmatrix} E^1_{\sigma(k+1)} \\ 0 \end{bmatrix} x(k+2) = \begin{bmatrix} A^1_{\sigma(k+1)} \\ A^2_{\sigma(k+1)} \end{bmatrix} x(k+1) + \begin{bmatrix} B^1_{\sigma(k+1)} \\ B^2_{\sigma(k+1)} \end{bmatrix} u(k+1).$$

Hence, we have

$$\begin{bmatrix} E^1_{\sigma(k)} \\ A^2_{\sigma(k+1)} \end{bmatrix} x(k+1) = \begin{bmatrix} A^1_{\sigma(k)} \\ 0 \end{bmatrix} x(k) + \begin{bmatrix} B^1_{\sigma(k)} \\ 0 \end{bmatrix} u(k) - \begin{bmatrix} 0 \\ B^2_{\sigma(k+1)} \end{bmatrix} u(k+1),$$

which results in (31), by left-multiplying with the inverse of $\begin{bmatrix} E^1_{\sigma(k)} \\ A^2_{\sigma(k+1)} \end{bmatrix}$. □

7.3 Unification of *E*-matrix approach

Since we have already established that solvability for all switching signals and all inputs requires jointly index-1, we know that $\operatorname{rank} E_i = r$ for some $r \in \mathbb{N}$ independent from the mode. Hence, it is possible to find invertible matrices P_i, Q_i such that

$$P_i E_i Q_i = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}. \tag{32}$$

For an arbitrary $\sigma : \mathbb{N} \to \mathtt{M}$ we consider the time-dependent coordinate transformation

$$\widehat{x}(k) := \begin{pmatrix} \widehat{x}^1(k) \\ \widehat{x}^1(k) \end{pmatrix} := Q_{\sigma(k-1)} x(k)$$

with arbitrarily chosen $\sigma(-1) \in \mathtt{M}$. Then the switched system (1) is equivalent to

$$P_{\sigma(k)} E_{\sigma(k)} Q_{\sigma(k)} \widehat{x}(k+1) = P_{\sigma(k)} A_{\sigma(k)} Q_{\sigma(k-1)} \widehat{x}(k) + P_{\sigma(k)} B_{\sigma(k)} u(k).$$

With

$$P_i A_i Q_j := \begin{bmatrix} A^1_{i,j} & A^2_{i,j} \\ A^3_{i,j} & A^4_{i,j} \end{bmatrix} \quad \text{and} \quad P_i B_i = \begin{bmatrix} B^1_i \\ B^2_i \end{bmatrix} \tag{33}$$

we therefore have the equivalent switched system

$$\begin{aligned} \widehat{x}(k+1) &= A^1_{\sigma(k),\sigma(k-1)} \widehat{x}^1(k) + A^2_{\sigma(k),\sigma(k-1)} \widehat{x}^2(k) + B^1_{\sigma(k)} u(k) \\ 0 &= A^3_{\sigma(k),\sigma(k-1)} \widehat{x}^1(k) + A^4_{\sigma(k),\sigma(k-1)} \widehat{x}^2(k) + B^2_{\sigma(k)} u(k). \end{aligned}$$

or, after applying a time shift to the second equation,

$$\begin{aligned} \begin{bmatrix} I & 0 \\ A^3_{\sigma(k+1),\sigma(k)} & A^4_{\sigma(k+1),\sigma(k)} \end{bmatrix} \widehat{x}^1(k+1) &= \begin{bmatrix} A^1_{\sigma(k),\sigma(k-1)} & A^2_{\sigma(k),\sigma(k-1)} \\ 0 & 0 \end{bmatrix} \widehat{x}(k) \\ &\quad + \begin{bmatrix} B^1_{\sigma(k)} \\ 0 \end{bmatrix} u(k) - \begin{bmatrix} 0 \\ B^2_{\sigma(k+1)} \end{bmatrix} u(k+1) \end{aligned}$$

It is easily seen, that the switched system (1) is jointly index-1 if, and only if, $\operatorname{rank} E_i$ is constant and the matrices $A^4_{i,j}$ are all invertible. Hence the following solvability characterization holds:

