Model reduction of singular switched systems in discrete time

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***Abstract*— Based on our recently established solution characterization of switched singular descriptor systems in discrete time, we propose a time-varying balanced truncation method. For that we consider the switched system on a finite time interval and define corresponding time-varying reachability and observability Gramians. We then show that these capture essential quantitative information about reachable and observable state directions. Based on these Gramians we formulate a time-varying balanced truncation method resulting in a fully-time varying linear system with possible varying state dimensions. We illustrate this method with a small dynamic Leontief model, where we can reduce the size to one third without altering the input-output behavior significantly. We also show that the method is suitable for a medium size random descriptor system (100×100) resulting in a time-varying system of less then a tenth of the size where the outputs of the original and reduced system are indistinguishable.**

I. Introduction

We consider singular switched linear systems in discrete time given by

$$\begin{aligned} E_{\sigma(k)}x(k+1) &= A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \\ y(k) &= C_{\sigma(k)}x(k) + D_{\sigma(k)}u(k), \end{aligned} \tag{1}$$

where $x(k) \in \mathbb{R}^n$ is the state at time instant $k \in \mathbb{N}$ and $\sigma : \mathbb{N} \to \mathtt{M} = \{1, 2, \cdots, \mathtt{m}\}$, $\mathtt{m} \in \mathbb{N}$, is the switching signal defined on a bounded time interval $[k_0, k_f] := \{k_0, k_0 + 1, \ldots, k_f\}$ of interest, $E_i, A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$, $C_i \in \mathbb{R}^{p\times n}$ and $D_i \in \mathbb{R}^{p\times m}$ are the system's matrices with $i \in \mathtt{M}$ and E_i in general singular. In the context of model reduction, we are interested in the input-output behavior of the system and thus, w.l.o.g., we assume in the following that $x(0) = 0$.

Sytems governed by (1) appear in numerous practical applications, such as circuit simulation, computational electromagnetics, fluid dynamics, and mechanical and chemical engineering; see [10], [18].

Our goal is to reduce the size of the system description without significantly altering the input-output response of the overall system, commonly known as Model Order Reduction (MOR). It typically addresses systems described by large-scale sets of ordinary differential (in continuous time) or difference equations (in discrete time) as well as differential or difference-algebraic equations, also known as singular systems. In this context, the primary goal of MOR is to develop a compact reduced-order model with a significantly smaller number of states, while approximating the true system response according to well-defined criteria. Here, reduction is needed to replace the original large-scale system descriptions with accurate and robust reduced-order models. In other words, given a full-order model of a dynamic system, the aim of model reduction is to find a reduced-order model such that the input-output behavior of both models remains closely aligned in an appropriate sense.

In parallel to MOR, switched systems have been widely studied over the last decades for modeling in applications such as mechanical and aeronautical systems, power converters, and the automotive industry (see e.g. [9], [15]). MOR methods for switched ordinary (non-singular) systems have been developed in [12], [5] for the continuous-time case and [2], [1], [13], [3] for the discrete-time case. The existing methods use the concept of balanced realization (see also [14], [17], [4], [8]). A balanced truncation method is also recently proposed in [6] for model reduction of singular linear switched systems in continuous time. The resulting reduced model is no longer a switched system but is fully time-varying, which is less practical for the continuous-time case. However, in discrete time, such a fully time-varying reduced model may still be feasible for practical applications, which motivated us to apply this time-varying balanced truncation approach to system (1).

In this paper, we propose a method inspired by the well-known balanced truncation method, wherein suitable reachability and observability Gramians are first introduced to quantify how easily or difficultly certain state directions can be reached or observed. Next, coordinate transformations are applied to balance these Gramians, revealing which state directions are simultaneously hard to reach and observe. Finally, the reduction removes the state directions that are most difficult to reach and observe. Here, for system (1), we focus our attention to the case that the reduction process is tailored to the specific switching signal, which allows for a more efficient reduction but requires complete knowledge of the switching signal. Furthermore, the reduction is considered on a finite time interval of interest, which on one hand is in most practical situations more relevant than the

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This work was partially carried out while all four authors were guest researchers at the VIASM, Hanoi, Vietnam.

consideration of an infinite time interval and on the other hand does not require any stability assumptions on the system.

