Funnel control

Origin and recent advances

Stephan Trenn

Jan C. Willems Center for Systems and Control
University of Groningen, Netherlands

Control Lab Guest Lecture, University of Naples Federico II, 20 November 2024

Who am I?

Stephan Trenn https://stephantrenn.net

Associate Professor for Systems & Control, Bernoulli Institute
Programme Director for master degrees Math., Applied Math., Systems & Control

- studied Mathematics and Computer Science in Ilmenau, Germany
- six month Erasmus student in Southampton, UK
- PhD 2009 in mathematical control theory in Ilmenau
- Postdoc (9 month) at University of Illinois, Urbana-Champaign, USA
- Postdoc (17 month) at University of Würzburg, Germany
- Assistant Professor for Math. Control Theory (2011 - 2017), Kaiserslautern, Germany

Research:

- Switched systems
- Differential-algebraic equations (DAEs)
- Funnel control

Systems & Control in Groningen

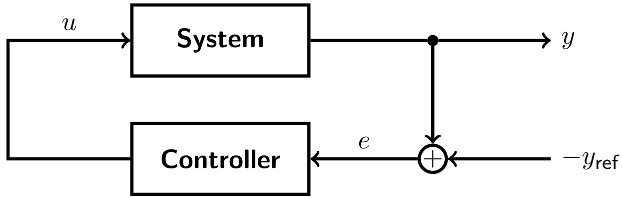
History

- Internationally leading in mathematical systems theory since 1970
- Rapidly growing engineering institute (ENTEG) in the last decades
- Jan C. Willems Center bundles current research activities (15 staff members + about 50 PhD students)

Global Ranking of Academic Subjects (GRAS) 2022

- RUG is ranked 15th worldwide in subject Automation and Control
- Best ranked in the EU!
- Within RUG, Automation and Control is best ranked subject

Control Task

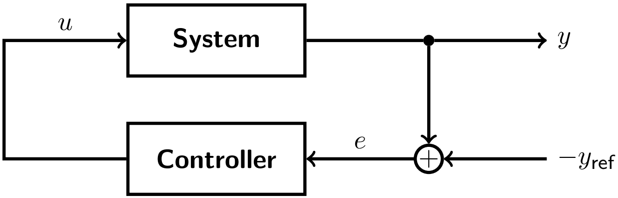


Goal: Output tracking $y(t) \approx y_{\mathrm{ref}}(t)$

Applications

- Flying to the moon
- Robotics
- (Adaptive) cruise control in cars
- Chemical processes

Control Task

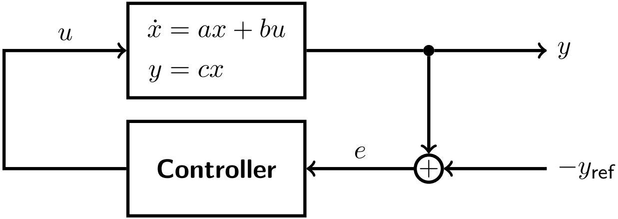


Goal: Output tracking $y(t) \approx y_{\text{ref}}(t)$

Challenge

- no exact knowledge of system model
- no future knowledge or model for reference signal

The scalar linear case with $y_{\text{ref}} = 0$

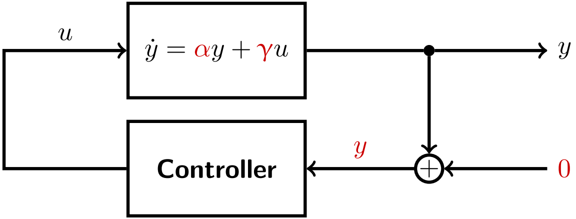


Assumptions

- Known model structure
- Known sign of *high frequency gain* $\gamma := cb$, assume $\gamma > 0$
- $y_{\text{ref}} = 0$

Unknown system parameters α and γ

The scalar linear case with $y_{\text{ref}} = 0$



Goal

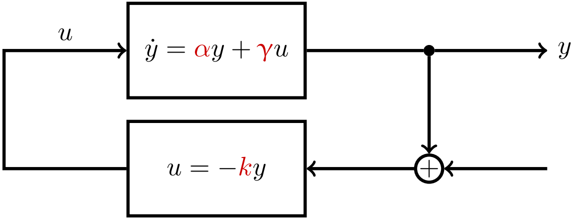
Design output feedback u such that $y(t) \to 0$ as $t \to \infty$

If we would know α, γ, how would we choose u?

Goal: $\dot{y} \overset{!}{=} -\lambda y \quad \rightsquigarrow \quad$ achievable with $u = -ky$ and $k := \frac{\alpha+\lambda}{\gamma}$

In general, with $u = -ky$ we have $\dot{y} = (\alpha - \gamma k)y$

The scalar linear case with $y_{\text{ref}} = 0$



Hence we have arrived at our first high gain control result:

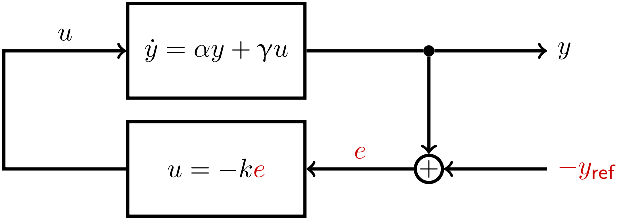
Theorem

The proportional negative feedback

$$u = -ky$$

achieves convergence for all $k > \frac{\alpha}{\gamma}$.

What happens for $y_{\text{ref}} \neq 0$?



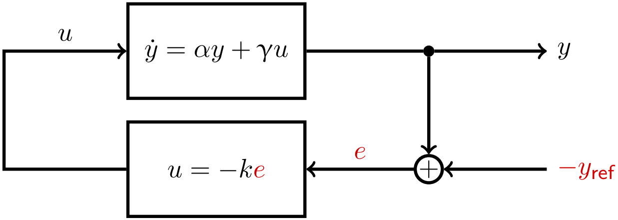
Error dynamics: $\dot e = \ \dots = (\alpha - \gamma k)e + \alpha y_{\text{ref}} - \dot y_{\text{ref}}$

Equilibrium for constant y_{ref}:

$$0 = (\alpha - \gamma k)e + \alpha y_{\text{ref}} \quad \Longleftrightarrow \quad e = \frac{\alpha}{\gamma k - \alpha} y_{\text{ref}}$$

$\leadsto$ no convergence to zero anymore

What happens for $y_{\mathrm{ref}} \neq 0$?



