Stabilization of switched DAEs via fast switching

Stephan Trenn

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

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Switched DAEs

Switched linear DAE (differential algebraic equation)

(swDAE) $\boxed{E_{\sigma(t)}\dot{x}(t) = A_{\sigma(t)}x(t)}$ or short $E_\sigma \dot{x} = A_\sigma x$

with

- switching signal $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{m}\}$
  - piecewise constant, right-continuous
  - locally finitely many jumps
- matrix pairs $(E_1, A_1), \ldots, (E_\mathtt{m}, A_\mathtt{m})$
  - $E_p, A_p \in \mathbb{R}^{n\times n}$, $p = 1, \ldots, \mathtt{m}$
  - (E_p, A_p) regular, i.e. $\det(sE_p - A_p) \not\equiv 0$

Main motivation

Modeling of electrical circuits

Special features

- Changing algebraic constraints
- Induced jumps
  $\to$ consistency projectors Π_p
- Dirac impulses possible

Question

$E_p\dot{x} = A_p x$ asymp. stable $\forall p$ $\overset{?}{\Longrightarrow}$ $E_\sigma \dot{x} = A_\sigma x$ asymp. stable $\forall \sigma$

Example 1: jumps and stability

Example 1a:

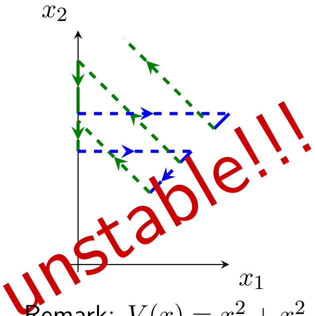
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

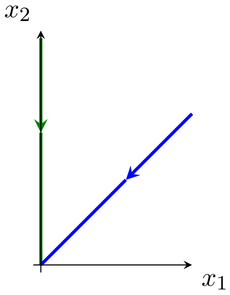
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

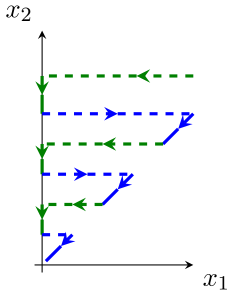
Example 1b:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$

$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$

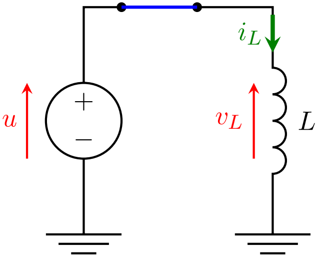


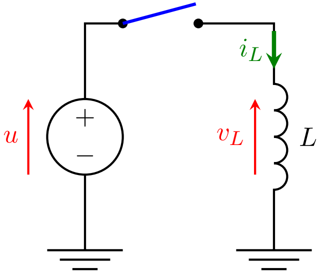




Remark: $V(x) = x_1^2 + x_2^2$ is Lyapunov function for all subsystem

Example 2: impulses in solutions



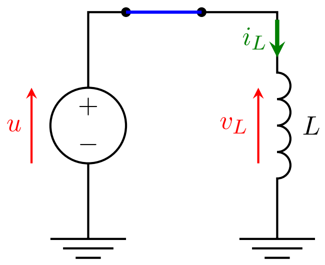


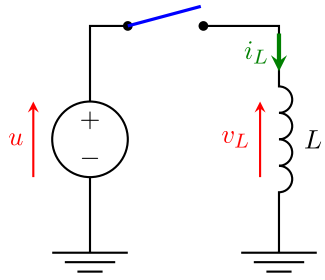
constant input: $\dot{u} = 0$

inductivity law: $L\frac{\mathsf{d}}{\mathsf{d}t} i_L = v_L$

switch dependent: $0 = v_L - u$ $\qquad$ $0 = i_L$

Example 2: impulses in solutions





$$x = [u, i_L, v_L]^\top$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ -1 & 0 & 1\end{bmatrix}x$$