Proposition 7.4 (cf. [21, Thm. 5.1]) *The switched system (1) is (locally uniquely causally) solvable for all inputs u and all switching signals σ if, and only if,* $\operatorname{rank} E_i = r_i = r$ *(i.e. E_i have a constant rank) and $A^4_{i,j}$ in (33) is invertible. In that case, x is a solution of (1) if, and only if, the initial value $x_0 = x(0)$ satisfies $A_{\sigma(0)} x_0 + B_{\sigma(0)} u(0) \in \operatorname{im} E_{\sigma(0)}$ and*

$$\begin{aligned} x(k+1) &= \widehat{\Phi}_{\sigma(k+1),\sigma(k),\sigma(k-1)} x(k) \\ &\quad + \widehat{\Psi}^c_{\sigma(k+1),\sigma(k),\sigma(k-1)} u(k) + \widehat{\Psi}^a_{\sigma(k+1),\sigma(k)} u(k+1), \end{aligned} \tag{34}$$

where

$$\begin{aligned} \widehat{\Phi}_{i,j,\ell} &:= Q_j \begin{bmatrix} A^1_{j,\ell} & A^2_{j,\ell} \\ -(A^4_{i,j})^{-1} A^3_{i,j} A^1_{j,\ell} & -(A^4_{i,j})^{-1} A^3_{i,j} A^2_{j,\ell} \end{bmatrix} Q_\ell^{-1}, \\ \widehat{\Psi}^c_{i,j} &:= Q_j \begin{bmatrix} B^1_j \\ -(A^4_{i,j})^{-1} A^3_{i,j} B^1_j \end{bmatrix}, \qquad \widehat{\Psi}^a_{i,j,\ell} := Q_j \begin{bmatrix} 0 \\ -(A^4_{i,j})^{-1} B^2_i \end{bmatrix}. \end{aligned}$$

Remark 7.5 Apparently, the surrogate system (34) does not only depend on the modes active at $k+1$ and k but also on the mode at $k-1$. This is somewhat unintuitive, in particular, when considering the system only on $[k_0, k_0+1]$ because the solution formula should be independent of the mode in the past. This was also highlighted in [21, Rem. 5.2] and with our result, we can now confirm that indeed the dependence on $\sigma(k-1)$ is an artifact of the specific method and not a fundamental property of the system.

8 Applications

8.1 Dynamic Leontief model example

Consider the switched case of the discrete dynamic Leontief model of a multisector economy given by (cf. [17])

$$C_{\sigma(k)}x(k+1) = (I - L_{\sigma(k)} + C_{\sigma(k)})x(k) - d(k) \tag{35}$$

where $x(k)$ is the vector of output levels at the time period $k = 0, 1, \ldots$, $d(k)$ is the vector of final demands, L_i is the Leontief input–output matrix, and C_i is the capital coefficient matrix, $i \in \mathtt{M}$. When the market and technology do not change over time, the matrices L_i and C_i are known and time-invariant [17]. Otherwise, those matrices change. We assume here that the changes on those matrices are due to some disturbances, and a switching signal $\sigma : \mathbb{N} \to \mathtt{M}$ rules the changing of L_i and C_i. The following data are taken from [17] where the mode-0 corresponds to the original (or nominal) data and the other modes correspond to its variations:

$$L_0 = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.40 & 0.10 & 0.50 \\ 0.30 & 0.50 & 0.20 \end{bmatrix}, \quad L_1 = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.40 & 0.10 & 0.50 \\ 0 & 0 & 0 \end{bmatrix}, \quad L_2 = L_0,$$
$$C_0 = \begin{bmatrix} 0.30 & 0.40 & 0.45 \\ 0 & 0 & 0 \\ 0.60 & 0.80 & 0.90 \end{bmatrix}, \quad C_1 = C_0, \quad C_2 = \begin{bmatrix} 0.30 & 0.40 & 0.5 \\ 0 & 0 & 0 \\ 0.60 & 0.80 & 0.90 \end{bmatrix}.$$

For the following discussion, all computations are exact and are done via MATLAB's symbolic toolbox, however, for simplicity, we show the results with only two decimals. Those matrices give us the system (1) with $\mathtt{M} = \{0, 1, 2\}$, $E_i = C_i$, $B_i = I$,

$$A_0 = \begin{bmatrix} 1.00 & 0.10 & 0.15 \\ -0.40 & 0.90 & -0.50 \\ 0.30 & 0.30 & 1.70 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 1.00 & 0.10 & 0.15 \\ -0.40 & 0.90 & -0.50 \\ 0.60 & 0.80 & 1.90 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.00 & 0.10 & 0.20 \\ -0.40 & 0.90 & -0.50 \\ 0.30 & 0.30 & 1.70 \end{bmatrix},$$