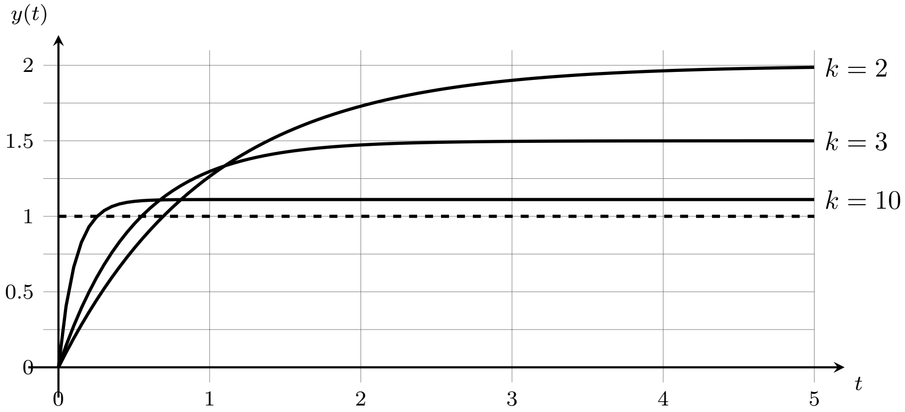
In general: Practical tracking with high gain control:

Theorem

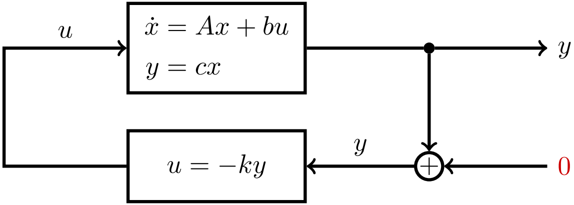
If y_{ref} and $\dot{y}_{\mathrm{ref}}$ are bounded, then

$$\forall \varepsilon > 0\ \exists K > 0\ \forall k > K: \quad \limsup_{t\to\infty} |e(t)| < \varepsilon$$

Example $\alpha = 1$, $\gamma = 1$, $y_{\mathsf{ref}} \equiv 1$



Higher order linear case with $y_{\text{ref}} = 0$



$A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}^{1\times n}$ unknown

Definition (Relative degree)

$r \in \{1, 2, \ldots, n\}$ is relative degree of system (A, b, c) $:\Longleftrightarrow$

(i) $\forall i \in \{0, \ldots, r-2\}: \quad cA^i b = 0$

(ii) $cA^{r-1} b \neq 0$

In particular, (A, b, c) has relative degree one $:\Longleftrightarrow \quad \gamma := cb \neq 0$

What is the meaning of the relative degree?

Frequency domain interpretation

Transfer function $c(sI - A)^{-1}b =: \frac{p(s)}{q(s)}$, then $r = \deg(q(s)) - \deg(p(s))$

Interpretation in time-domain:

Theorem (Byrnes-Isidori form)

(A, b, c) has relative degree $r \in \{1, \ldots, n\}$ if and only if there exists a coordinate transformation T such that $\binom{\eta}{z} = Tx$ *such that* $y = \eta_1$, $\dot{y} = \eta_2$, $\ldots$, $y^{(r-1)} = \eta_r$

$$TAT^{-1} = \begin{bmatrix} \begin{smallmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{smallmatrix} & 0 \\ a_{11} & a_{12} \\ A_{21} & A_{22} \end{bmatrix}, Tb = \begin{bmatrix} 0 \\ \gamma \\ 0 \end{bmatrix}, CT^{-1} = [1, 0, \ldots, 0],$$

with $a_{11} \in \mathbb{R}^{1\times r}$, $a_{12} \in \mathbb{R}^{1\times(n-r)}$, $A_{21} \in \mathbb{R}^{(n-r)\times r}$, $A_{22} \in \mathbb{R}^{(n-r)\times(n-r)}$

What is the meaning of the relative degree?

Frequency domain interpretation

Transfer function $c(sI-A)^{-1}b =: \frac{p(s)}{q(s)}$, then $r = \deg(q(s)) - \deg(p(s))$

Interpretation in time-domain:

Theorem (Byrnes-Isidori form)

(A,b,c) *has relative degree* $r \in \{1,\ldots,n\}$ *if and only if there exists a coordinate transformation* T *such that* $\binom{\eta}{z} = Tx$ *such that* $y = \eta_1$, $\dot{y} = \eta_2$, …, $y^{(r-1)} = \eta_r$

$$y^{(r)} = a_{12}\begin{pmatrix} y \\ \vdots \\ y^{(r-1)} \end{pmatrix} + a_{22}z + \gamma u$$

$$\dot{z} = A_{21}\begin{pmatrix} y \\ \vdots \\ y^{(r-1)} \end{pmatrix} + A_{22}z$$

Zero dynamics

$$\dot{x} = Ax + bu$$
$$y = cx$$

$$y^{(r)} = a_{12}\begin{pmatrix} y \\ \vdots \\ y^{(r-1)} \end{pmatrix} + a_{22}z + \gamma u$$
$$\dot{z} = A_{21}\begin{pmatrix} y \\ \vdots \\ y^{(r-1)} \end{pmatrix} + A_{22}z$$

Question

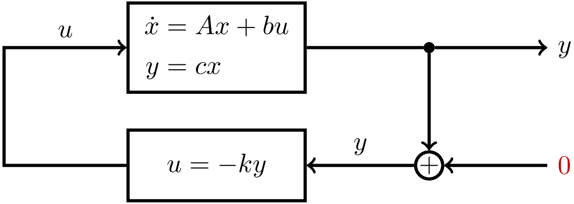
Which input u is needed to keep output y identically zero?

Byrnes-Isidori form for identically zero output:

$$0 = a_{22}z + \gamma u$$
$$\dot{z} = A_{22}z \quad \longleftarrow \text{zero dynamics}$$

Answer: $u(t) = -\frac{1}{\gamma}a_{22}e^{A_{22}t}z(0) \to \infty$ if A_{22} has "bad" eigenvalues!

