Utilizing discontinuous piecewise Lyapunov function for a biological PWA system and extending the analysis via input-to-state type stability

Radu Cimpean and Stephan Trenn

***Abstract*— We study a piecewise affine (PWA) model of the spiking of neurons located in a subsystem of the olfactory system. Our long-term goal is to understand the stability properties of a network of such neurons and as a first step towards this goal we consider the simplest case of the interaction between two types of neurons (excitatory and inhibitory). Due to the discontinuous nature of the PWA model, it is challenging to find a continuous Lyapunov function, we therefore utilize a recently proposed constructive method to find a discontinuous Lyapunov function. In order to utilize this method, it is necessary to define a suitable polyhedral partition of the state-space and carefully investigate the dynamics at the boundaries. As an additional step, we propose extending the analysis by employing tools from the input-to-state (ISS) stability framework.**

I. Introduction

In the biological field, there has been a vivid interest in studying interactions between neurons and currently this analysis has been expanding through the use of techniques from different fields such as neuromorphic computing [1], [2], graph theory [3], [4], dynamical systems [5], [6] and systems and control [7], [8].

For this paper, the focus will be on the olfactory system, and more precisely on the interactions between neurons located in the olfactory bulb, a component responsible for receiving information and further relaying it to the higher areas of the brain for identification [9], [10]. For simplification, we will consider just two main types of neurons: excitatory ones, named mitral cells (MC) and inhibitory ones, called granule cells/interneurons (IN). The ODE system proposed in [10], which for this paper we will refer to as the Wanner-Friedrich (WF) model, encompasses the aforementioned interactions between neurons in a threshold-linear network frame, which is a special instance of a piecewise-affine (PWA) system. To better understand the complex dynamical interactions within such a network of neurons, we consider, as a first step, the stability property of a simple network consisting of an excitatory neuron interacting with an inhibitory neuron.

The aim of this paper is to prove the global asymptotic stability (GAS) of the scalar autonomous WF model by using a discontinuous piecewise Lyapunov function, introduced in [11]. Moreover, we will offer an extension pathway on how to deal with inputs once they are considered, using input-to-state stability (ISS). The novelty involving this approach comes from two aspects: 1) we suggest a way to deal with larger neural networks, which can be decomposed into smaller components that can be easier dealt with and hence offer a toolkit for testing stability; 2) we plan to develop a theoretical framework for PWA systems involving the cone copositive method in order to translate ISS Lyapunov function requirements into linear matrix inequalities (LMIs), which, given feasibility, would lead to certifying the systems as ISS.

The modeling framework of PWA systems [12], [13] or more generally, hybrid dynamical systems, has already been used in biological regulatory networks [14] and one of the earliest conceptualizations of this approach for control theory was offered by [12]. Among the extensively studied properties of PWA systems, stability is of particular relevance to this work and has been studied in the following sources [11], [15], [16], [17], [18], [19], [20], [21]. There has also been some interest and use of the piecewise affine model approach for analysing biological systems [22]. This formulation presents the advantage that it captures the linearity of the behaviour of the system given a proper split of the state space. Along with this partition, several properties of the PWA systems have been studied in biological networks [22], [23], [24], [25], [26], [27]. In terms of determining a piecewise Lyapunov function, certain similarities in our proposed approach can be seen in [28], [29], [30], [31] including some of the building blocks for the formulation of the LMIs. These similarities include the piecewise affine nature of the dynamics, with threshold-based switching between states, and the presence of feedback mechanisms. Despite the fact that we take inspiration from [16] in describing our methodology, several important distinctions should be noted. The biological mechanisms underlying the nonlinearities differ substantially, leading to distinct mathematical structures. The WF model reflects the neuronal interaction with graded firing rate, continuous responses to inputs, whereas the gene transcription models and more explicitly the ones present in [29], use binary switching, where genes are expressed or not. Biologically, the gene model characterizes transcription networks that exhibit behavior similar to logical circuits, for instance toggle switches and bistability. Furthermore, the threshold mechanisms play distinct roles: in the WF model, thresholds switch the effect of excitation/inhibition, while in the toggle-switch they delineate domains of attraction. Having the biological models described above, the resulting LMIs are constructed with different objectives. For the gene network, the Lyapunov candidate ensures convergence to one of two stable equilibria and requires continuity on certain

R.Cimpean is with the Bernoulli Institute and Groningen Cognitive Systems and Materials Center (CogniGron), Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands (phone: +31 6 19721054, r.cimpean@rug.nl)

S.Trenn is with the Bernoulli Institute, Faculty of Science and Engineering, University of Groningen, 9747 AG Groningen, The Netherlands, s.trenn@rug.nl

region boundaries, specifically, those belonging to cycles or where sliding modes can occur, as discussed in [29]. Importantly, continuity is not required on all boundaries; crossing boundaries that do not form cycles and can be certified to lack sliding modes allow discontinuities, as it will be seen in the formulation for the WF model. In our case, allowing the Lyapunov function to be piecewise discontinuous across such boundaries reflects the system's alternating neural dynamics and is consistent with existing approaches cf. [29].

The structure of this paper is as follows. We introduce the concept of stability in the context of a piecewise affine system and the steps towards building the piecewise Lyapunov function, along with the analysis on the WF model. Once the analysis is performed and the building blocks are obtained, we offer the result of the computational implementation of these outputs and thus provide a stability certificate for the scalar case of the biological model, along with a graphical representation of the level sets for certain regions. Furthermore, we introduce concepts from input-to-state (ISS) theory, and we try to offer a perspective for extending the cone copositive method, once inputs are considered. This may also allow to study larger networks by viewing them as interconnected small networks and using versions of the small-gain-theorem (e.g. [32]) to conclude overall stability of the complex network.

II. Stability of a PWA system

A. PWA system analysis for the Wanner-Friedrich olfactory bulb model

When working within a PWA system framework, stability analysis becomes a more complicated issue due to the partition of the state space into regions which exhibit different behaviors. According to [16], the Lyapunov method can be still employed by resorting to the piecewise quadratic Lyapunov functions for each region, combining these results and then providing the stability conditions in terms of cone-constrained inequalities. These inequalities are being formulated and solved with the help of LMIs. Solving these inequalities leads to suitable local Lyapunov function candidates, which together with suitable boundary conditions, ensure asymptotic stability. According to [11], [16], several ingredients will be introduced to perform the actual steps that underline the aforementioned analysis.