and

$$\ker E_0 = \operatorname{im} \begin{bmatrix} 1.00 & 1.13 \\ -0.75 & 0.00 \\ 0.00 & -0.75 \end{bmatrix}, \quad \ker E_1 = \operatorname{im} \begin{bmatrix} 1.00 & 1.13 \\ -0.75 & 0.00 \\ 0.00 & -0.75 \end{bmatrix}, \quad \ker E_2 = \operatorname{im} \begin{bmatrix} 1.00 \\ -0.75 \\ 0.00 \end{bmatrix},$$
$$\mathcal{S}_0 = \operatorname{im} \begin{bmatrix} 1.00 \\ 1.08 \\ 1.14 \end{bmatrix}, \quad \mathcal{S}_1 = \operatorname{im} \begin{bmatrix} 1.00 \\ 0.77 \\ 0.59 \end{bmatrix}, \quad \mathcal{S}_2 = \operatorname{im} \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \\ -0.80 & 1.80 \end{bmatrix}.$$

First, note that each mode is index-1 since $\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^3$ for each $i = 0, 1, 2$, i.e., each mode as an individual system is solvable.

8.1.1 Solvability for arbitrary switching signals

Geometric checking shows that the family of matrix triplets $\{(E_i, A_i, B_i)\}_{i \in \mathtt{M}_1}$ with $\mathtt{M}_1 = \{0, 1\}$ is jointly index-1 since $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^3$ for all $i, j \in \mathtt{M}_1$. Therefore, all switched systems composed of only mode-0 and mode-1 are solvable for all switching signals. However, adding mode-2 into the family of the matrix triplets gives us a non-jointly index-1 system because $\ker E_0 \cap \mathcal{S}_2 \neq \{0\}$ also $\ker E_1 \cap \mathcal{S}_2 \neq \{0\}$.

8.1.2 Solvability for given mode sequences

From the observation above, in order to have a solvable system, having mode transitions inside $\mathtt{M}_1$ is always possible. On the other hand, the mode transitions $0 \to 2$ and $1 \to 2$ do not lead to a (uniquely) solvable switched system. Furthermore, it can be verified that

$$\widehat{\mathcal{R}}_2 + \widehat{\mathcal{S}}_0 \supsetneq \ker E_2 \oplus \mathcal{S}_0 \text{ and } \widehat{\mathcal{R}}_2 + \widehat{\mathcal{S}}_1 \supsetneq \ker E_2 \oplus \mathcal{S}_1,$$

hence the mode transitions $2 \to 0$ and $2 \to 1$ are also not allowed (i.e. it is not sequentially index-1 also not switched index-1 for the corresponding switching signals containing those mode transitions). The system is then not solvable for arbitrary inputs and furthermore, also not solvable for a given input because $\mathcal{R}_i = \mathbb{R}^3$ for all $i \in \mathtt{M}$ and

$$\dim(\ker E_2 \oplus \mathcal{S}_0) = \dim(\ker E_2 \oplus \mathcal{S}_1) = 2 < \dim \mathcal{R}_i,$$

i.e., the solvability condition w.r.t. a given input (22) can never be satisfied with any input sequence.

8.1.3 One-step maps

For this part, consider only the mode transition $0 \to 1$. With $E_0^+ = \begin{bmatrix} -0.12 & 0.00 & 0.21 \\ 0.13 & 0.00 & 0.00 \\ 0.10 & 0.00 & -0.05 \end{bmatrix}$, we get the one-step maps for the four approaches

$$\Phi_{1,0} = \begin{bmatrix} 0.17 & 0.17 & 0.97 \\ 0.13 & 0.13 & 0.75 \\ 0.10 & 0.10 & 0.57 \end{bmatrix}, \qquad \widetilde{\Phi}_{1,0} = \begin{bmatrix} 0.17 & 0.17 & 0.97 \\ 0.13 & 0.13 & 0.75 \\ 0.10 & 0.10 & 0.57 \end{bmatrix},$$

$$\bar{\Phi}_{1,0} = \begin{bmatrix} 0.35 & 0.47 & 0.53 \\ 0.27 & 0.36 & 0.41 \\ 0.21 & 0.28 & 0.31 \end{bmatrix}, \qquad \widehat{\Phi}_{1,0} = \begin{bmatrix} 0.35 & 0.47 & 0.53 \\ 0.27 & 0.36 & 0.41 \\ 0.21 & 0.28 & 0.31 \end{bmatrix},$$