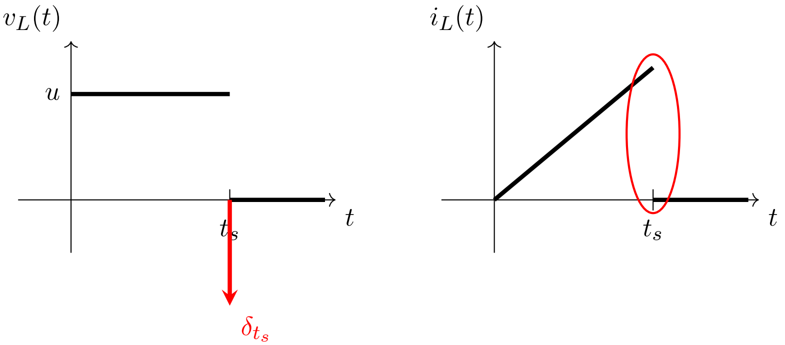
$$\begin{bmatrix}1 & 0 & 0\\ 0 & L & 0\\ 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}x$$

Solution of example

$L\frac{\mathrm{d}}{\mathrm{d}t}i_L = v_L, \quad 0 = v_L - u$ or $0 = i_L$

u constant, $i_L(0) = 0$

switch at $t_s > 0$: $\sigma(t) = \begin{cases} 1, & t < t_s \\ 2, & t \geq t_s \end{cases}$



Observations from examples

Solutions

- modes have constrained dynamics: consistency spaces
- switches $\Rightarrow$ inconsistent initial values
- inconsistent initial values $\Rightarrow$ jumps in x

Stability

- common Lyapunov function not sufficient
- stability depends on jumps

Impulses

- switching $\Rightarrow$ Dirac impulse in solution x
- Dirac impulse = infinite peak $\Rightarrow$ instability

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Solutions for unswitched DAEs

Consider $E\dot{x} = Ax$.

Theorem (Weierstrass 1868)

(E, A) *regular* $\iff$ $\exists S, T \in \mathbb{R}^{n\times n}$ *inv.:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right),$$

N *nilpotent*, $T = [V, W]$

Corollary (for regular (E, A))

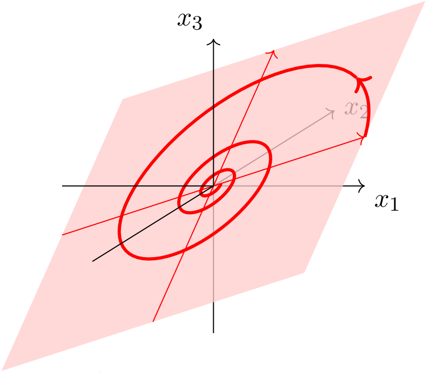
x *solves* $E\dot{x} = Ax$ $\iff$

$$x(t) = Ve^{Jt}v_0$$

$V \in \mathbb{R}^{n\times n_1}$, $J \in \mathbb{R}^{n_1\times n_1}$, $v_0 \in \mathbb{R}^{n_1}$.

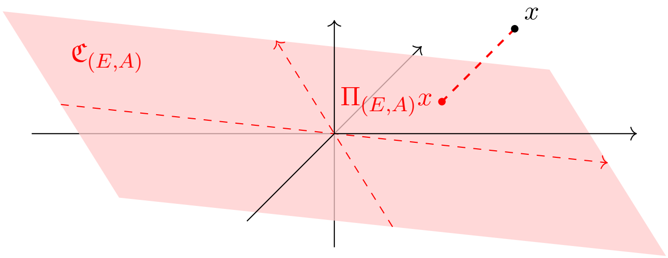
Consistency space: $\mathfrak{C}_{(E,A)} := \operatorname{im} V$

$$(E, A) = \left(\begin{bmatrix} 0 & 4 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -4\pi & -4 & 0 \\ -1 & 4\pi & 0 \\ -1 & -4 & 4 \end{bmatrix} \right)$$



$$V = \begin{bmatrix} 0 & 4 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \; J = \begin{bmatrix} -1 & -4\pi \\ \pi & -1 \end{bmatrix}$$