1) Define the PWA system and perform the polyhedral partition: We begin by defining the PWA system as

$$\dot{x} = A_s x + b_s \quad x \in X_s,\ s = 1 \dots S, \tag{1}$$

where $A_s \in \mathbb{R}^{n\times n}$, $b_s \in \mathbb{R}^n$, and $X = \bigcup_{s=1}^{S} X_s \subseteq \mathbb{R}^n$ is a polyhedral partition of the state space X, i.e. each X_s is a finite intersection of closed half-spaces. Furthermore, let

$$\hat{A}_s = \begin{pmatrix} A_s & b_s \\ 0 & 0 \end{pmatrix}.$$

We start implementing this PWA structure for the WF model from [10]. The adapted scalar version of this ODE system is given by

$$\begin{aligned} \dot{x}^{\mathrm{e}}(t) &= -\alpha x^{\mathrm{e}}(t) - \beta[x^{\mathrm{i}}(t) - \theta_{x^{\mathrm{i}}}]_+ + \sigma(t), \\ \dot{x}^{\mathrm{i}}(t) &= -\gamma x^{\mathrm{i}}(t) + \delta[x^{\mathrm{e}}(t) - \theta_{x^{\mathrm{e}}}]_+, \end{aligned} \tag{2}$$

where $x^{\mathrm{e}}(t), x^{\mathrm{i}}(t) \in \mathbb{R}$ are the firing rates of the excitatory/inhibitory neuron; $\theta_{x^{\mathrm{e}}}, \theta_{x^{\mathrm{i}}} > 0$ are firing thresholds; $\alpha, \beta, \gamma, \delta > 0$ are biological parameters; $\sigma(t)$ is an external input; $[.]_+$ denotes the half-wave rectification function (i.e. $[p]_+ = \max\{0, p\}$).

We indicate the following values based on [10] and the coding of the scalar version of the ODE system in Matlab: $\alpha = 1,\ \beta = 1.435,\ \gamma = 0.0125,\ \delta = 0.0063$. For this case study, only the autonomous version of (2) will be considered (i.e. $\sigma(t) = 0$) and thus formulated in the piecewise affine system of the form ($x = (x^{\mathrm{e}}, x^{\mathrm{i}})^T$)

$$\dot{x} = A_s x + b_s, \quad x \in X_s, \quad \bigcup_s X_s = \mathbb{R}^2.$$

Based on the the half-wave rectification function, in order to reflect the linear property of each case in the model, we began with the assumption that the state space should be split into four regions (I, II, III, IV), in the following format:

$$\dot{x} = A_{\mathrm{I}} x + b_{\mathrm{I}} = \begin{bmatrix} -\alpha & 0 \\ 0 & -\gamma \end{bmatrix} x, \qquad \begin{cases} x^{\mathrm{e}} \le \theta_{x^{\mathrm{e}}}, \\ x^{\mathrm{i}} \le \theta_{x^{\mathrm{i}}}, \end{cases}$$

$$\dot{x} = A_{\mathrm{II}} x + b_{\mathrm{II}} = \begin{bmatrix} -\alpha & 0 \\ \delta & -\gamma \end{bmatrix} x + \begin{pmatrix} 0 \\ -\delta\theta_{x^{\mathrm{e}}} \end{pmatrix}, \qquad \begin{cases} x^{\mathrm{e}} \ge \theta_{x^{\mathrm{e}}}, \\ x^{\mathrm{i}} \le \theta_{x^{\mathrm{i}}}, \end{cases}$$

$$\dot{x} = A_{\mathrm{III}} x + b_{\mathrm{III}} = \begin{bmatrix} -\alpha & -\beta \\ \delta & -\gamma \end{bmatrix} x + \begin{pmatrix} \beta\theta_{x^{\mathrm{i}}} \\ -\delta\theta_{x^{\mathrm{e}}} \end{pmatrix}, \qquad \begin{cases} x^{\mathrm{e}} \ge \theta_{x^{\mathrm{e}}}, \\ x^{\mathrm{i}} \ge \theta_{x^{\mathrm{i}}}, \end{cases}$$

$$\dot{x} = A_{\mathrm{IV}} x + b_{\mathrm{IV}} = \begin{bmatrix} -\alpha & -\beta \\ 0 & -\gamma \end{bmatrix} x + \begin{pmatrix} \beta\theta_{x^{\mathrm{i}}} \\ 0 \end{pmatrix}, \qquad \begin{cases} x^{\mathrm{e}} \le \theta_{x^{\mathrm{e}}}, \\ x^{\mathrm{i}} \ge \theta_{x^{\mathrm{i}}}. \end{cases}$$

This four region partition describes the stance that each of the two neurons has with respect to its threshold, namely $\theta_{x^{\mathrm{e}}} = 2$ and $\theta_{x^{\mathrm{i}}} = 50$, with values taken from [10]. Hence, for the region I, neither type of neuron has reached its threshold. In region II, the excitatory neuron is active, but the inhibitory one is below its threshold. Next, in region III both neurons are active, meaning they both surpassed their respective thresholds. Lastly, region IV covers the part where only the inhibitory neuron is active and above the threshold. In addition, to apply the method described in more detail in [11], we had to ensure that the boundaries of the proposed regions exhibited qualitatively uniform dynamical behavior. However, it was observed that the boundary between region II and region III did not exhibit a simple crossing behavior. In particular, at the point $\left(\frac{\gamma}{\delta}\theta_{x^{\mathrm{i}}} + \theta_{x^{\mathrm{e}}},\ \theta_{x^{\mathrm{i}}}\right)$, the direction of the vector field changed qualitatively: to the left of this point, the flow decreased, while to the right, there was a slight increase. This suggested that the boundary was not a standard transition region, but rather marked a change in flow behavior. This fact led us to introduce additional boundaries, guided by the observation of vector field turning behavior. Specifically, we examined points where one component of the derivative became zero, indicating a switch in the flow direction. For instance, within region IV, the trajectory exhibited a turning motion around a certain line before moving toward the point $(0, \theta_{x^{\mathrm{i}}})$, after which the flow decreased toward the equilibrium. By noticing these patterns, we proposed a further refinement in order to encapsulate similar behavior, which would lead to an easier path to finding the Lyapunov function. The aforementioned refinement resulted in a total of nine regions, as proposed in Figure 1. These nine regions maintain the description

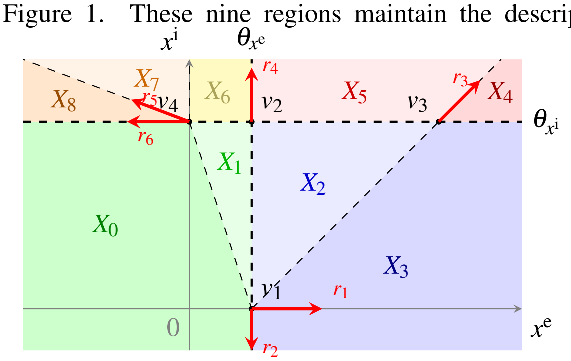

Fig. 1. Proposed state partition for the WF model (2); note that the scale was adjusted for illustration purposes.