$$\Psi^c_{1,0} = \begin{bmatrix} 0.00 & 0.00 & 0.57 \\ 0.00 & 0.00 & 0.44 \\ 0.00 & 0.00 & 0.34 \end{bmatrix}, \qquad \widetilde{\Psi}^c_{1,0} = \begin{bmatrix} 0.00 & 0.00 & 0.57 \\ 0.00 & 0.00 & 0.44 \\ 0.00 & 0.00 & 0.34 \end{bmatrix},$$

$$\bar{\Psi}^c_{1,0,0} = \begin{bmatrix} 0.38 & 0.35 & 0.38 \\ 0.29 & 0.27 & 0.29 \\ 0.22 & 0.21 & 0.22 \end{bmatrix}, \qquad \widehat{\Psi}^c_{1,0} = \begin{bmatrix} 0.38 & 0.35 & 0.38 \\ 0.29 & 0.27 & 0.29 \\ 0.22 & 0.21 & 0.22 \end{bmatrix},$$

$$\Psi^a_{1,0} = \begin{bmatrix} -0.80 & 0.24 & 0.40 \\ -0.04 & -0.73 & 0.02 \\ 0.57 & 0.49 & -0.28 \end{bmatrix}, \qquad \widetilde{\Psi}^a_{1,0,0} = \begin{bmatrix} -0.80 & 0.24 & 0.40 \\ -0.04 & -0.73 & 0.02 \\ 0.57 & 0.49 & -0.28 \end{bmatrix},$$

$$\bar{\Psi}^a_{1,0} = \begin{bmatrix} -0.80 & 0.24 & 0.40 \\ -0.04 & -0.73 & 0.02 \\ 0.57 & 0.49 & -0.28 \end{bmatrix}, \qquad \widehat{\Psi}^a_{1,0} = \begin{bmatrix} -0.80 & 0.24 & 0.40 \\ -0.04 & -0.73 & 0.02 \\ 0.57 & 0.49 & -0.28 \end{bmatrix}.$$

Not all corresponding one-step maps from the four approaches are the same, however, they give us the same solution trajectories for the same initial value and input. For the case with $u = 0$, this can be seen from the fact that for any initial value $x_{k_0} \in \mathbb{R}^n$ and $\operatorname{im} Y_{k_0} = \mathcal{S}_{\sigma(k_0)}$ we have that $\Phi_{i,j} Y_{k_0} x_{k_0} = \bar{\Phi}_{i,j} Y_{k_0} x_{k_0} = \widetilde{\Phi}_{i,j} Y_{k_0} x_{k_0} = \widehat{\Phi}_{i,j} Y_{k_0} x_{k_0}$.

8.2 Discretization of continuous-time switched singular systems

Consider a switched differential-algebraic equation (swDAE) [28] of the form

$$E_{\sigma(t)} \dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t), \tag{36}$$

where $\sigma : \mathbb{R} \to \mathtt{M}$ is the switching signal, $x : \mathbb{R} \to \mathbb{R}^n$ is the state and $u : \mathbb{R} \to \mathbb{R}^m$ is the input. We call (x, u) an (impulse-free) solution of the swDAE if Ex is absolutely continuous, u is locally integrable, and (36) holds for almost all $t \in \mathbb{R}$ (or on the interval of interest). It is well known (see e.g. [28]), that for the case of regular and index-1 matrix pairs (E_i, A_i), $i \in \mathtt{M}$, there is for every x_0, every input u and every switching signal a unique solution (on (t_0, ∞)) of the corresponding (inconsistent) initial value problem. We now want to investigate under which assumptions different numerical discretization methods lead to solvable switched singular systems in discrete time.