II. Preliminaries

Consider a general time-varying linear system in discrete time given by

$$\begin{aligned} x(k+1) &= A_k x(k) + B_k u(k), \quad x(k_0) = 0, \\ y(k) &= C_k x(k) + D_k u(k), \end{aligned} \tag{2}$$

where $k \in [k_0, k_f]$, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^p$ and A_k, B_k, C_k, D_k are matrices of appropriate size.

Definition 1 (Gramians): The (time-varying) *reachability Gramian* $P_k \in \mathbb{R}^{n\times n}$ for (2) is recursively given by $P_{k_0} := 0$ and for $k = k_0+1, k_0+2, \dots, k_f$

$$P_k := A_{k-1} P_{k-1} A_{k-1}^\top + B_{k-1} B_{k-1}^\top.$$

The (time-varying) *observability* Gramian $Q_k \in \mathbb{R}^{n\times n}$ is recursively given by $Q_{k_f} := C_{k_f}^\top C_{k_f}$ and for $k = k_f - 1, k_f - 2, \dots, 1$

$$Q_k := A_k^\top Q_{k+1} A_k + C_k^\top C_k.$$

Lemma 2: Consider the (time-varying) coordinate transformation $z_k = T_k x_k$, then the corresponding transformed system is given by

$$\begin{aligned} z(k+1) &= \bar{A}_k z(k) + \bar{B}_k u(k), \quad z(k_0) = 0, \\ y(k) &= \bar{C}_k z(k) + \bar{D}_k u(k), \end{aligned}$$

where $\bar{A}_k := T_{k+1}^{-1} A_k T_k$, $\bar{B}_k := T_{k+1}^{-1} B_k$, $\bar{C}_k = C_k T_k$, and $\bar{D}_k = D_k$. Furthermore, the corresponding Gramians $\bar{P}_k$ and $\bar{Q}_k$ satisfy

$$\bar{P}_k = T_k^{-1} P_k T_k^{-T} \quad \text{and} \quad \bar{Q}_k = T_k^\top Q_k T_k.$$

Proof: This follows in a straight-forward way by plugging in the corresponding definitions. ∎

With the above definition of the Gramians it can be shown that the Gramians are tightly connected with the reachability and observability properties of (2).

Theorem 3: Consider (2) with corresponding Gramians P_k and Q_k, then

$$\operatorname{im} P_k = \left\{ x_k \;\middle|\; \begin{array}{l} \exists u : [k_0, k-1] \to \mathbb{R}^m \text{ and solution } x \text{ of (2) s.t. } x(k) = x_k \end{array} \right\},$$

$$\ker Q_k = \left\{ x_k \;\middle|\; \begin{array}{l} 0 = y(k) = y(k+1) = \ldots = y(k_f) \\ \text{for solution of (2) on } [k, k_f] \text{ with} \\ u = 0 \text{ and } x(k) = x_k \end{array} \right\},$$

i.e. $\operatorname{im} P_k$ equals the reachability space $\mathcal{R}_k$ of (2) considered on the time interval $[k_0, k]$ and $\ker Q_k$ equals the unobservability space $\mathcal{U}_k$ of (2) considered on the time interval $[k, k_f]$.

Proof: It is easily seen that the reachability space satisfies $\mathcal{R}_{k+1} = A_k \mathcal{R}_k + \operatorname{im} B_k$. Since $\mathcal{R}_0 = \{0\} = \operatorname{im} P_0$, we just need to show that $\operatorname{im} P_{k+1} = \mathcal{R}_{k+1}$ under the inductive assumption that $P_k = \mathcal{R}_k$. For that consider any symmetric factorization $P_k = L_k L_k^\top$, then, utilizing the fact that for any matrix we have $\operatorname{im} M = \operatorname{im} M M^\top$,

$$\begin{aligned} \mathcal{R}_{k+1} &= A_k \operatorname{im} P_k + \operatorname{im} B_k = A_k \operatorname{im} L_k + \operatorname{im} B_k B_k^\top \\ &= \operatorname{im} A_k L_k (A_k L_k)^\top + \operatorname{im} B_k B_k^\top = \operatorname{im} P_{k+1}. \end{aligned}$$