High gain stabilization for r.d.-one systems



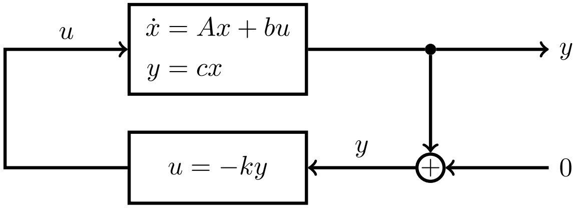
Assumptions:

- Relative degree $r = 1$ $\Leftrightarrow$ $\gamma := cb \neq 0$, in particular:

$$\text{System} \quad \Leftrightarrow \quad \begin{aligned} \dot{y} &= a_{11}y + a_{12}z + \gamma u \\ \dot{z} &= a_{21}y + A_{22}z \end{aligned}$$

- positive high frequency gain $\Leftrightarrow$ $\gamma > 0$
- stable zero-dynamics (minimum phase) $\Leftrightarrow$ A_{22} Hurwitz

High gain stabilization for r.d.-one systems



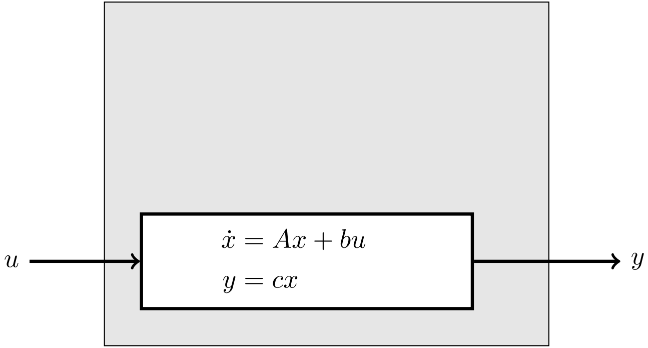
Theorem (High-gain stabilization)

$cb > 0$ and stable zero-dynamics

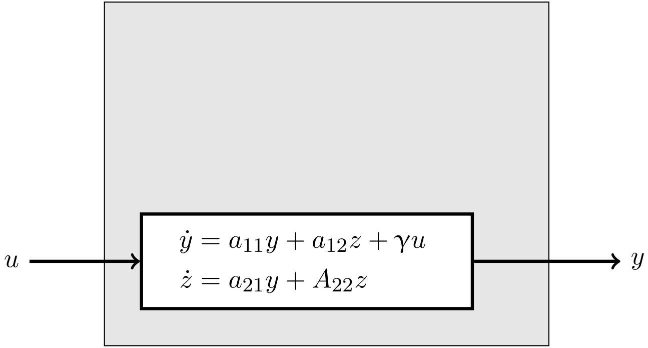
$$\Rightarrow \quad \exists K > 0 \; \forall\, k \geq K : \; \textit{Closed loop is asymptotically stable}$$

Key idea of proof: Show that $\begin{bmatrix} a_{11} - \gamma k & a_{12} \\ a_{21} & A_{22} \end{bmatrix}$ is Hurwitz for sufficiently large k.

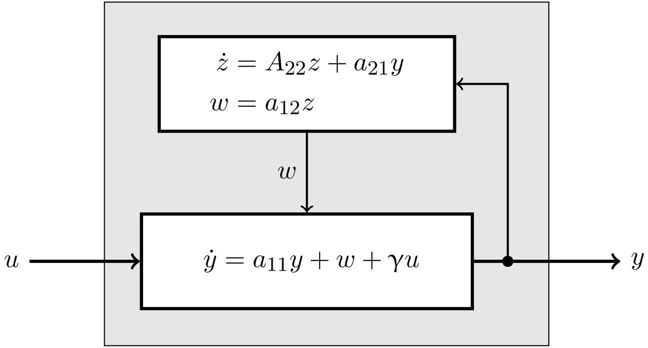
From linear to nonlinear systems



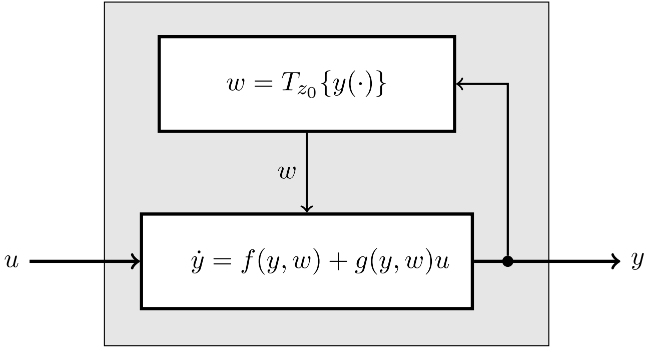
From linear to nonlinear systems



From linear to nonlinear systems



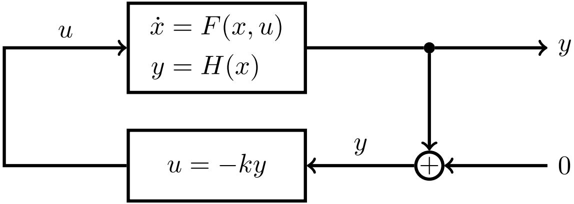
From linear to nonlinear systems



Assumptions:

- T_{z_0} is causal BIBO operator, i.e. $\exists \kappa(\cdot): \quad \|w\| \leq \kappa(\|y\|)$
- f and g continuous and $g > 0$

High gain stabilization for nonlinear systems



Theorem

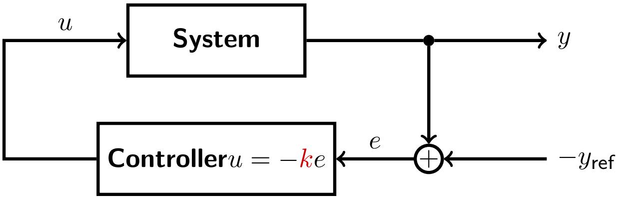
Assume there exists (nonlinear) coordinate transformation such that system is equivalent to

$$\dot{y} = f(y,w) + g(y,w)u, \quad w = T_{z_0}\{y(\cdot)\}$$

with f, g *continuous,* T_{z_0} *causal BIBO operator and* $g > 0$*, then*

$$\forall y(0)\ \forall z_0\ \exists K > 0\ \forall k \geq K : \quad \limsup_{t\to\infty} |y(t)| < \varepsilon$$

Summary high gain feedback



Goal: Output tracking

Challenge: Unknown system parameters

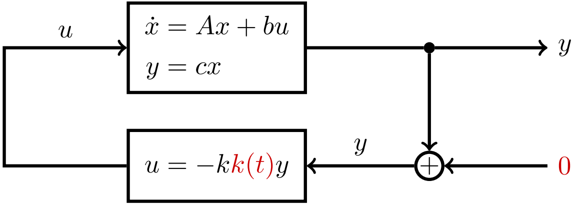
Structural assumptions

- Relative degree one with known sign of "high frequency gain"
- Stable zero dynamics

High gain feedback: $u = -ke$ "works" for sufficiently large gain $k > 0$

Remaining challenge: When is k sufficiently large?

Choosing gain adaptively, linear case



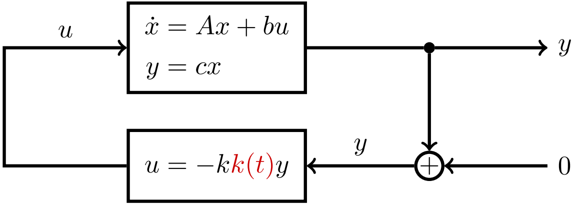
Theorem (High-gain stabilization)

$cb > 0$ *and stable zero-dynamics* $\Rightarrow \quad \exists K > 0 \; \forall \, k \geq K : y(t) \to 0$

Key idea

Why not make k time-varying with $\dot{k}(t) > 0$ as long as $y(t) > 0$?