8.2.1 (Semi-)Explicit Euler method

We start with the most canonical numerical discretization method, the explicit Euler method, which is based on the following approximation:

$$E_{\sigma(t_k)} \tfrac{x(t_k+h)-x(t_k)}{h} \approx A_{\sigma(t_k)} x(t_k) + B_{\sigma(t_k)} u(t_k),$$

where $t_k := t_0 + kh$ and $h > 0$ is the step-size. To avoid notational technicalities, we assume here that h is chosen such that t_k does not coincide with the switching times and discontinuities in u. The corresponding discrete-time switched system then takes the form (where with some abuse of notation we identify $\sigma(t_k), x(t_k), u(t_k)$ with $\sigma(k)$, $x(k)$, $u(k)$):

$$E_{\sigma(k)} x(k+1) = A^h_{\sigma(k)} x(k) + h B_{\sigma(k)} u(k),$$

where $A^h_i := E_i + hA_i$, $i \in \mathtt{M}$. Since $\operatorname{rank} E_i$ in general is mode-dependent, it immediately follows that an explicit Euler approximation does not result in a jointly index-1 discretization (because then $\operatorname{rank} E_i$ must be mode-independent). Furthermore, exploiting the row-reduction approach from Section 7.2 for jointly index-1, we immediately see that $\{(E_i, A^h_i)\}_{i\in\mathtt{M}}$ is jointly index-1 if, and only if, $\{(E_i, A_i)\}_{i\in\mathtt{M}}$ is jointly index-1. In that case, we can also see via the row-reduced approach that the consistency condition

$$0 = A^2_\sigma(t)x(t) + B^2_{\sigma(t)} u(t)$$

also remains valid in the discretized version, where this condition is simply multiplied by h (which however may lead to numerical issues for small h).

Furthermore, by applying a row reducing left multiplication $S_{\sigma(t)}$ of the swDAE (36) (which doesn't change the solution properties at all), we see that we can easily avoid the unnecessary multiplication of the algebraic constraint by h; this results in fact in the well known semi-explicit Euler (where it is usually assumed that $E_i = \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$, however this is not necessary here).

8.2.2 (Semi-)Implicit Euler

It is well known that in general, the explicit Euler method exhibits some stability issues and therefore the implicit Euler method has been proposed, which for the swDAE (36) is based on the following approximation:

$$E_{\sigma(t_{k+1})} \tfrac{x(t_{k+1})-x(t_k)}{h} \approx A_{\sigma(t_{k+1})} x(t_{k+1}) + B_{\sigma(t_{k+1})} u(t_{k+1}).$$

This results in the discretization

$$E^h_{\sigma(k+1)}x(k+1) = E_{\sigma(k+1)}x(k) + hB_{\sigma(k+1)}u(k+1), \tag{37}$$

where $E^h_i := E_i + hA_i$ which is invertible for all sufficiently small $h > 0$ by regularity of the matric pairs (E_i, A_i) (and hence E^h_i is invertible for all but finitely many $h > 0$). Consequently, there are no issues concerning the solvability of the discretized system. Furthermore, by applying mode-wise row operations to obtain a row-reduced form, we immediately see, that the consistency conditions for $x(k+1)$ are preserved. However, one major disadvantage is the non-strict causality w.r.t. the input, which is in general not the expected solution behavior.

It is also possible to consider a semi-implicit method [30], where an approximation of the half point $x(t_k + h/2)$ either forward from $x(t_k)$ or backwards from $x(t_k + h)$ leads to the following discretization

$$E^h_{\sigma(k+1)}x(k+1) = A^h_{\sigma(k)}x(k) + B^h_{\sigma(k+1),\sigma(k)} \begin{pmatrix} u(k) \\ u(k+1) \end{pmatrix}, \tag{38}$$

where $E^h_i := E_i + \frac{h}{2}A_i$, $A^h_j := E_j + \frac{h}{2}A_j$ and $B^h_{i,j} := \frac{h}{2}[B_j, -B_i]$. Here E^h_i is always invertible for sufficiently small $h > 0$ because (E_i, A_i) is assumed to be regular. Consequently, the switched system (38) is always solvable. However, it is not clear whether the algebraic constraints are (exactly or approximately) satisfied, which is a big disadvantage of this method. Furthermore, similarly to the fully implicit method, the dependence on the input is non-strictly causal, which is not necessarily the expected solution behavior.

Finally, Backward Differentiation Formulae (BDF) are an established numerical method to approximate solutions of singular systems in continuous time [31] and can be seen as multi-step generalizations of the implicit Euler method. As such, they will result in higher order switched singular systems, of which the solution theory is not fully established; this is a topic of future research.

9 Conclusion and Outlooks

Two solvability notions have been introduced for inhomogeneous singular linear switched systems: solvability (initial values are dependent on the input) and strong solvability (initial values and inputs are independent of each other).