of the behavior of the two neurons with respect to their threshold and reflect the four region format as follows: I (X_0 and X_1); II (X_2 and X_3); III (X_4 and X_5); IV (X_6, X_7 and X_8). After establishing the PWA system in terms of the regions, we described them in terms of vertices and rays, thus obtaining the cones and ray matrices. In Figure 1, we have the following vertices and rays:

$$v_1 = \begin{pmatrix} \theta_{x^{\mathrm{e}}} \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} \theta_{x^{\mathrm{e}}} \\ \theta_{x^{\mathrm{i}}} \end{pmatrix},\ v_3 = \begin{pmatrix} \frac{\gamma}{\delta}\theta_{x^{\mathrm{i}}} + \theta_{x^{\mathrm{e}}} \\ \theta_{x^{\mathrm{i}}} \end{pmatrix},\ v_4 = \begin{pmatrix} 0 \\ \theta_{x^{\mathrm{i}}} \end{pmatrix},$$

$$r_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\ r_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix},\ r_3 = \begin{pmatrix} \delta \\ \gamma \end{pmatrix},\ r_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, r_5 = \begin{pmatrix} -\beta \\ 1 \end{pmatrix},\ r_6 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}.$$

The selection of vertices and rays was guided by the geometry of the partition, with threshold-based coordinates and behavioral transitions informing their placement and connections to capture consistent neuronal dynamics within regions.

2) Express the polyhedron partitions in terms of the sum of convex hulls and cones: To perform this step, we introduce the ν-representation of the polyhedron as given in [11].

Definition 1. The ν-representation of a polyhedron X in $\mathbb{R}^n$ is

$$X = \mathrm{conv}\{v_1, v_2, \ldots, v_\lambda\} + \mathrm{cone}\{r_1, r_2, \ldots, r_\rho\} \tag{3}$$

for a finite number $\lambda \geq 1$ of vertices $\{v_l\}_{l=1}^{\lambda}$ and a finite number $\rho \geq 0$ of rays $\{r_l\}_{l=1}^{\rho}$, $v_l, r_l \in \mathbb{R}^n$. Here conv$\{...\}$ denotes the convex hull and cone$\{\ldots\}$ the smallest enclosing convex cone (with the convention that cone$\{\} = \{0\}$).

With the aforementioned building blocks in the form of vertices and rays, we expressed each proposed region of the WF model in the ν-representation form: $X_0 = \mathrm{conv}\{v_1, v_4\} + \mathrm{cone}\{r_2, r_6\}$, $X_1 = \mathrm{conv}\{v_1, v_2, v_4\}$, $X_2 = \mathrm{conv}\{v_1, v_2, v_3\}$, $X_3 = \mathrm{conv}\{v_1, v_3\} + \mathrm{cone}\{r_1, r_2\}$, $X_4 = \mathrm{conv}\{v_3\} + \mathrm{cone}\{r_1, r_3\}$, $X_5 = \mathrm{conv}\{v_2, v_3\} + \mathrm{cone}\{r_3, r_4\}$, $X_6 = \mathrm{conv}\{v_2, v_4\} + \mathrm{cone}\{r_4\}$, $X_7 = \mathrm{conv}\{v_4\} + \mathrm{cone}\{r_4, r_5\}$, $X_8 = \mathrm{conv}\{v_4\} + \mathrm{cone}\{r_5, r_6\}$.

3) Perform conic homogenization and provide the cones and ray matrices:

Definition 2. Consider a polyhedron $X \subset \mathbb{R}^n$ with representation (3). For each vertex $v_l \in \mathbb{R}^n$, its vertex homogenization $\bar{v}_l \in \mathbb{R}^{n+1}$ is defined as $\bar{v}_l = \mathrm{col}(v_l, 1) \in \mathbb{R}^{n+1}$, where col$(\cdot)$ indicates a vector obtained by stacking the arguments over each other. For each ray $r_l \in \mathbb{R}^n$, its direction homogenization $\bar{r}_l \in \mathbb{R}^{n+1}$ is defined as $\bar{r}_l = \mathrm{col}(r_l, 0) \in \mathbb{R}^{n+1}$.

Since we utilized a cone copositive approach, we considered for all regions X (given by a ν-representation), which contain the origin, the following cone:

$$\mathscr{C}_X = \mathrm{cone}\{\{v_l\}_{l=1}^{\lambda}, \{r_l\}_{l=1}^{\rho}\}$$

with corresponding ray matrix

$R = \begin{bmatrix} v_1 & \ldots & v_\lambda & r_1 & \ldots r_\rho \end{bmatrix} \in \mathbb{R}^{n \times (\lambda+\rho)}$; whereas for regions X, which do not contain the origin, we considered the homogenized cone:

$$\mathscr{C}_{\hat{X}} = \mathrm{cone}\{\{\bar{v}_l\}_{l=1}^{\lambda}, \{\bar{r}_l\}_{l=1}^{\rho}\}$$

with $\bar{v}_l$ and $\bar{r}_l$ given as in Definition 2, the corresponding ray matrix is given by

$$\hat{R} = \begin{bmatrix} v_1 & \ldots & v_\lambda & r_1 & \ldots & r_\rho \\ 1 & \ldots & 1 & 0 & \ldots & 0 \end{bmatrix},$$

with $\hat{R} \in \mathbb{R}^{(n+1)\times(\lambda+\rho)}$. Having given above the ν-representation form of our biological model, applying this definition is straightforward, thus being skipped.

4) Define the piecewise quadratic Lyapunov candidate for each region: Let

$$V_s(x) = x^T P_s x + 2\nu_s^T x + \omega_s,\ x \in X_s,\ s = 1 \ldots S, \tag{4}$$

be a piecewise quadratic function, with $\{P_s\}_{s=1}^{S}$ a family of symmetric matrices in $\mathbb{R}^{n\times n}$, $\{\nu_s\}_{s=1}^{S}$ a family of real vectors in $\mathbb{R}^n$ and $\{\omega_s\}_{s=1}^{S}$ a family of real scalars. Denote by

$$\hat{P}_s = \begin{pmatrix} P_s & \nu_s \\ \nu_s^T & \omega_s \end{pmatrix}, \tag{5}$$

which will be used further in defining the LMIs. With respect to the partition, there is also a case to be made for when 0 belongs to the region or not. Hence, let Σ_0 denote the subset of indices such that $0 \in X_s$ and $\bar{\Sigma}_0 = \Sigma \setminus \Sigma_0$ denote the subset of indices such that $0 \notin X_s$. Moreover, if $0 \in X_s$, then $\omega_s = 0$. The goal in the following is to find a family of Lyapunov functions $\{V_s\}_{s=1}^{S}$ such that, firstly, in each region X_s the function V_s is a local Lyapunov function, i.e. it is positive definite and decreases along solutions and, secondly, the values at the boundaries between regions are compatible with the solution behavior.