Choosing gain adaptively, linear case



Theorem (Adaptive High-Gain Feedback, Byrnes & Willems 1984)

$cb > 0$ *and stable zero-dynamics* $\Rightarrow$

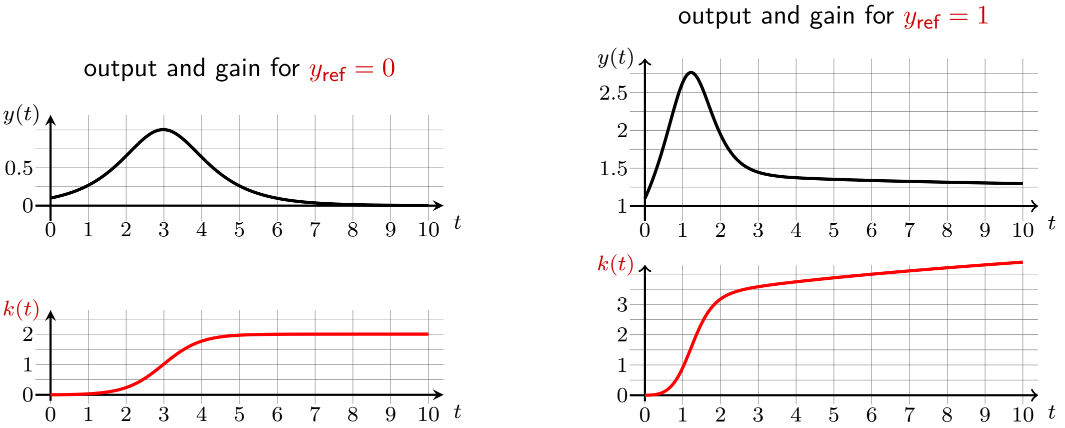
$$\dot{k}(t) = y(t)^2 \text{ makes closed loop asymptotically stable}$$

and $k(\cdot)$ *remains bounded*

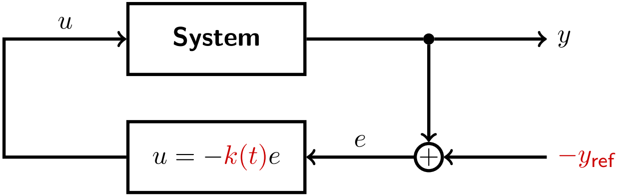
Boundedness of $k(t) = \int_0^t y(s)^2 \, \mathrm{d}s$ follows from final exponential decay of y.

Simulations

$$\dot{y} = y + u, \quad u(t) = -k(t)(y(t) - y_{\text{ref}}(t)), \quad \dot{k} = (y - y_{\text{ref}})^2$$



High gain adaptive control and tracking?

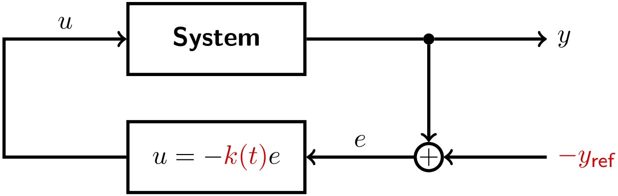


Unbounded gain

For $y_{\text{ref}} \neq 0$ the adaptation rule $\dot{k} = e^2$ leads to unbounded gain.

Recall: Constant gain for $y_{\text{ref}} \neq 0$ only leads to practical tracking, i.e. $e(t) \not\to 0$

High gain adaptive control and tracking?

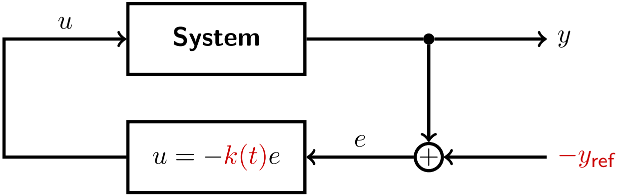


How to prevent unbounded growth?

Stop increasing gain when error is sufficiently small, e.g. via

$$\dot{k}(t) = \begin{cases} 0 & |e(t)| \leq \lambda \\ |e(t)|(|e(t)| - \lambda) & |e(t)| > \lambda \end{cases}$$

High gain adaptive control and tracking?



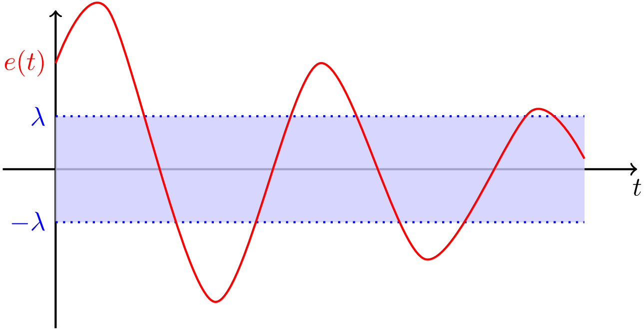
Theorem (λ-tracking, Ilchmann & Ryan 1994)

Assume r.d.-one with "$\gamma > 0$", stable zero-dynamics and $y_{\text{ref}}, \dot{y}_{\text{ref}}$ bounded. For $\lambda > 0$ consider

$$\dot{k}(t) = \begin{cases} 0, & |e(t)| \leq \lambda, \\ |e(t)|\big(|e(t)| - \lambda\big), & |e(t)| > \lambda. \end{cases}$$

Then the closed loop is practically stable, i.e. $\limsup_{t\to\infty} |e(t)| \leq \lambda$.

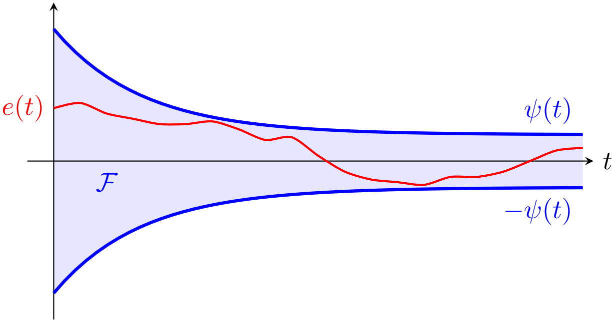
Remaining problems of λ-tracker



Problems:

- No guarantees when $|e(t)| \leq \lambda$
- No bounds on transient behaviour
- Monotonically growing $k(\cdot)$ $\Rightarrow$ Measurement noise unnecessarily amplified