We see that the unobservable space satisfies $\mathcal{U}_{k_f} = \ker C_{k_f}$ and $\mathcal{U}_k = \ker C_k \cap A_k^{-1} \mathcal{U}_{k+1}$. Since for general matrices M we have that $\ker M = (\operatorname{im} M^\top)^\perp$, we see that $\ker Q_{k_f} = (\operatorname{im} C^\top)^\perp = (\operatorname{im} Q_{k_f})^\perp = \ker Q_{k_f}$ and using the induction assumption $\ker Q_{k+1} = \mathcal{U}_{k+1}$ together with the symmetric factorization $Q_{k+1} = R_{k+1}^\top R_{k+1}$, we arrive at the following:

$$\begin{aligned} \ker Q_k &= (\operatorname{im} A_k^\top R_{k+1}^\top R_{k+1} A_k + \operatorname{im} C_k^\top C_k)^\perp \\ &= \ker R_{k+1} A_k \cap \ker C_k = A_k^{-1} \ker R_{k+1} \cap \ker C_k \\ &= A_k^{-1} \ker \mathcal{U}_{k+1} \cap \ker C_k = \mathcal{U}_k. \end{aligned}$$

∎

Furthermore, the Gramians do not only provide qualitative information about reachability and observability, but also quantitative information (which is crucial for the upcoming model reduction method) as highlighted in the following Theorem.

Theorem 4: For all $x_k \in \operatorname{im} P_k$ we have that[1]

$$\begin{aligned} &x_k^\top P_k^\dagger x_k \\ &= \min \left\{ \sum_{\ell=k_0}^{k-1} u(\ell)^\top u(\ell) \;\middle|\; \begin{array}{l} u \text{ is s.t. solution } x \\ \text{of (2) satisfies } x(k) = x_k \end{array} \right\}, \end{aligned}$$

i.e. the reachability Gramian provides information about the minimal control energy required to reach a (reachable) state. Furthermore

$$x_k^\top Q_k x_k = \sum_{\ell=k}^{k_f} y(\ell)^\top y(\ell),$$

where $y(k), y(k+1), \dots, y(k_f)$ is the output of (2) considered on $[k, k_f]$ with $u = 0$ and $x(k) = x_k$, i.e. the observability Gramian provides information about the output energy visible from the given initial state.

Proof: For $k > \ell \geq k_0$, define the fundamental matrices $\Phi(k, \ell)$ of (2) as

$$\Phi(k, \ell) := A_{k-1} A_{k-2} \cdots A_{\ell+1} A_\ell, \quad \Phi(\ell, \ell) := I.$$

Then, since $x(k_0) = 0$, we have

$$x(k) = \sum_{\ell=k_0}^{k-1} \Phi(k, \ell+1) B_\ell u(\ell). \tag{3}$$

Furthermore, we see easily by induction that

$$P_k = \sum_{\ell=k_0}^{k-1} \Phi(k, \ell+1) B_\ell B_\ell^\top \Phi(k, \ell+1)^\top,$$

$$Q_k = \sum_{\ell=k}^{k_f} \Phi(\ell, k)^\top C_\ell^\top C_\ell \Phi(\ell, k).$$

Consider the control $\hat{u}$ on $[k_0, k]$ defined by

$$\hat{u}(\ell) = B_\ell^\top \Phi(k, \ell+1)^\top P_k^\dagger x_k.$$

[1]Here $P^\dagger$ denotes any symmetric pseudo-inverse of P_k with $P_k P_k^\dagger P_k = P_k$, for example the well known Moore-Penrose pseudo-inverse.