5) Introduce suitable LMI conditions locally: To ensure that V_s is a Lyapunov function on region X_s the following set of LMIs are proposed in [11, Lem. 2]:

- For all $s \in \Sigma_0$

$$\begin{aligned} &R_s^T P_s R_s - N_s \succeq 0 \\ &2\nu_s^T R_s e_i \geq 0,\ i = 1, \ldots, \lambda_s + \rho_s \\ &-R_s^T(A_s^T P_s + P_s A_s)R_s - M_s \succeq 0 \\ &-2\nu_s^T A_s R_s e_i \geq 0,\ i = 1, \ldots, \lambda_s + \rho_s \end{aligned} \tag{6}$$

- For all $s \in \bar{\Sigma}_0$

$$\begin{aligned} &\hat{R}_s^T \hat{P}_s \hat{R}_s - N_s \succeq 0 \\ &-\hat{R}_s^T(\hat{A}_s^T \hat{P}_s + \hat{P}_s \hat{A}_s)\hat{R}_s - M_s \succeq 0 \end{aligned} \tag{7}$$

where N_s, M_s are unknown symmetric entrywise positive matrices. If the set of LMIs have a solution $\{P_s, \nu_s, \omega_s, N_s, M_s\}_{s\in\Sigma}$, then the quadratic functions given by (4) are indeed local Lyapunov functions for the PWA system (1). For the WF model, this means that we have to set up one set of LMIs of the form (6) for region X_0 and eight sets of LMIs of the form (7) for the remaining regions $X_1, X_2, \ldots, X_8$.

6) Determine the type of boundary and further formulate suitable LMI conditions: In order to ensure asymptotic stability, we need to determine what type of boundaries the PWA system exhibits and such a characterization allows us to further indicate operative conditions in terms of LMIs. According to [11, Def. 23, Cor. 30], we have to distinguish between the following boundary types: unreachable, crossing, Caratheodory, and other types, including sliding boundaries. By our specific choice of partitioning, we ensured that all boundaries of the WF model are crossing boundaries and therefore we do not need to provide further details about the other type of boundaries. Furthermore, since the WF is actually a locally Lipschitz continuous PWA system, the existence and uniqueness of classical solutions (in the sense of Caratheodory) are guaranteed and the definition of a crossing boundary, given in [11], simplifies to the following:

Definition 3. For a PWA system (1) a boundary $X_{\mathscr{B}} \subseteq X$ between two or more regions is called a crossing boundary if for all x in the relative interior of $X_{\mathscr{B}}$ the set of modes from which x can be reached with a solution is disjoint from the set of modes which the solution starting at x can continue in.

In the case of planar PWA systems, boundaries are either the intersection of two regions or a single point (i.e. the boundaries of boundaries). For single point boundaries, it suffices to show that these are not equilibria to conclude that they are crossing boundaries; for the WF model this is indeed the case. Hence, it remains to verify the following twelve boundaries, where we use the notation $X_{ij} := X_i \cap X_j$ and also provide a ν-representation and a corresponding normal vector η_{ij} for each boundary: $X_{10} = \mathrm{conv}\{v_4, v_1\}$, $\eta_{10} = \begin{pmatrix} -1 & -\frac{\theta_{x\mathrm{e}}}{\theta_{x\mathrm{i}}} \end{pmatrix}^\top$, $X_{21} = \mathrm{conv}\{v_1, v_2\}$, $\eta_{21} = \begin{pmatrix} -1 & 0 \end{pmatrix}^\top$, $X_{32} = \mathrm{conv}\{v_2, v_3\}$, $\eta_{32} = \begin{pmatrix} -1 & \frac{\delta}{\gamma} \end{pmatrix}^\top$, $X_{34} = \{v_3\} + \mathrm{cone}\{r_1\}$, $\eta_{34} = \begin{pmatrix} 0 & 1 \end{pmatrix}^\top$, $X_{45} = \{v_3\} + \mathrm{cone}\{r_3\}$, $\eta_{45} = \begin{pmatrix} 1 & -\frac{\delta}{\gamma} \end{pmatrix}^\top$ $X_{56} = \{v_2\} + \mathrm{cone}\{r_4\}$, $\eta_{56} = \begin{pmatrix} -1 & 0 \end{pmatrix}^\top$, $X_{67} = \{v_4\} + \mathrm{cone}\{r_4\}$, $\eta_{67} = \begin{pmatrix} -1 & 0 \end{pmatrix}^\top$, $X_{78} = \{v_4\} + \mathrm{cone}\{r_5\}$, $\eta_{78} = \begin{pmatrix} 1 & -\frac{\beta}{\alpha} \end{pmatrix}^\top$, $X_{80} = \{v_4\} + \mathrm{cone}\{r_6\}$, $\eta_{80} = \begin{pmatrix} 0 & -1 \end{pmatrix}^\top$, $X_{61} = \mathrm{conv}\{v_2, v_4\}$, $\eta_{61} = \begin{pmatrix} 0 & -1 \end{pmatrix}^\top$, $X_{52} = \mathrm{conv}\{v_2, v_3\}$, $\eta_{52} = \begin{pmatrix} 0 & -1 \end{pmatrix}^\top$, $X_{30} = \{v_1\} + \mathrm{cone}\{r_2\}$, $\eta_{30} = \begin{pmatrix} -1 & 0 \end{pmatrix}^\top$.

It can be easily seen, that a solution of the PWA system crosses a boundary X_{ij} at the point x in the relative interior of X_{ij} from region X_i to X_j if

$$\eta_{ij}^\top (A_i x + b_i) > 0 \wedge \eta_{ij}^\top (A_j x + b_j) > 0.$$

In case of a continuous PWA system, as it is for the WF model, only one of the above two conditions needs to be checked. However, this condition still has to hold for infinitely many x and cannot be checked directly. Nonetheless, similar to [11, Lem. 32] we can reduce the condition to a finite check:

Lemma 1.
1) $X_{ij} = \mathrm{conv}\{x_{ij}^l, x_{ij}^r\}$ is a crossing boundary if either $(A_i x_{ij}^l + b_i)^T \eta_{ij} \geq 0$ and $(A_i x_{ij}^r + b_i)^T \eta_{ij} > 0$ or $(A_i x_{ij}^l + b_i)^T \eta_{ij} > 0$ and $(A_i x_{ij}^r + b_i)^T \eta_{ij} \geq 0$.
2) $X_{ij} = \{x_{ij}\} + \mathrm{cone}\{r_{ij}\}$ is a crossing boundary if either $(A_i x_{ij} + b_i)^T \eta_{ij} > 0$ and $(A_i r_{ij})^T \eta_{ij} \geq 0$ or $(A_i x_{ij} + b_i)^T \eta_{ij} \geq 0$ and $(A_i r_{ij})^T \eta_{ij} > 0$.