The funnel as time-varying error bound

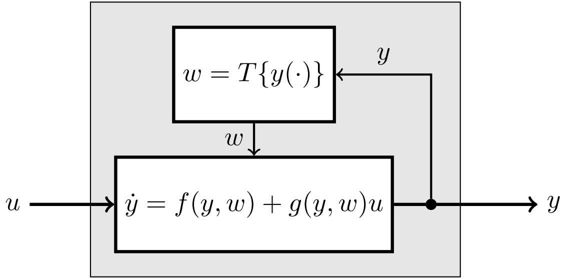


$$\mathcal{F} = \mathcal{F}(\psi) := \{(t,e) \mid |e| < \psi(t)\}$$

Idea: $k(t)$ large $\iff$ Distance of $e(t)$ to funnel boundary small

$\rightsquigarrow$ **Funnel gain:** $k(t) = \dfrac{1}{\psi(t) - |e(t)|}$

Funnel controller works



System class

Equivalent to structure left:

- T is causal and BIBO
- f, g continuous
- $g > 0$

Theorem (Ilchmann, Ryan, Sangwin 2002)

Assume $y_{\text{ref}}, \dot{y}_{\text{ref}}, \psi, \dot{\psi}$ *bounded*, $\liminf_{t\to\infty} \psi(t) > 0$ *and* $|e(0)| < \psi(0)$ *where* $e := y - y_{\text{ref}}$. *Then*

$$u(t) = -k(t)e(t) \quad \text{with} \quad k(t) = \frac{1}{\psi(t) - |e(t)|}$$

ensures that $e(t)$ *remains within funnel* $\mathcal{F}(\psi)$ *while* $k(t)$ *remains bounded.*

Proof

Step 1: Existence of solution

- Standard ODE theory: solution of closed loop exists on $[0,\omega)$ for $\omega \in (0,\infty]$
- Choose $\omega > 0$ maximal
- If $\omega < \infty$ then "$|e(\omega)| = \psi(\omega)$"

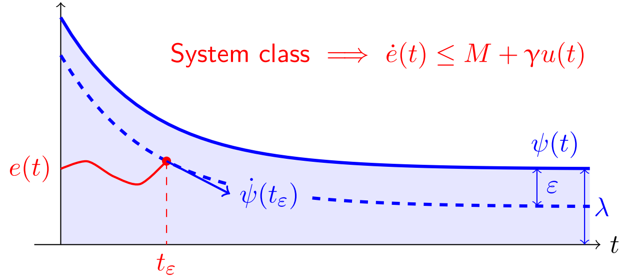
Step 2: We show that $\omega < \infty$ implies $|e(t)| - \psi(t) > \varepsilon$ for some $\varepsilon > 0$
Error dynamics are given by

$$\dot{e} = f(y,w) - \dot{y}_{\text{ref}} + g(y,w)u$$

Step 2a: Boundedness of e, y, and w

$e(t)$ within funnel for $t \in [0,\omega)$ *(domain of ODE)*
$\Rightarrow$ e bounded on $[0,\omega)$ *(because ψ is bounded)*
$\Rightarrow$ y bounded on $[0,\omega)$ *(because y_{ref} is bounded)*
$\Rightarrow$ w bounded on $[0,\omega)$ *(because T is BIBO)*
$\Rightarrow$ $f(y,w)$ bounded and $g(y,w)$ bounded away from zero on $[0,\omega)$ *(continuity)*
$\Rightarrow$ $\dot{e}(t) \leq M + \gamma u(t)$ if $u(t) < 0$ and $\dot{e}(t) \geq -M + \gamma u(t)$ if $u(t) > 0$

Step 2b: Funnel invariant (case $e(t) > 0$)



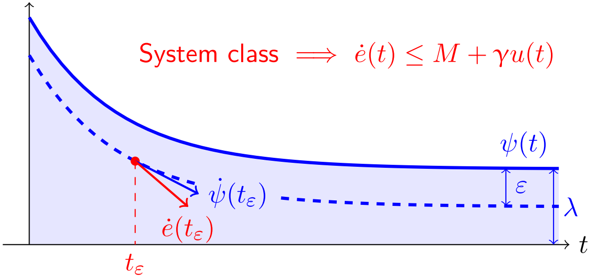
Assumptions: $\varepsilon < \psi(0) - e(0)$ $\quad$ $\varepsilon < \lambda/2$ $\quad$ $\psi(t) \geq \lambda$

$$e(t_\varepsilon) = \psi(t_\varepsilon) - \varepsilon \quad \implies \quad k(t_\varepsilon) = \frac{1}{\psi(t_\varepsilon) - |e(t_\varepsilon)|} = \frac{1}{\varepsilon}$$

$$\implies \quad u(t_\varepsilon) = -k(t_\varepsilon)e(t_\varepsilon) \leq -\frac{1}{\varepsilon}\frac{\lambda}{2}$$

$$\implies \quad \dot{e}(t_\varepsilon) \leq M - \frac{\gamma\lambda}{2\varepsilon}$$

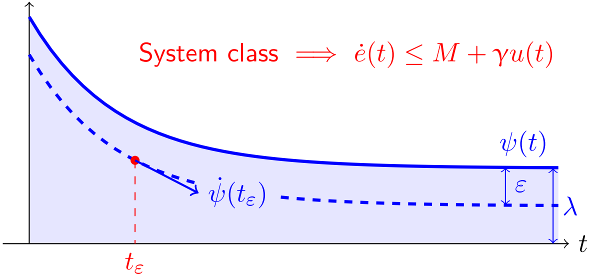
Step 2b: Funnel invariant (case $e(t) > 0$)



Assume $\dot\psi(t) > -\Psi$ and $\varepsilon \leq \dfrac{\gamma\lambda}{2(\Psi + M)}$ we have

$$\dot e(t_\varepsilon) \leq M - \frac{\gamma\lambda}{2\varepsilon} \leq -\Psi < \dot\psi(t_\varepsilon)$$

Step 2b: Funnel invariant (case $e(t) > 0$)



Consequence: For sufficiently small $\varepsilon > 0$,

$$\mathcal{F}_\varepsilon := \{(t,e) \mid |e(t)| < \psi(t) - \varepsilon\}$$

is positively invariant, i.e.

$$(0, e(0)) \in \mathcal{F}_\varepsilon \quad \Rightarrow \quad (t, e(t)) \in \mathcal{F}_\varepsilon \ \forall t \geq 0$$

and $\omega < \infty$ impossible!