Then with this control, system (2) has the solution x satisfying $x(k_0) = 0, x(k) = x_k$. Indeed, plugging this input into (3) we get

$$\begin{aligned} x(k) &= \sum_{\ell=k_0}^{k-1} \Phi(k,\ell+1) B_\ell B_\ell^\top \Phi(k,\ell+1)^\top P_k^\dagger x_k \\ &= P_k P_k^\dagger x_k. \end{aligned}$$

By assumption, $x_k \in \operatorname{im} P_k$, i.e. there exists $z_k \in \mathbb{R}^n$ such that $x_k = P_k z_k$ and hence $P_k P_k^\dagger x_k = P_k P_k^\dagger P_k z_k = P_k z_k = x_k$.

Now we define a scalar product for controls u, v on $[k_0, k]$:

$$\langle u, v \rangle = \sum_{\ell=k_0}^{k} u(\ell)^\top v(\ell).$$

Then, for arbitrary u for which the solution of system (2) satisfies $x(k_0) = 0, x(k) = x_k$, we have

$$\begin{aligned} \langle u, \hat{u} \rangle &= \sum_{\ell=k_0}^{k} u(\ell)^\top \hat{u}(\ell) \\ &= \sum_{\ell=k_0}^{k} u(\ell)^\top B_\ell^\top \Phi(k,\ell+1)^\top P_k^\dagger x_k \\ &= \left(\sum_{\ell=k_0}^{k} \Phi(k,\ell+1) B_\ell u(\ell) \right)^\top P_k^\dagger x_k \\ &\overset{(3)}{=} x_k^\top P_k^\dagger x_k. \end{aligned} \tag{4}$$

This implies that $\langle u - \hat{u}, \hat{u} \rangle = \langle u, \hat{u} \rangle - \langle \hat{u}, \hat{u} \rangle = x_k^\top P_k^\dagger x_k - x_k^\top P_k^\dagger x_k = 0$ and

$$\begin{aligned} \langle u, u \rangle &= \langle \hat{u}, \hat{u} \rangle + 2\langle u - \hat{u}, \hat{u} \rangle + \langle u - \hat{u}, u - \hat{u} \rangle \\ &= \langle \hat{u}, \hat{u} \rangle + \langle u - \hat{u}, u - \hat{u} \rangle \\ &\geq \langle \hat{u}, \hat{u} \rangle \\ &= x_k^\top P_k^\dagger x_k. \end{aligned}$$

This concludes the first part of this theorem.

Now, for $x(k) = x_k$ and $u = 0$, we have $x(\ell) = \Phi(\ell, k) x_k$. This implies that $y(\ell) = C_\ell \Phi(\ell, k) x_k$. Therefore,

$$\begin{aligned} \sum_{\ell=k}^{k_f} y(\ell)^\top y(\ell) &= \sum_{\ell=k}^{k_f} x_k^\top \Phi(\ell,k)^\top C_\ell^\top C_\ell \Phi(\ell,k) x_k \\ &= x_k^\top Q_k x_k. \end{aligned}$$

The proof is complete. ∎

Definition 5: The system (2) is called *balanced* if all Gramians are diagonal and for all $k \in [k_0, k_f]$ there exists diagonal matrices Σ_k, Σ_k^r, Σ_k^o such that

$$P_k = \operatorname{diag}(\Sigma_k, \Sigma_k^r, 0, 0) \text{ and } Q_k = \operatorname{diag}(\Sigma_k, 0, \Sigma_k^o, 0),$$

where the corresponding block diagonal matrices in P_k and Q_k have equal sizes (0×0 is allowed). Furthermore, it is assumed that all diagonal entries of $\Sigma_k, \Sigma_k^r, \Sigma_k^o$ are positive and for Σ_k are ordered from largest to smallest.

Theorem 6: There always exists a time-dependent coordinate transformation such that (2) becomes balanced.