Applying this check to all of the above twelve boundaries we can indeed conclude that all boundaries are crossing boundaries. To ensure that the local Lyapunov functions decrease along the boundaries, we impose the following additional LMIs:

$$\hat{R}_{ij}^\top (\hat{P}_i - \hat{P}_j) \hat{R}_{ij} - N_{ij} \succeq 0 \tag{8}$$

for each of the twelve boundaries X_{ij} with $(i, j) \in \{(1,0), (2,1), (3,2), (3,4), (4,5), (5,6), (6,7), (7,8), (8,0), (6,1), (5,2), (3,0)\}$.

Note that it is important to keep track of how the solution crosses the boundary in order to set up the correct LMI. In our approach, we have used the convention that the normal vector η_{ij} points in the direction of region j and therefore Lemma 1 then ensures that indeed the solution crosses the boundary X_{ij} from region X_i to X_j; in particular, the order of the index in the boundary is important: while $X_i \cap X_j = X_j \cap X_i$ the LMIs for X_{ij} and X_{ji} are different.

7) Feasibility and GAS: Given the LMIs constraints provided in (6), (7) and with the additional boundary conditions from (8), we can conclude that if a feasible solution exists, then the PWA system (1) is globally asymptotically stable, with a piecewise quadratic Lyapunov function.

B. Stability certificate for WF model

Using the Matlab environment together with YALMIP and the SeDuMi solver, we implemented the above LMIs and a feasible solution was found, thus certifying stability of the WF model. The found feasible solution resulted in the Lyapunov function shown in (9).

This Lyapunov function is discontinuous along the boundaries. We also tried to find a continuous Lyapunov function; however, for the considered partition the corresponding more restrictive LMIs were not feasible. Although refining the partition may allow the construction of a continuous Lyapunov function, employing a discontinuous Lyapunov function typically simplifies the search, as it involves fewer constraints. For illustration purposes, we indicate several levels sets based on (9) for regions X_0 and X_1 (Fig.2).

We observe that the level sets from X_1 that cross into the region X_0 decrease, thus confirming the desired properties of a piecewise Lyapunov function. We have also tried to reduce the regions from nine to six, meaning we defined I (X_0); II (X_1 and X_2); III (X_3 and X_4); IV (X_5), together with demarcating their respective boundaries. When seeking a feasible solution, the solver indicated an ill-posed problem, hence supporting the initial nine region split.

$$V(x) = \begin{cases} 0.2990(x^{\mathrm{e}})^2 + 0.0005(x^{\mathrm{i}})^2 - 0.0018x^{\mathrm{e}}x^{\mathrm{i}}, & \text{for region } 0 \\ -0.0941(x^{\mathrm{e}})^2 + 0.0017(x^{\mathrm{i}})^2 - 0.0090x^{\mathrm{e}}x^{\mathrm{i}} + 1.4758x^{\mathrm{e}} - 0.0256x^{\mathrm{i}} + 0.8133, & \text{for region } 1 \\ 0.0023(x^{\mathrm{i}})^2 - 0.0010x^{\mathrm{e}}x^{\mathrm{i}} + 0.0674x^{\mathrm{e}} - 0.1126x^{\mathrm{i}} + 5.2495, & \text{for region } 2 \\ 0.0980(x^{\mathrm{e}})^2 + 0.3909(x^{\mathrm{i}})^2 - 0.3898x^{\mathrm{e}}x^{\mathrm{i}} - 0.303x^{\mathrm{e}} + 0.2757x^{\mathrm{i}} + 8.4278, & \text{for region } 3 \\ 0.0651(x^{\mathrm{e}})^2 + 65.5738(x^{\mathrm{i}})^2 + 13.098x^{\mathrm{e}}x^{\mathrm{i}} - 668.036x^{\mathrm{e}} - 6953.2x^{\mathrm{i}} + 184400, & \text{for region } 4 \\ 0.00047(x^{\mathrm{e}})^2 + 4.9741(x^{\mathrm{i}})^2 + 0.097x^{\mathrm{e}}x^{\mathrm{i}} - 4.9024x^{\mathrm{e}} - 487.798x^{\mathrm{i}} + 11964.56, & \text{for region } 5 \\ 0.4893(x^{\mathrm{e}})^2 + 4.4419(x^{\mathrm{i}})^2 + 0.878x^{\mathrm{e}}x^{\mathrm{i}} - 45.7598x^{\mathrm{e}} - 438.449x^{\mathrm{i}} + 10825, & \text{for region } 6 \\ 0.0474(x^{\mathrm{e}})^2 + 3.9357(x^{\mathrm{i}})^2 + 0.6714x^{\mathrm{e}}x^{\mathrm{i}} - 35.9002x^{\mathrm{e}} - 389.3846x^{\mathrm{i}} + 9635.5, & \text{for region } 7 \\ 2.1469(x^{\mathrm{e}})^2 + 4.729(x^{\mathrm{i}})^2 + 4.7084x^{\mathrm{e}}x^{\mathrm{i}} - 237.733x^{\mathrm{e}} - 470.2282x^{\mathrm{i}} + 11690, & \text{for region } 8 \end{cases} \tag{9}$$

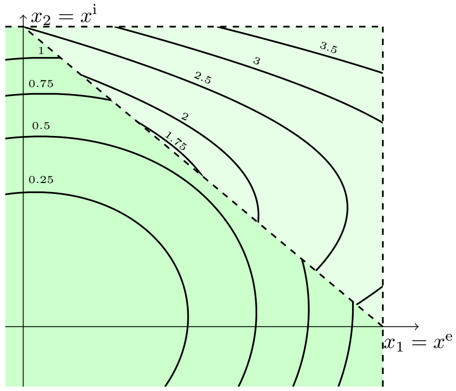

Fig. 2. Level sets for X_0 and X_1

III. ISS

A. ISS definition and corresponding LMIs

In the previous section, we have obtained a Lyapunov function, hence establishing GAS. By achieving this result, in the autonomous case, several steps can be taken forward. We could proceed by trying to scale further to a network of neurons and attempt to obtain a similar result. However, adding just one neuron to the model increases the difficulty in terms of defining the activation regions and then determining the boundaries and ensuring that each boundary is a crossing boundary. Therefore, we propose a new avenue to tackle the higher dimension interactions by using concepts stemming from input-to-state stability (ISS).

Definition 4. Systems of the form

$$\dot{x} = f(x,u) \quad x(0) = x_0, \tag{10}$$

where $x(t) \in \mathbb{R}^n$, $u \in \mathscr{U} = L^\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $x_0 \in \mathbb{R}^n$ are called input-to-state stable (ISS), if (10) is forward complete and there exist $\beta \in \mathscr{KL}$ and $\gamma \in \mathscr{K}$ such that for all $x_0 \in \mathbb{R}^n$, all $u \in \mathscr{U}$ and all $t \geq 0$ the following holds

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(||u||_\infty), \tag{11}$$

where β describes the transient behavior of the system and γ is called gain and shows the influence of the input on the system [33]. For a better understanding of the following function classes $\mathscr{K}, \mathscr{KL}, \mathscr{K}_\infty$, we reference [34, Def. 3.3, Def. 3.4, Lem. 3.2]. Next, we introduce the ISS Lyapunov function definition.