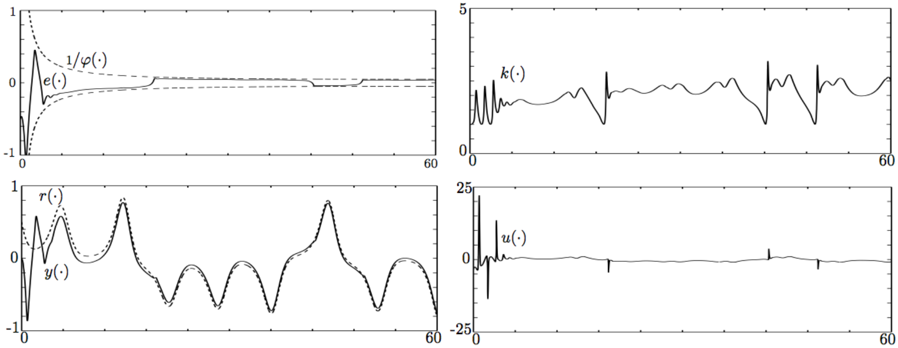
Extensions of funnel controller

- Asymptotic tracking (Lee & Trenn 2019)
- Multi-Input Multi-Output (MIMO) (already in Ilchmann et al. 2002)
- Higher relative degree (Ilchmann et al. 2007, Berger et al. 2018)
- Input saturation (Ilchmann et al. 2004, Hopfe et al. 2010)
- Bang-Bang funnel control (Liberzon & Trenn 2013)
- Funnel synchronization for multi-agent systems (Shim & Trenn 2015)
- For DAE-systems (Berger 2016)

Relative degree two via backstepping

For rel. deg. two systems, Funnel Controller is given by (Ilchmann et al. 2007):

$$
\begin{aligned}
u(t) &= -k(t)e(t) - (\|e(t)\|^2 + k(t)^2)k(t)^4(1 + \|\xi(t)\|^2)(\xi(t) + k(t)e(t)) \\
k(t) &= 1/(1 - \varphi(t)^2\|e(t)\|^2) \\
\dot{\xi}(t) &= -\xi(t) + u(t)
\end{aligned}
$$



Taken from: Ilchmann, Ryan, Townsend 2007, SICON

Alternative Approach for relative degree two

Use two funnels, one for error and one for derivative of error

Simple Control Law

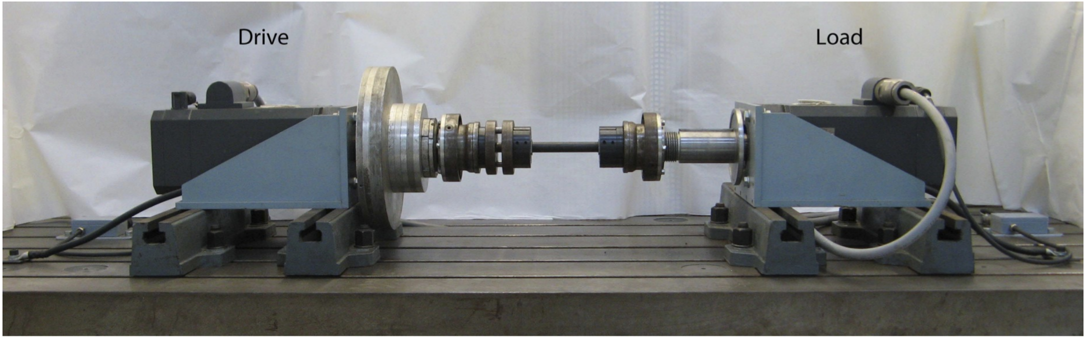
$$u(t) = -k_0(t)^2 e(t) - k_1(t)\dot{e}(t)$$
$$k_i(t) = \frac{1}{\psi_i(t) - |e(t)|}, \quad i = 0, 1$$

System class: $\ddot{y}(t) = f(p_f(t), T_f\{y, \dot{y}\}(t)) + g(p_g(t), T_g\{y, \dot{y}\}(t))u(t)$

Theorem (Hackl et al. 2012)

The above Funnel Controller for relative-degree-two-systems works (under mild assumptions on ψ_0 and ψ_1).

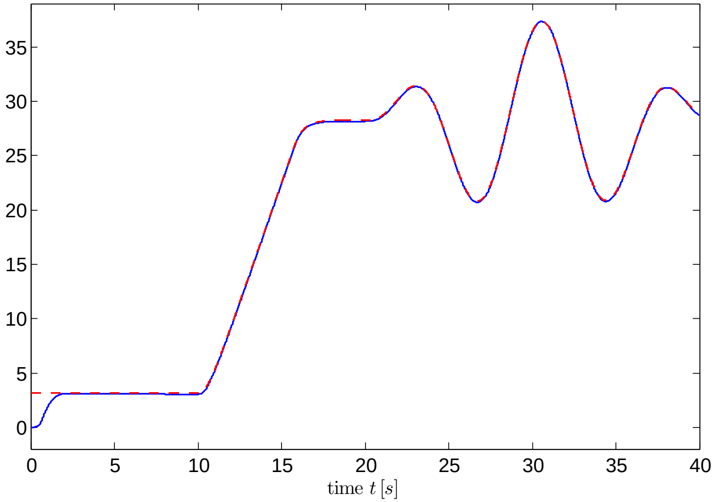
Experimental verification



$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \gamma \end{bmatrix} \big(u(t) + u_L(t) - (Tx_2)(t)\big),$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t),$$

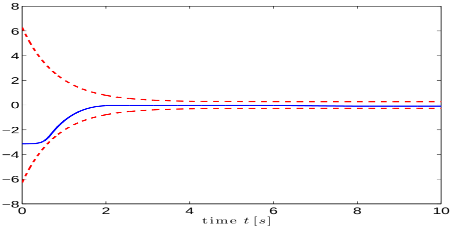
x_1: angle of rotating machine, $x_2 = \dot{x}_1$: angular velocity
u_L: unknown (bounded) load
$T : \mathbb{C}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \to \mathbb{L}^{\infty}_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R})$ friction operator

Tracking control in experiment

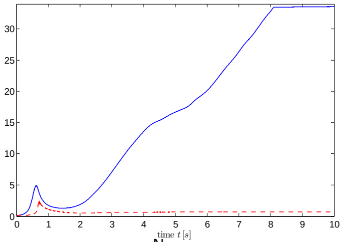


Measured angle $y(t)$ in rad, reference angle $y_{\text{ref}}(t)$ in rad

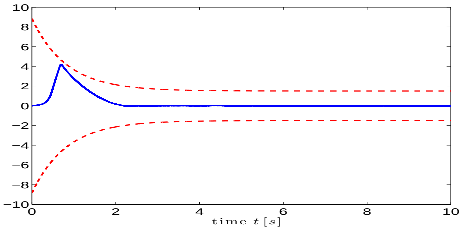
Experiment: Error, gains, input



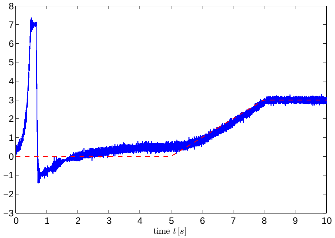
$e(t)$ in rad, $1/\varphi_0(t)$



$k_0(t)$ in $\frac{\text{Nm}}{\text{rad}}$, $k_1(t)$ in $\frac{\text{Nms}}{\text{rad}}$