Proof: In view of Lemma 2 it suffices to consider the coordinate transformation T_k for each k individually (although the resulting transformed system matrices will depend on the coordinate transformation at T_k and T_{k+1}, the transformed Gramians will only depend on T_k). Hence the problem can be reduced to the following question: Given two symmetric positive definite matrices P and Q find a coordinate transformation T such that $T^{-1} P T^{-T}$ and $T^\top Q T$ have the block structured as in Definition 5. The existence of such a transformation is well known, see e.g. [19, Thm. 7.5], whose proof is also constructive and can be used for the implementation of the reduction algorithm. ∎

III. Model reduction procedure

The model reduction procedure consists of four main steps:

Step 1: Obtain surrogate system

We assume that the switched singular system (1) is solvable for all input signals, which can be characterized via the recently introduced notion of *switched index-1 w.r.t the considered switching signal* σ for the family $\{(E_i, A_i, B_i)\}$, for details see [16]; therein, an equivalent surrogate system is given, which takes the form

$$\begin{aligned} x(k+1) &= \Phi_{\sigma(k+1),\sigma(k)} x(k) \\ &\quad + \Psi^c_{\sigma(k+1),\sigma(k)} u(k) + \Psi^a_{\sigma(k+1),\sigma(k)} u(k+1) \\ y(k) &= C_{\sigma(k)} x(k) + D_{\sigma(k)} u(k). \end{aligned}$$

This can be rewritten as

$$\begin{aligned} x(k+1) &= \widetilde{A}_k x(k) + \widetilde{B}_k \widetilde{u}(k), \\ y(k) &= \widetilde{C}_k x(k) + \widetilde{D}_k \widetilde{u}(k), \end{aligned} \tag{5}$$

where $\widetilde{u}(k) := \left[\begin{smallmatrix} u(k) \\ u(k+1) \end{smallmatrix} \right]$ and

$$\begin{aligned} \widetilde{A}_k &:= \Phi_{\sigma(k+1),\sigma(k)}, & \widetilde{B}_k &:= [\Psi^c_{\sigma(k+1),\sigma(k)}, \Psi^a_{\sigma(k+1),\sigma(k)}], \\ \widetilde{C}_k &:= C_{\sigma(k)}, & \widetilde{D}_k &:= [D_{\sigma(k)}, 0], \end{aligned}$$

Step 2: Calculate Gramians for surrogate system

With $P_{k_0} = 0$, define P_k recursively for $k = k_0+1, k_0+2, \ldots, k_f$ as follows

$$P_k := \widetilde{A}_{k-1} P_{k-1} \widetilde{A}_{k-1}^\top + \widetilde{B}_{k-1} \widetilde{B}_{k-1}^\top,$$

and with $Q_{k_f+1} := 0$, defined Q_k recursively for $k = k_f, k_f - 1, \ldots, k_0$

$$Q_k := \widetilde{A}_k^\top Q_{k+1} \widetilde{A}_k + \widetilde{C}_k^\top \widetilde{C}_k.$$

Remark 7: It should be noted that the calculated reachability Gramians for the surrogate systems are not necessarily exactly related to the reachability properties of the original singular switched system. This is because for the surrogate system it is assumed that the inputs $\widetilde{u}_k$ and $\widetilde{u}_{k+1}$ are independent of each other, however, from the definition it is clear that they are in fact *not*

fully independent because the last components of $\widetilde{u}_k$ are always equal to the first components of $\widetilde{u}_{k+1}$. This means in particular, that the actual reachability spaces of the original singular switched system are in general smaller than the reachability spaces of the surrogate system. However, in the context of model reduction this means that we overestimate the reachability properties of the original systems and may not reduce as much as possible.

Step 3: Calculate time-varying balancing transformation

For $k \in [k_0, k_f]$ find a balancing coordinate transformation T_k, i.e. it holds for all $k \in [k_0, k_f]$ that

$$\begin{aligned} T_k^{-1} P_k T_k^{-\top} &= \operatorname{diag}(\Sigma_k, \Sigma_k^r, 0, 0), \\ T_k^{\top} Q_k T_k &= \operatorname{diag}(\Sigma_k, 0, \Sigma_k^o, 0). \end{aligned} \tag{6}$$

These balancing transformations can be calculated using the constructive proof of [19, Thm. 7.5].