Definition 5. A continuous function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called an ISS Lyapunov function for (10), if there exist $\alpha_1, \alpha_2 \in \mathscr{K}_\infty, \alpha \in \mathscr{K}_\infty$, and $\xi \in \mathscr{K}$ such that

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \quad \forall x \in \mathbb{R}^n,$$

and for any $x(t) \in \mathbb{R}^n$, $u \in \mathscr{U}$, the following inequality holds:

$$\dot{V}_u(x) \leq -\alpha(V(x)) + \xi(||u||_\infty),$$

where $\dot{V}_u(x) = \nabla V(x) f(x,u)$.

From [33, Th. 2.12], the existence of an ISS Lyapunov function indicates that the system is also ISS. Next, the ISS Lyapunov functions have the added benefit that together with a condition called small-gain, certify that we ensure stability of interconnection of the subsystems. In particular, under a robust small-gain condition, one can guarantee the existence of a common ISS Lyapunov function for the interconnected system [33]. If we consider the 1-1 neuron interaction as such a subsystem and we can find an ISS Lyapunov function, we could therefore couple several such subsystems and then provided a suitable small-gain condition equivalent, we could establish the stability for the network. Since we will work within a PWA structure, we will also indicate an updated version of ISS, called "practical ISS" (input-to-state practical stability or in short ISpS) [35], for which we have the following alteration of (11)

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(||u||_\infty) + c,$$

where $c \geq 0$ is a constant. Having these concepts introduced, we pursue the matter by trying to offer the set-up for the LMI for the PWA system. Firstly, we consider an ISS Lyapunov function restricted to the regions provided by the PWA formulation. Therefore, we begin adapting the ideas from [16] and [11] to extend the cone copositive method. We proceed to consider the stability problem for the PWA system

$$\dot{x} = A_s x + B_s u + b_s, \tag{12}$$

for which the terms are as in (1), with the additional $B_s \in \mathbb{R}^{n \times m}$ as the input matrix. In terms of (4), we make use of the following formulation

$$V_s(x) = \begin{bmatrix} x^T & 1 \end{bmatrix} \begin{bmatrix} P_s & \nu_s \\ \nu_s^T & \omega_s \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} = \hat{x}^T \hat{P}_s \hat{x}. \tag{13}$$

We notice that the format of (4) remains unchanged, and therefore the ideas from [16] and [11] about this part are kept in the current analysis, with a small change: we require that all the elements of matrix N_s should be greater than $\varepsilon_s > 0$, rather than just positive, to satisfy the ISS condition.

Next, we need to compute $\dot{V}_{u,s}(x)$. We obtain

$$\dot{V}_{u,s}(x) = \begin{bmatrix} x \\ u \\ 1 \end{bmatrix}^T \begin{bmatrix} A_s^T P_s + P_s A_s & P_s B_s & P_s b_s + A_s^T \nu_s \\ B_s^T P_s & 0 & B_s^T \nu_s \\ b_s^T P_s + \nu_s^T A_s & \nu_s^T B_s & 2\nu_s^T b_s \end{bmatrix} \begin{bmatrix} x \\ u \\ 1 \end{bmatrix}. \tag{14}$$

Next, we aim to match the required conditions for the existence of an ISS Lyapunov function. To do so in a more general framework, we first adopt the conditions for input-to-state practical stability (ISpS), and then reduce to the ISS case by setting the constant bounding term to zero (i.e., $\Theta_s^* = 0$). Our goal is to bound $\dot{V}_{u,s}(x)$, given by (14), as follows:

$$\dot{V}_{u,s}(x) \le -\lambda_s^* V_s(x) + \Gamma_s^* |u|^2 + \Theta_s^*, \quad \forall x \in X_s \tag{15}$$

where $\lambda_s^* \in \mathbb{R}_+, \Gamma_s^*, \Theta_s^* \in \mathbb{R}_+ \cup \{0\}$. The following proposed lemma, offers the key ingredients towards the ISS analysis of PWA systems through PWQ- Lyapunov function by using the cone copositive method.

Lemma 2. Consider the system (12) with the polyhedra $\{X_s\}_{s=1}^S$ expressed as (3), the ray matrices $\{R_s\}_{s\in\Sigma_0}$ with $R_s \in \mathbb{R}^{n\times(\lambda_s+\rho_s)}$, the homogenized ray matrices $\{\hat{R}_s\}_{s\in\bar{\Sigma}_0}$ with $\hat{R}_s \in \mathbb{R}^{(n+1)\times(\lambda_s+\rho_s)}$, and define the matrices

$$\hat{A}_s = \begin{pmatrix} A_s & b_s \\ 0 & 0 \end{pmatrix}$$

with $s \in \Sigma_0 \cup \bar{\Sigma}_0$. Consider the following LMI conditions:

- For all $s \in \Sigma_0$

$$\begin{aligned} &R_s^T P_s R_s - N_s \succeq 0;\ N_s^{ij} \ge \varepsilon_s \\ &2\nu_s^T R_s e_i \ge 0,\ i = 1,\dots,\lambda_s+\rho_s \end{aligned} \tag{16}$$

- For all $s \in \bar{\Sigma}_0$

$$\hat{R}_s^T \hat{P}_s \hat{R}_s - N_s \succeq 0;\ N_s^{ij} \ge \varepsilon_s \tag{17}$$

where $\hat{P}_s \in \mathbb{R}^{(n+1)\times(n+1)}$ are symmetric matrices in the form (5), and N_s, M_s are symmetric (entrywise) positive matrices of appropriate dimensions and ε_s is a small positive value. Moreover, consider the following LMI condition

$$\begin{bmatrix} A_s^T P_s + P_s A_s + \lambda_s^* P_s & P_s B_s & P_s b_s + \lambda_s^* \nu_s + A_s^T \nu_s \\ B_s^T P_s & \text{-}\Gamma_s^* I_{\lambda_s+\rho_s} & B_s^T \nu_s \\ b_s^T P_s + \lambda_s^* \nu_s^T + \nu_s^T A_s & \nu_s^T B_s & 2\nu_s^T b_s + \lambda_s^* \omega_s \text{-} \Theta_s^* \end{bmatrix} \preceq 0.$$

If for our choices of $\lambda_s^* \in \mathbb{R}_+$ and $\varepsilon_s > 0$ the above given LMI conditions produce a feasible solution $\{P_s, \nu_s, \omega_s, N_s, M_s, \Gamma_s^*, \Theta_s^*\}_{s\in\Sigma}$, then the resulting family of Lyapunov candidate (13) consists of local ISS-Lyapunov functions in the sense that it is positive definite on X_s and (15) holds.