The characterizations have been fully established. With respect to switching signals, the characterizations have been formulated for fixed switching signals, fixed mode sequences with arbitrary switching times, and arbitrary switching signals whereas with respect to input sequences, the characterizations have been formulated for a given input sequence and arbitrary inputs.

Six (strict) index-1 notions of switched, sequentially, and jointly index-1 have been introduced, and they are necessary and sufficient conditions for the (strong) solvability w.r.t. fixed switching signals, fixed mode sequences, and arbitrary switching signals, respectively.

For every solvable system, a surrogate system has been established, and the explicit solution formula of the original singular system can then be written in its original state. Furthermore, in terms of inputs, the solutions of solvable systems are causal (the current input can affect the current state) whereas the solutions of strongly solvable systems are strictly causal (the current input cannot affect the current state).

In future works, we will exploit the surrogate systems to further study their original solvable singular switched systems. These future studies include model reduction, controllability analysis, stability characterizations, and control designs, among others. Another important future research topic is the consideration of numerical aspects and the robust implementation of analysis and simulation tools, in particular for large-scale systems.

Acknowledgements. The first and fourth authors were supported by RPI Universitas Diponegoro Research Grant contract no. 973-31/UN7.D2/PP/IX/2024. The manuscript was in large parts prepared while the first, second, third, and fifth authors were invited guest researchers at the Vietnam Institute for Advanced Study in Mathematics (VIASM), Hanoi, Vietnam. The authors would like to thank the financial and organizational support of the Vietnam Institute for Advanced Study in Mathematics (VIASM), which provides a fruitful research environment and hospitality. The first author also would like to thank Universitas Diponegoro for the partial funding for the visit to VIASM through the WCU program 2024.

Declarations

Not applicable

References

[1] Hu, W., Ding, Z., Chen, X., Long, X.: Controllability analysis of flyback converter with constant power load based on switched linear systems. Energy Reports **8**, 747–754 (2022) https://doi.org/10.1016/j.egyr.2022.08.018

[2] Liao, F., Zhu, Y., Zhou, D.: Observer-based fault estimation for a class of discrete-time switched affine systems: An application to the DC-DC converter. Journal of the Franklin Institute **358**(15), 7992–8011 (2021) https://doi.org/10.1016/j.jfranklin.2021.08.001

[3] Wang, S., Bi, Y., Bi, J., Wang, X., Qi, W., Cong, S., Sheng, Z.: Robust control of sampled-data system subject to input time-delay. In: 2016 35th Chinese Control Conference (CCC), pp. 2957–2960 (2016). https://doi.org/10.1109/ChiCC.2016.7553813

[4] Tang, T.-T., Shi, T., Zhang, C.: Reliable model predictive control for linear sampled-data system with input constraints. In: 2018 5th International Conference on Information, Cybernetics, and Computational Social Systems (ICCSS), pp. 399–403 (2018). https://doi.org/10.1109/ICCSS.2018.8572365

[5] Etienne, L., Girard, A., Greco, L.: Stability and stabilizability of discrete-time dual switching systems with application to sampled-data systems. Automatica **100**, 388–395 (2019) https://doi.org/10.1016/j.automatica.2018.11.014

[6] Kunkel, P., Mehrmann, V.: Differential-Algebraic Equations. Analysis and Numerical Solution. EMS Publishing House, Zürich, Switzerland (2006). https://doi.org/10.4171/017

[7] Yang, C., Zhang, Q., Huang, S.: Input-to-state stability of a class of Luré descriptor systems. Int. J. Robust. Nonlinear Control **23**(12), 1324–1337 (2013) https://doi.org/10.1002/rnc.2821

[8] Haohui, L., Yun, T.: The voltage stability of a DAE model for singlemachine infinite bus system. Automation of Electric Power Systems **24**, 11–15 (2000)

[9] Belov, A.A., Andrianova, O.G., Kurdyukov, A.P.: Control of discrete-time descriptor systems. Cham: Springer International Publishing **39** (2018) https://doi.org/10.1007/978-3-319-78479-3

[10] Schmidt, T.: Parameterschätzung bei Mehrkörpersystemen mit Zwangsbedingungen. VDIVerlag, Dusseldorf (1994)

[11] Duan, G.-R.: Analysis and Design of Descriptor Linear Systems. Springer, New York, NY (2010)

[12] Craig, J.J.: Introduction to Robotics: Mechanics and Control, 3rd edn. Pearson Education International, India (2005)