Step 4: Decide on reduction size and obtain reduced model

Denoting the diagonal entries of Σ_k as $s_1, s_2, \ldots, s_{n_k}$ and given a desired threshold $\varepsilon_k > 0$ for $k = k_0, k_0 + 1, \ldots, k_f$ choose the minimal value $r_k \in \mathbb{N}$ such that $s_{r_k+1} < \varepsilon_k$ (where $s_{n_k+1} := 0$). Alternatively, a desired reduction size r_k can be chosen directly. Now define the left- and right-projection matrices Π_k^l and Π_k^r as the first r_k rows of T_k^{-1} and the first r_k columns of T_k, respectively. The reduced switched system is then given by

$$\begin{aligned} \hat{x}(k+1) &= \hat{A}_k x(k) + \hat{B}_k \begin{bmatrix} u(k) \\ u(k+1) \end{bmatrix}, \\ \hat{y} &= \hat{C}_k x(k) + D_k u(k), \end{aligned}$$

where

$$\hat{A}_k := \Pi_{k+1}^l \widetilde{A}_k \Pi_k^r, \ \hat{B}_k := \Pi_{k+1}^l \tilde{B}_k, \ \hat{C}_k := \widetilde{C}_k \Pi_k^r.$$

Remark 8 (Error bounds): It is well known that the classical (non-time-varying) balanced truncation method can provide a-priori error bounds. In the continuous time case some error bounds can also be obtained for time-varying balanced truncation, see [11], however, we are not aware of any error bounds available for the discrete-time case with possibly singular Gramians. Nevertheless, in view of the clear energy-interpretation of the Gramians as established in Theorem 4 it is our strong believe that error bounds in terms of the neglected singular values s_k, $k > r_k$, can be established, however this is ongoing research.

IV. Simulations

A. Dynamic Leontief model example

Consider the switched dynamic Leontief model

$$x(k) = \mathfrak{L}_{\sigma(k)} x(k) + \mathfrak{C}_{\sigma(k)} \big(x(k+1) - x(k)\big) + d(k), \tag{7}$$

where $x(k)$ is the vector of output levels at the time $k = k_0, k_1, \ldots, k_f$, $d(k)$ is the vector of final demands (the input in this model), $\sigma : \mathbb{N} \to \mathtt{M}$ is the switching signal, $\mathfrak{L}_i$ is the Leontief input–output matrix, and $\mathfrak{C}_i$ is the capital coefficient matrix, $i \in \mathtt{M}$. After rearranging terms, (7) takes the form (1) with $E_i = \mathfrak{C}_i$, $A_i = I - \mathfrak{L}_i + \mathfrak{C}_i$ and $B_i = I$. For illustration we consider (7) with data from [7] for mode 1 and for the second mode, we assume a complete one-sided decoupling of the third sector from the other two sectors (e.g. by suddenly stopping supplying goods to the other sectors in response to politically imposed export restrictions):

$$\begin{aligned} \mathfrak{L}_1 &= \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.40 & 0.10 & 0.50 \\ 0.30 & 0.50 & 0.20 \end{bmatrix}, \quad \mathfrak{L}_2 = \begin{bmatrix} 0.30 & 0.30 & 0.30 \\ 0.40 & 0.10 & 0.50 \\ 0 & 0 & 0 \end{bmatrix}, \\ \mathfrak{C}_1 &= \begin{bmatrix} 0.30 & 0.40 & 0.45 \\ 0 & 0 & 0 \\ 0.60 & 0.80 & 0.90 \end{bmatrix}, \quad \mathfrak{C}_2 = \mathfrak{C}_1. \end{aligned}$$

As an output we choose the total production, i.e. $y(k) = C_{\sigma(k)} x(k)$ with $C_1 = C_2 = [1, 1, 1]$.