Consistency projector



Definition (Consistency projectors for regular (E,A))

Let $S,T \in \mathbb{R}^{n\times n}$ be invertible with $(SET, SAT) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} J & 0 \\ 0 & I \end{smallmatrix}\right]\right)$:

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Lyapunov functions for regular (E, A)

Definition (Lyapunov function for $E\dot{x} = Ax$)

$Q = Q^\top > 0$ on $\mathfrak{C}_{(E,A)}$ and $P = P^\top > 0$ solutions of

$$A^\top PE + E^\top PA = -Q \quad \text{(generalize Lyapunov equation)}$$

Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} : x \mapsto (Ex)^\top PEx$

V monotonically decreasing along solutions:

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} V(x(t)) &= (Ex(t))^\top PE\dot{x}(t) + (E\dot{x}(t))^\top PEx \\ &= x(t)^\top E^\top PAx(t) + x(t)^\top A^\top PEx(t) \\ &= -x(t)^\top Qx(t) < 0 \end{aligned}$$

Theorem (Owens & Debeljkovic 1985)

$E\dot{x} = Ax$ *asymptotically stable* $\iff$ $\exists$ *Lyapunov function*

Intermediate summary: Problems and their solutions

Consider again switched DAE

$$E_\sigma \dot{x} = A_\sigma x \qquad \textbf{(swDAE)}$$

1. Stability criteria for single DAEs $E_p \dot{x} = A_p x$
   $\Rightarrow$ Lyapunov functions
2. **No classical solutions for switched DAEs**
   $\Rightarrow$ Allow for jumps in solutions
3. How does inconsistent initial value "jump" to consistent one?
   $\Rightarrow$ Consistency projectors $\Pi_{(E_1,A_1)}, \ldots, \Pi_{(E_\mathtt{m},A_\mathtt{m})}$
4. Differentiation of jumps
   $\Rightarrow$ Space of Distributions as solution space
5. **Multiplication with non-smooth coefficients**
   $\Rightarrow$ Space of piecewise-smooth distributions
   $\Rightarrow$ Existence and uniqueness of (distributional) solutions

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Asymptotic stability and impulse free solutions

Definition (Asymptotic stability of switched DAE)

(swDAE) asymptotically stable $:\Leftrightarrow$ x is impulse free* and $x(t\pm) \to 0$ for $t \to \infty$

* i.e. $x[t] = 0$ $\forall t \in \mathbb{R}$; however jumps in x are still allowed

Let $\Pi_p := \Pi_{(E_p, A_p)}$ be the consistency projector of (E_p, A_p)

Impulse freeness condition

(IFC): $\forall p, q \in \{1, \ldots, \mathtt{m}\}: \quad E_q(I - \Pi_q)\Pi_p = 0$

Theorem (Trenn 2009)

(IFC) $\quad\Longleftrightarrow\quad$ *all solutions of* $E_\sigma \dot{x} = A_\sigma x$ *are impulse free* $\forall \sigma$

Sufficient conditions for impulse freeness

Index 1: $E_q(I - \Pi_q) = 0$ or Same consistency spaces: $(I - \Pi_q)\Pi_p = 0$

Stability for arbitrary switching

Consider **(swDAE)** with:

($\exists V_p$): $\forall p \in \{1, \ldots, \mathtt{m}\}$ $\exists$ Lyapunov function V_p for (E_p, A_p)

i.e. each DAE $E_p\dot{x} = A_p x$ is asymptotically stable

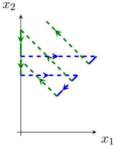
Lyapunov jump condition

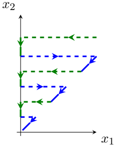
(LJC): $\forall p, q \in \{1, \ldots, \mathtt{m}\}$ $\forall x \in \mathfrak{C}_{(E_p, A_p)}$: $\quad V_q(\Pi_q x) \leq V_p(x)$