We need to further adapt the conditions from the above stated lemma to the definition of ISS Lyapunov function in order to be able to certify that (13) is ISS. We will need to further show that there exist class $\mathscr{K}_\infty$-functions α_1 and α_2 such that (13) satisfies:

$$\alpha_1(|\hat{x}|) \le V_s(x) \le \alpha_2(|\hat{x}|),$$

where $\hat{x}$ represents a component-wise transformation of x and also $\dot{V}_{u,s}(x)$ satisfies:

$$\dot{V}_{u,s}(x) \le -\alpha(V_s(x)) + \xi(\|u\|_\infty),$$

where α is a class $\mathscr{K}$-function and ξ is a function of the L_∞-norm of u. Additional aspects that can be transformed via the cone copositive method into LMI conditions are subject to further work.

B. Application to WF model

We could consider ISS for the WF model w.r.t. to the input σ, however, qualitatively, we know that a nonzero input pushes away the equilibrium from zero and for now we want to primarily understand that once the input vanishes, the state returns back to the origin. The challenge is to answer this question for a large network and we expect that the stability property depends on the coupling strength between the different neurons. In principle, the construction of a piecewise Lyapunov function from the previous section can be extended to arbitrary high dimension, but it quickly becomes challenging to set up all regions and necessary refinement. Instead, we propose to use the cone copositive approach to first establish ISS for a small sub-network (here the 1-1 case studied in the previous section) and then consider the coupling with the remaining network as additional inputs to this small network and the corresponding ISS property thereof. Let us materialize this conceptual idea by considering 2 excitatory and 2 inhibitory neurons. Now, let us group them as subsystems, rather than based on the type and then we get the following couplings

$$\begin{bmatrix} \dot{x}_1^{\mathrm{e}} \\ \dot{x}_1^{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \text{-}\alpha & 0 \\ 0 & \text{-}\gamma \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{e}} \\ x_1^{\mathrm{i}} \end{bmatrix} + \begin{bmatrix} \text{-}\beta_1 & 0 \\ 0 & \delta_1 \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{i}} - \theta_{x^{\mathrm{i}}} \\ x_1^{\mathrm{e}} - \theta_{x^{\mathrm{e}}} \end{bmatrix}_+ + \begin{bmatrix} \text{-}\beta_2 & 0 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} x_2^{\mathrm{i}} - \theta_{x^{\mathrm{i}}} \\ x_2^{\mathrm{e}} - \theta_{x^{\mathrm{e}}} \end{bmatrix}_+$$

$$\begin{bmatrix} \dot{x}_2^{\mathrm{e}} \\ \dot{x}_2^{\mathrm{i}} \end{bmatrix} = \begin{bmatrix} \text{-}\alpha & 0 \\ 0 & \text{-}\gamma \end{bmatrix} \begin{bmatrix} x_2^{\mathrm{e}} \\ x_2^{\mathrm{i}} \end{bmatrix} + \begin{bmatrix} \text{-}\beta_4 & 0 \\ 0 & \delta_4 \end{bmatrix} \begin{bmatrix} x_2^{\mathrm{i}} - \theta_{x^{\mathrm{i}}} \\ x_2^{\mathrm{e}} - \theta_{x^{\mathrm{e}}} \end{bmatrix}_+ + \begin{bmatrix} \text{-}\beta_3 & 0 \\ 0 & \delta_3 \end{bmatrix} \begin{bmatrix} x_1^{\mathrm{i}} - \theta_{x^{\mathrm{i}}} \\ x_1^{\mathrm{e}} - \theta_{x^{\mathrm{e}}} \end{bmatrix}_+ .$$

We can see from the coupling that the output of one of the subsystems will appear as input for the other subsystem. Testing the above given lemma should offer an ISS Lyapunov function for each of the subsystems and with the step further of formulating an adapted small-gain theorem, which will be the focus of further research, we are on the path of ensuring the stability of networks of neurons. This offers us also the chance to explore and assess aspects from excitatory-inhibitory neuron models which rely on weighted interaction in a way that the analysis provided by the current cone-copositive tools on autonomous systems restrict, especially when the system has to be considered with inputs. Further investigation into these extensions will be conducted in subsequent work.

IV. Conclusions

In the pursuit of trying to understand the interactions between neurons, we have focused on a specific biological subsystem called the olfactory bulb, that belongs to the olfactory system. From the field related literature, we selected the WF model, that encapsulates the interaction between excitatory and inhibitory neurons. Considering the scalar autonomous version of the model, we proceeded in studying the stability by means of Lyapunov functions. Due to the nature of the system, we approached the case from a piecewise affine system perspective. We have seen that by expressing the WF model as a PWA system with a polyhedral partition, we could make use of the cone copositive method to find a discontinuous Lyapunov function certifying asymptotic stability for the autonomous case. Several aspects that needed to be highlighted revolve around the suitable partitioning of the state space, which was achieved when carefully taking into account the thresholds and also the points where a certain change in the vector field direction occurred. Moreover, establishing the boundaries and properly classifying them was another aspect that we carefully dealt with. Lastly, the previous work on formulating the LMIs that lead to a possibility of a discontinous piecewise Lyapunov function made the restrictions less conservative than if we required also continuity. We moved further with trying to scale the analysis to a larger network of excitatory and inhibitory neurons that exist in the olfactory bulb. Therefore, starting from the proven GAS for the neuronal subsystem, we proceeded to extend the approach to finding now Lyapunov functions which certify ISS. First, we provided some theoretical base for the ISS part that would be required in the extension of the cone copositive method for PWA systems. We indicated some sufficient conditions given by Lemma 2 that would be a first step towards certifying ISS, and more precisely ISpS, for a PWA system. Next, by considering the definition of ISS Lyapunov, we would need to further provide suitable conditions that would ensure we have an ISS Lyapunov, thus proving the system is ISS. Moving a step further, with additional work into adapting the small-gain theorem for the stability of interconnected ISS system or ISpS for our case, we could reach the desired result of ensuring the stability property for a network of neurons.

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References

[1] C. D. Schuman, T. E. Potok, R. M. Patton, J. D. Birdwell, M. E. Dean, G. S. Rose, and J. S. Plank, "A survey of neuromorphic computing and neural networks in hardware," *arXiv preprint arXiv:1705.06963*, 2017.

[2] M. Davies, A. Wild, G. Orchard, Y. Sandamirskaya, G. A. F. Guerra, P. Joshi, P. Plank, and S. R. Risbud, "Advancing neuromorphic computing with loihi: A survey of results and outlook," *Proceedings of the IEEE*, vol. 109, no. 5, pp. 911–934, 2021.