The family of matrix triplets $\{(E_i, A_i, B_i)\}_{i \in \{0,1\}}$ is jointly index-1 [16] with the corresponding surrogate system (5) given by:

$\Phi_{1,1} \approx$ $\begin{bmatrix} 0.26 & 0.11 & 0.57 \\ 0.28 & 0.12 & 0.62 \\ 0.29 & 0.13 & 0.65 \end{bmatrix}$	$\Phi_{1,2} \approx$ $\begin{bmatrix} 0.35 & 0.27 & 0.64 \\ 0.38 & 0.29 & 0.68 \\ 0.40 & 0.31 & 0.72 \end{bmatrix}$	$\Phi_{2,2} \approx$ $\begin{bmatrix} 0.50 & 0.39 & 0.91 \\ 0.39 & 0.30 & 0.70 \\ 0.30 & 0.23 & 0.53 \end{bmatrix}$
$\Psi^c_{1,1} \approx$ $\begin{bmatrix} 0.16 & 0 & 0.32 \\ 0.17 & 0 & 0.35 \\ 0.18 & 0 & 0.37 \end{bmatrix}$	$\Psi^c_{1,2} \approx$ $\begin{bmatrix} 0.16 & 0 & 0.32 \\ 0.17 & 0 & 0.35 \\ 0.18 & 0 & 0.37 \end{bmatrix}$	$\Psi^c_{2,2} \approx$ $\begin{bmatrix} 0.23 & 0 & 0.46 \\ 0.18 & 0 & 0.35 \\ 0.13 & 0 & 0.27 \end{bmatrix}$
$\Psi^a_{1,1} \approx$ $\begin{bmatrix} -0.74 & 0.32 & 0.37 \\ -0.04 & -0.73 & 0.02 \\ 0.53 & 0.44 & -0.26 \end{bmatrix}$	$\Psi^a_{1,2} \approx$ $\begin{bmatrix} -0.74 & 0.32 & 0.37 \\ -0.04 & -0.73 & 0.02 \\ 0.53 & 0.44 & -0.26 \end{bmatrix}$	$\Psi^a_{2,2} \approx$ $\begin{bmatrix} -0.80 & 0.24 & 0.40 \\ -0.04 & -0.73 & 0.02 \\ 0.57 & 0.49 & -0.28 \end{bmatrix}$

With the switching signal $\sigma(k) = 1$ for $k \in [0, 4]$ and $\sigma(k) = 2$ for $k \in [5, 9]$, we arrive at the time-varying linear system (2) where $A_k = \Phi_{\sigma(k+1),\sigma(k)}$, $B_k = [\Psi^c_{\sigma(k+1),\sigma(k)}, \Psi^a_{\sigma(k+1),\sigma(k)}]$, $C_k = C_{\sigma}(k)$. We apply our proposed reduction method to reduce this system to a scalar time-varying system $r = 1$. Then the reduced system is given by

k	1	2	3	4	5	6	7	8	9
$\hat{A}_k$	0	0.85	-0.91	-0.94	-0.97	-0.97	0.95	-0.89	
$\hat{C}_k$	0	0.31	0.51	-0.62	0.78	-1.10	1.54	2.18	-3.28

and

$$\begin{aligned} \hat{B}_1 &= \begin{bmatrix} 1.02 & 0 & 2.05 & 0.63 & 1.51 & -0.33 \end{bmatrix}, \\ \hat{B}_2 &= \begin{bmatrix} 0.83 & 0 & 1.67 & 0.51 & 1.22 & -0.25 \end{bmatrix}, \\ \hat{B}_3 &= \begin{bmatrix} -0.73 & 0 & -1.46 & -0.44 & -1.06 & 0.22 \end{bmatrix}, \\ \hat{B}_4 &= \begin{bmatrix} 0.67 & 0 & 1.34 & 0.18 & 0.58 & -0.09 \end{bmatrix}, \\ \hat{B}_5 &= \begin{bmatrix} -0.48 & 0 & -0.96 & -0.12 & -0.40 & 0.06 \end{bmatrix}, \\ \hat{B}_6 &= \begin{bmatrix} 0.35 & 0 & 0.70 & 0.06 & 0.23 & -0.03 \end{bmatrix}, \\ \hat{B}_7 &= \begin{bmatrix} 0.25 & 0 & 0.49 & 0.01 & 0.14 & -0.00 \end{bmatrix}, \\ \hat{B}_8 &= \begin{bmatrix} -0.16 & 0 & -0.33 & 0.08 & 0 & -0.04 \end{bmatrix}. \end{aligned}$$

The corresponding output comparison for some randomly chosen input is shown in Fig. 1.