[13] Brenan, K.E., Campbell, S.L., Petzold, L.R.: Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations. North-Holland, Amsterdam (1989)

[14] Newcomb, R.W., Dziurla, B.: Some circuits and systems applications of semistate theory. Circuits Systems Signal Process **8**, 235–260 (1989)

[15] Tuan, L.A., Phat, V.N.: Existence of solutions and finite-time stability for nonlinear singular discrete-time neural networks. Bulletin of the Malaysian Mathematical Sciences Society **42**, 2423–2442 (2019) https://doi.org/10.1007/s40840-018-0608-y

[16] Campbell, S.L.: Nonregular singular dynamic Leontief systems. Econometrica **47**(6), 1565–1568 (1979) https://doi.org/10.2307/1914020

[17] Jódar, L., Merello, P.: Positive solutions of discrete dynamic Leontief input–output model with possibly singular capital matrix. Mathematical and Computer Modelling **52**(7), 1081–1087 (2010) https://doi.org/10.1016/j.mcm.2010.02.043

[18] Haimes, Y.Y., Jiang, P.: Leontief-based model of risk in complex interconnected

infrastructures. Journal of Infrastructure Systems **7**(1), 1–12 (2001) https://doi.org/10.1061/(ASCE)1076-0342(2001)7:1(1)

[19] Kerimkhulle, S., Alimova, Z., Slanbekova, A., Baizakov, N., Azieva, G., Koishybayeva, M.: The use Leontief input-output model to estimate the resource and value added. In: 2022 International Conference on Smart Information Systems and Technologies (SIST), pp. 1–5 (2022). https://doi.org/10.1109/SIST54437.2022.9945746

[20] Sun, Z., Ge, S.S.: Switched Linear Systems. Communications and Control Engineering. Springer, London (2005). https://doi.org/10.1007/1-84628-131-8

[21] Anh, P.K., Linh, P.T., Thuan, D.D., Trenn, S.: The one-step-map for switched singular systems in discrete-time. In: Proc. 58th IEEE Conf. Decision Control (CDC) 2019, pp. 605–610 (2019). https://doi.org/10.1109/CDC40024.2019.9030154

[22] Sutrisno, Trenn, S.: Switched linear singular systems in discrete time: Solution theory and observability notions. Systems & Control Letters **183**(105674), 1–11 (2024) https://doi.org/10.1016/j.sysconle.2023.105674

[23] Brüll, T.: Existence and uniqueness of solutions of linear variable coefficient discrete-time descriptor systems. Linear Algebra and its Applications **431**(1), 247–265 (2009) https://doi.org/10.1016/j.laa.2009.02.019

[24] Penrose, R.: A generalized inverse for matrices. Proc. Camb. Phil. Soc. **51**, 406–413 (1955) https://doi.org/10.1017/S0305004100030401

[25] Golub, G.H., Van Loan, C.F.: Matrix Computations, 4th edn. Johns Hopkins studies in the mathematical sciences. The Johns Hopkins University Press, Baltimore (2013)

[26] Berger, T., Ilchmann, A., Trenn, S.: The quasi-Weierstraß form for regular matrix pencils. Linear Algebra Appl. **436**(10), 4052–4069 (2012) https://doi.org/10.1016/j.laa.2009.12.036

[27] Anh, P.K., Linh, P.T., Thuan, D.D., Trenn, S.: Stability analysis for switched discrete-time linear singular systems. Automatica **119**(109100) (2020) https://doi.org/10.1016/j.automatica.2020.109100

[28] Trenn, S.: Switched differential algebraic equations. In: Vasca, F., Iannelli, L. (eds.) Dynamics and Control of Switched Electronic Systems - Advanced Perspectives for Modeling, Simulation and Control of Power Converters, pp. 189–216. Springer, London (2012). Chap. 6. https://doi.org/10.1007/978-1-4471-2885-4_6

[29] Hossain, M.S., Trenn, S.: Model reduction for switched differential-algebraic equations with known switching signal. DAE Panel **3** (2025) https://doi.org/10.52825/dae-p.v3i.688

[30] Meijer, T.J., Dolk, V.S., Chong, M.S., Heemels, W.P.M.H.: Robust Observer Design for Polytopic Discrete-Time Nonlinear Descriptor Systems (2022). https://arxiv.org/abs/2207.04290

[31] Ascher, U.M., Petzold, L.R.: Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations. SIAM Publications, Philadelphia (1998)