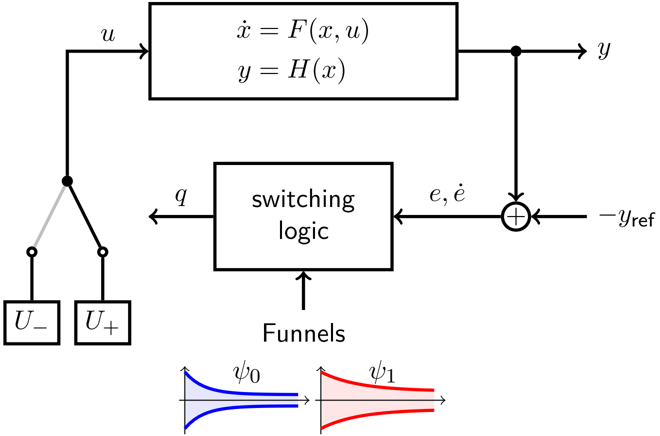


$\dot{e}(t)$ in rad/s, $1/\varphi_1(t)$

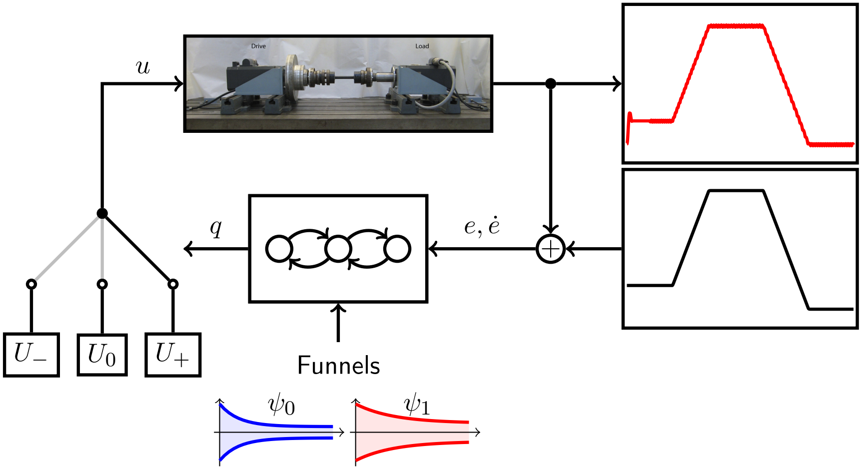


$u(t)$ in Nm, $u_L(t)$ in Nm

Bang-Bang Funnel Control



Bang-Bang Funnel Control



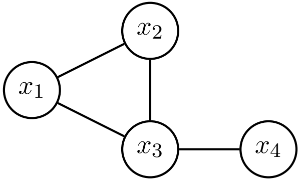
Funnel synchronization - setup

Given

- N agents with individual n-dimensional dynamics:

$$\dot{x}_i = f_i(t, x_i) + u_i$$

- undirected connected coupling-graph $G = (V, E)$
- local feedback $u_i = \gamma_i(x_i, x_{\mathcal{N}_i})$



$$u_1 = \gamma_1(x_1, x_2, x_3)$$
$$u_2 = \gamma_2(x_2, x_1, x_3)$$
$$u_3 = \gamma_3(x_3, x_1, x_2, x_4)$$
$$u_4 = \gamma_4(x_4, x_3)$$

Desired

Control design for practical synchronization

$$x_1 \approx x_2 \approx \ldots \approx x_n$$

A „high-gain“ result

Let $\mathcal{N}_i := \{j \in V \mid (j,i) \in E\}$ and $d_i := |\mathcal{N}_i|$ and $\mathcal{L}$ be the Laplacian of G.

Diffusive coupling

$$u_i = -k \sum_{j \in \mathcal{N}_i} (x_i - x_j) \quad \text{or, equivalently,} \quad u = -k\, \mathcal{L}\, x$$

Theorem (Practical synchronization, KIM et al. 2013)

Assumptions: G connected, all solutions of average dynamics

$$\dot{s}(t) = \frac{1}{N} \sum_{i=1}^{N} f_i(t, s(t))$$

remain bounded. Then $\forall \varepsilon > 0\ \exists K > 0\ \forall k \geq K$*: Diffusive coupling results in*

$$\boxed{\limsup_{t\to\infty} \|x_i(t) - x_j(t)\| < \varepsilon \quad \forall i,j \in V}$$

Remarks on high-gain result

Common trajectory

It even holds that

$$\limsup_{t\to\infty} |x_i(t) - s(t)| < \varepsilon/2,$$

where $s(\cdot)$ solves $\quad \dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \quad s(0) = \frac{1}{N}\sum_{i=1}^{N} x_i.$

Independent of coupling structure and amplification k.

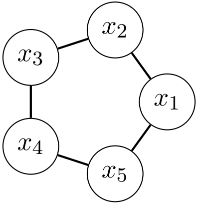
Error feedback

With $e_i := x_i - \overline{x}_i$ and $\overline{x}_i := \frac{1}{d_i}\sum_{j\in\mathcal{N}_i} x_j$ diffusive coupling has the form

$$u_i(t) = -ke_i(t)$$

Attention: $e_i \neq x_i - s$, in particular, agents do not know „limit trajectory“ $s(\cdot)$

Example (taken from Kim et al. 2015)



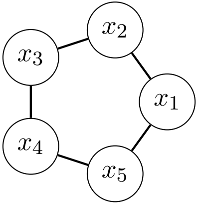
Simulations in the following for $N = 5$ agents with dynamics

$$f_i(t, x_i) = (-1 + \delta_i)x_i + 10\sin t + 10m_i^1 \sin(0.1t + \theta_i^1) + 10m_i^2 \sin(10t + \theta_i^2),$$

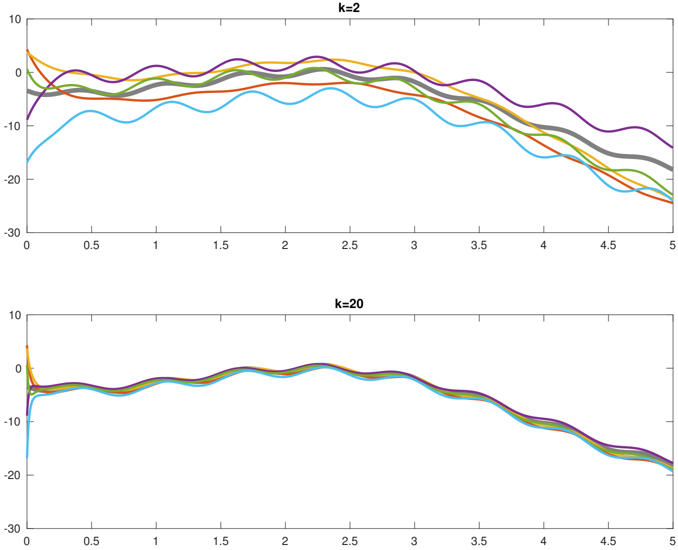
with randomly chosen parameters $\delta_i, m_i^1, m_i^2 \in \mathbb{R}$ and $\theta_i^1, \theta_i^2 \in [0, 2\pi]$.