B. Large scale academic example

We also illustrate the performance of our proposed reduction method for a large scale random descriptor system (1), where $n = 100$, $\operatorname{rank} E_i = 50$, $i \in \mathtt{M} = \{1, 2\}$, $m = p = 1$. We consider the time interval $[k_0, k_f] =$

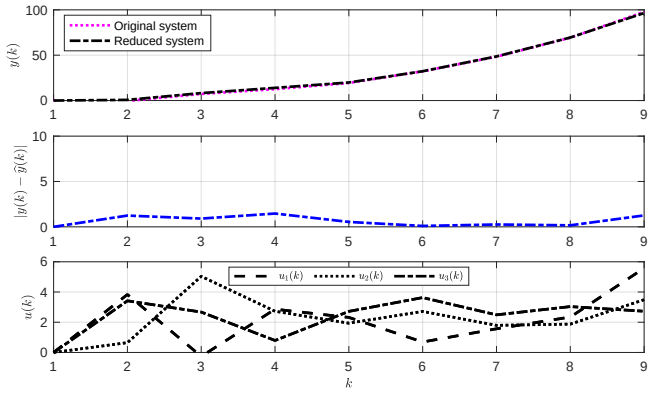

Fig. 1. Comparison of output of a Leontief model with the output of the reduced model (top figure) and the output error (middle figure) for a randomly chosen input (bottom figure)

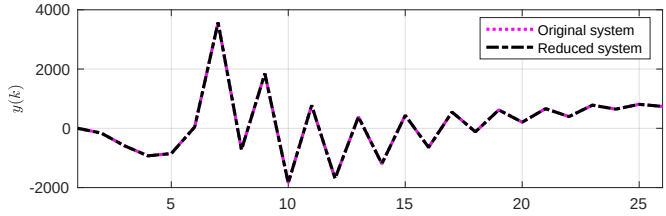

Fig. 2. Comparison of the output of a random descriptor system and the output of the reduced system.

$[1, 26]$ and the switching signal σ with switching sequence $(1, 2, 1, 2, 1)$ and switching times $(6, 11, 16, 21)$. The threshold for the singular values of the balanced Gramians is chosen to be $\varepsilon_k = 0.1$ for all $k \in [k_0, k_f]$ and the corresponding time-varying balanced truncation methods results in a reduced model with sizes

$$(0, 2, 4, 6, 6, 6, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 6, 6, 5, 4, 3, 2, 1).$$

For some randomly chosen input the corresponding original output and the output of the reduced system is shown in Figure 2.

V. Conclusion and outlook

We have presented a comprehensive model reduction method for switched singular systems in discrete time. We consider the (realistic) situation of approximating the input-output behavior on a *finite* time interval and we make the (somewhat restrictive) assumption that the switching signal is completely known on that interval. This allows to treat the switched system as an instance of a time-varying linear system and to introduce time-varying reachability and observability Gramians (the former defined forward in time, the latter backwards in time). These Gramians are defined in terms of the recently introduced surrogate system of the singular switched system; they capture the essential quantitative reachability and observability properties which then form the basis of our time-varying balanced truncation method.

There are still a couple of remaining issues. The first one is the dependence of our method on the knowledge of the switching signal, which in some situations (e.g. when switching is due to scheduled maintenance) is not a limitation, but in other situations (e.g. when the switching is due to faults or is itself considered an input signal) is significantly restricting the applicability of our method. However, the reachability and observability properties strongly depend on the switching signal, hence any method valid for arbitrary switching signals will not result in the best possible reduction for individual switching signals. It is the topic of future research to investigate this trade-off between an effective reduction tailored to specific switching signals and a less effective switching-signal-independent reduction method. Another open issue is the establishment of an error bound, which is available for the classical balanced truncation (time-invariant and consideration of an infinite time interval), but it is not fully clear whether such an error bound can be derived for our method as well. Finally, numerical experiments reveal that for large scale examples (1000 or more states) numerical issues may arise (e.g. the calculate Gramians are not positive definite anymore due to numerical inaccuracies); additionally for such large scale problems a naive implementation of our method may not be feasible anymore and would require some taylored code optimization.

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