[3] T. Gedeon, "Global dynamics of neural nets with infinite gain," *Physica D: Nonlinear Phenomena*, vol. 146, no. 1-4, pp. 200–212, 2000.

[4] J. C. Reijneveld, S. C. Ponten, H. W. Berendse, and C. J. Stam, "The application of graph theoretical analysis to complex networks in the brain," *Clinical neurophysiology*, vol. 118, no. 11, pp. 2317–2331, 2007.

[5] P. Dayan and L. F. Abbott, *Theoretical neuroscience: computational and mathematical modeling of neural systems*. MIT press, 2005.

[6] T. Gedeon, "Attractors in continuous–time switching networks," *Communications on Pure and Applied Analysis*, vol. 2, no. 2, pp. 187–209, 2003.

[7] S. Díaz-Seoane, A. B. Blas, and A. F. Villaverde, "Controllability and accessibility analysis of nonlinear biosystems," *Computer Methods and Programs in Biomedicine*, vol. 242, p. 107837, 2023.

[8] J. E. Lewis and L. Glass, "Nonlinear dynamics and symbolic dynamics of neural networks," *Neural Computation*, vol. 4, no. 5, pp. 621–642, 1992.

[9] Z. Li and J. Hopfield, "Modeling the olfactory bulb and its neural oscillatory processings," *Biological cybernetics*, vol. 61, no. 5, pp. 379–392, 1989.

[10] A. A. Wanner and R. W. Friedrich, "Whitening of odor representations by the wiring diagram of the olfactory bulb," *Nature neuroscience*, vol. 23, no. 3, pp. 433–442, 2020.

[11] R. Iervolino, S. Trenn, and F. Vasca, "Asymptotic stability of piecewise affine systems with filippov solutions via discontinuous piecewise lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 66, no. 4, pp. 1513–1528, 2020.

[12] F. J. Christophersen, *Piecewise affine systems*. Springer, 2007.

[13] D. Leenaerts and W. M. Van Bokhoven, *Piecewise linear modeling and analysis*. Springer Science & Business Media, 2013.

[14] V. Noel, S. Vakulenko, and O. Radulescu, "Algorithm for identification of piecewise smooth hybrid systems: application to eukaryotic cell cycle regulation," in *Algorithms in Bioinformatics: 11th International Workshop, WABI 2011, Saarbrücken, Germany, September 5-7, 2011. Proceedings 11*. Springer, 2011, pp. 225–236.

[15] J. M. Gonçalves, "Regions of stability for limit cycles of piecewise linear systems," in *42nd IEEE International Conference on Decision and Control (IEEE Cat. No. 03CH37475)*, vol. 1. IEEE, 2003, pp. 651–656.

[16] R. Iervolino, D. Tangredi, and F. Vasca, "Lyapunov stability for piecewise affine systems via cone-copositivity," *Automatica*, vol. 81, pp. 22–29, 2017.

[17] P. Petsagkourakis, W. P. Heath, and C. Theodoropoulos, "Stability analysis of piecewise affine systems with multi-model predictive control," *Automatica*, vol. 111, p. 108539, 2020.

[18] M. Kantner, "Robust stability of piecewise linear discrete time systems," in *Proceedings of the 1997 American Control Conference (Cat. No. 97CH36041)*, vol. 2. IEEE, 1997, pp. 1241–1245.

[19] D. Mignone, G. Ferrari-Trecate, and M. Morari, "Stability and stabilization of piecewise affine and hybrid systems: An lmi approach," in *Proceedings of the 39th IEEE Conference on Decision and Control (Cat. No. 00CH37187)*, vol. 1. IEEE, 2000, pp. 504–509.

[20] L. B. Groff, G. Valmórbida, and J. M. G. da Silva, "Stability analysis of piecewise affine discrete-time systems," in *2019 IEEE 58th Conference on Decision and Control (CDC)*. IEEE, 2019, pp. 8172–8177.

[21] L. Rodrigues, "Stability analysis of piecewise-affine systems using controlled invariant sets," *Systems & control letters*, vol. 53, no. 2, pp. 157–169, 2004.

[22] N. Augier and A. G. Yabo, "Time-optimal control of piecewise affine bistable gene-regulatory networks," *International Journal of Robust and Nonlinear Control*, vol. 33, no. 9, pp. 4967–4988, 2023.

[23] F. Blanchini, E.-S. Hana, G. Giordano, and E. D. Sontag, "Control-theoretic methods for biological networks," in *2018 IEEE conference on decision and control (CDC)*. IEEE, 2018, pp. 466–483.

[24] E. Farcot, "Geometric properties of a class of piecewise affine biological network models," *Journal of Mathematical Biology*, vol. 52, pp. 373–418, 2006.

[25] E. Farcot and J.-L. Gouzé, "A mathematical framework for the control of piecewise-affine models of gene networks," *Automatica*, vol. 44, no. 9, pp. 2326–2332, 2008.

[26] J.-i. Imura, K. Kashima, M. Kusano, T. Ikeda, and T. Morohoshi, "Piecewise affine systems approach to control of biological networks," *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences*, vol. 368, no. 1930, pp. 4977–4993, 2010.

[27] J.-L. Gouzé and T. Sari, "A class of piecewise linear differential equations arising in biological models," *Dynamical systems*, vol. 17, no. 4, pp. 299–316, 2002.

[28] M. Pasquini and D. Angeli, "On piecewise quadratic lyapunov functions for piecewise affine models of gene regulatory networks," in *2018 IEEE Conference on Decision and Control (CDC)*. IEEE, 2018, pp. 1071–1076.

[29] ——, "On convergence for piecewise affine models of gene regulatory networks via a lyapunov approach," *IEEE Transactions on Automatic Control*, vol. 65, no. 8, pp. 3333–3348, 2020.

[30] ——, "Study of piecewise multi-affine models for genetic regulatory networks via a lyapunov approach: an lmi framework," *IFAC-PapersOnLine*, vol. 53, no. 2, pp. 16 739–16 744, 2020.

[31] ——, "On convergence for hybrid models of gene regulatory networks under polytopic uncertainties: a lyapunov approach," *Journal of Mathematical Biology*, vol. 83, no. 6, p. 64, 2021.

[32] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth, "An ISS small gain theorem for general networks," *Math. Control Signals Syst.*, vol. 19, no. 2, pp. 93–122, 2007.

[33] A. Mironchenko, "Input-to-state stability of infinite networks," in *Input-to-State Stability: Theory and Applications*. Springer, 2023, pp. 47–49.

[34] H. K. Khalil, *Nonlinear Systems*, 2nd ed. Upper Saddle River, NJ: Prentice-Hall, 1996.

[35] E. Sontag and Y. Wang, "New characterizations of input-to-state stability," *IEEE Transactions on Automatic Control*, vol. 41, no. 9, pp. 1283–1294, 1996.