Parameters identical in all following simulations, in particular $\delta_2 > 1$, hence agent 2 has unstable dynamics (without coupling).

Example (taken from KIM et al. 2015)



$$u = -k\ \mathcal{L}\ x$$



gray curve:

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t))$$

$$s(0) = \tfrac{1}{N}\sum_{i=1}^{N} x_i(0)$$

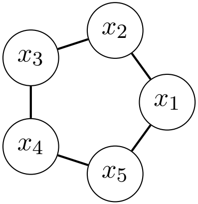
Funnel synchronization: Initial idea

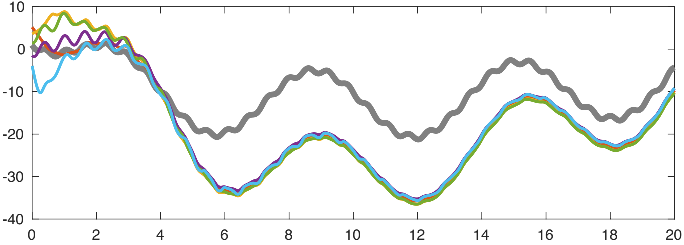
Reminder diffusive coupling: $u_i = -k_i e_i$ with $e_i = x_i - \overline{x}_i$.

Combine diffusive coupling with Funnel Controller

$$u_i(t) = -k_i(t)\, e_i(t) \quad \text{with} \quad k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

First simulations



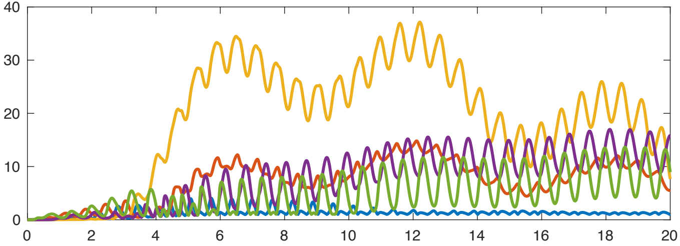


$$u_i(t) = -k_i(t)e_i(t)$$

$$k_i(t) = \frac{1}{\psi(t) - |e_i(t)|}$$

$$\psi(t) = \underline{\psi} + (\overline{\psi} - \underline{\psi})e^{-\lambda t}$$

$$\overline{\psi} = 20, \ \underline{\psi} = 1, \ \lambda = 1$$



Observations from simulations

Funnel synchronization seems to work

- errors remain within funnel
- practical synchronizations is achieved
- limit trajectory does not coincide with solution $s(\cdot)$ of

$$\dot{s}(t) = \frac{1}{N}\sum_{i=1}^{N} f_i(t, s(t)), \qquad s(0) = \tfrac{1}{N}\sum_{i=1}^{N} x_i(0).$$

What determines the new limiting trajectory?

- Coupling graph?
- Funnel shape?
- Gain function?

Diffusive coupling revisited

Diffusive coupling for weighted graph

$$u_i = -k \sum_i^N \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_i^N k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j)$$

where $\alpha_{ij} = \alpha_{ji} \in \{0,1\}$ is the weight of edge (i,j)

Conjecture

If $k_{ij} = k_{ji}$ are all sufficiently large, then practical synchronization occurs with desired limit trajectory s of average dynamics.

Proof technique from Kim et al. 2013 should still work in this setup.

Edgewise Funnel synchronization

Diffusive coupling → edgewise Funnel synchronization

$$u_i = -\sum_{i}^{N} k_{ij} \cdot \alpha_{ij} \cdot (x_i - x_j) \quad \longrightarrow \quad u_i = -\sum_{i}^{N} k_{ij}(t) \cdot \alpha_{ij} \cdot (x_i - x_j)$$

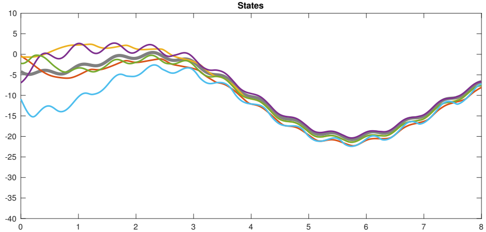
Edgewise error feedback

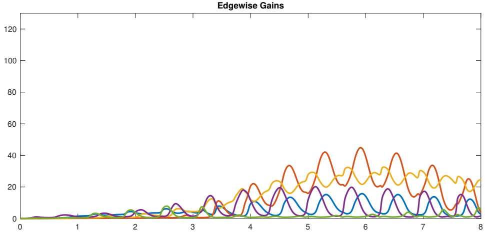
$$k_{ij}(t) = \frac{1}{\psi(t) - |e_{ij}|}, \quad \text{with} \quad e_{ij} := x_i - x_j$$

Properties:

- Decentralized, i.e. u_i only depends on state of neighbors
- Symmetry, $k_{ij} = k_{ji}$
- Laplacian feedback, $u = -\mathcal{L}_K(t,x)x$

Simulation (from Trenn 2017)

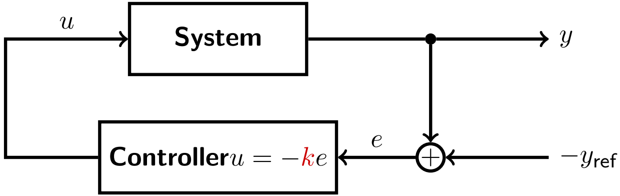




Properties

+ Synchronization occurs
+ Predictable limit trajectory (given by average dynamics)
+ Local feedback law
+ Convergence recently proved (Lee et al. 2023)

Summary high gain feedback and funnel control



Goal: Output tracking

Challenge: Unknown system parameters

Structural assumptions

- Relative degree one with known sign of "high frequency gain"
- Stable zero dynamics

High gain feedback: $u = -ke$ "works" for sufficiently large gain $k > 0$

Funnel gain: $k(t) = \frac{1}{\psi(t) - |e(t)|}$ achieves tracking with prescribed perfomance