Theorem (Liberzon & Trenn 2009)

(IFC) $\wedge$ **($\exists V_p$)** $\wedge$ **(LJC)** $\implies$ **(swDAE)** *asymtotically stable* $\forall \sigma$

Examples 1a and 1b fulfill **(IFC)** and **($\exists V_p$)**, but only 1b fulfills **(LJC)**





Slow switching

Consider the set of switching signals with dwell time $\tau > 0$:

$$\Sigma^\tau := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, \mathtt{m}\} \;\middle|\; \begin{array}{l} \forall \text{ switching times} \\ t_i \in \mathbb{R}, i \in \mathbb{Z} : \\ t_{i+1} - t_i \geq \tau \end{array} \right\}.$$

Theorem (Liberzon & Trenn 2009)

$\exists \tau > 0$: **(IFC)** $\wedge$ **($\exists V_p$)** $\implies$ **(swDAE)** *asymptotically stable* $\forall \sigma \in \Sigma^\tau$

Examples 1a and 1b both fulfill **(IFC)** and **($\exists V_p$)**
$\Rightarrow$ both examples are asymptotically stable for slow switching

Remark

Result also holds for average dwell time.

Generalization to nonlinear switched DAEs

Previous results can be generalized to nonlinear switched DAEs:

$$E_\sigma(x)\dot{x} = f_\sigma(x)$$

where **(IFC)** has to be replaced by suitable nonlinear version, e.g. [Liberzon & Trenn 2012]:

$$\forall p, q \in \{1, \ldots, \mathtt{m}\}\ \forall x_0^- \in \mathfrak{C}_p\ \exists \text{ unique } x_0^+ \in \mathfrak{C}_q:\ x_0^+ - x_0^- \in \ker E_q(x_0^+)$$

where $\mathfrak{C}_p$ is the consistency manifold of $E_p(x)\dot{x} = f_p(x)$

Problem

Above **(IFC)** not invariant under nonlinear coordinate transformation!
A proper nonlinear generalization was recently published [Chen & Trenn 2023]

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Commutativity and stability of switched ODEs

Theorem (Narendra and Balakrishnan 1994)

Consider switched ODE

(swODE) $\dot{x} = A_\sigma x$

with A_p *Hurwitz,* $p \in \{1, 2, \ldots, \mathtt{m}\}$ *and commuting* A_p*, i.e.*

$$[A_p, A_q] := A_p A_q - A_q A_p = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{m}\} \tag{C}$$

$\Rightarrow$ **(swODE)** *asymptotically stable* $\forall \sigma$.

Proof idea: Consider switching times $t_0 < t_1 < \ldots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}(t-t_k)} e^{A_{p_{k-1}}(t_k - t_{k-1})} \cdots e^{A_{p_1}(t_2-t_1)} e^{A_{p_0}(t_1-t_0)} x_0 \\ &\overset{(C)}{=} e^{A_1 \Delta t_1} e^{A_2 \Delta t_2} \cdots e^{A_{\mathtt{m}} \Delta t_{\mathtt{m}}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

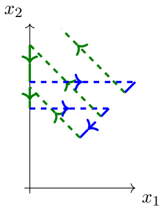
Generalization to **(swDAE)**

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Generalization - Questions

- Which matrices have to commute?
- What about the jumps?

Example 1a:
$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\right)$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$



$[A_1, A_2] = 0$, but unstable for fast switching

The matrix A^{diff}

Let (E,A) regular with $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)$, N nilpotent

consistency projector: $\Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Definition (differential "projector")

$$\Pi^{\text{diff}}_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

Lemma (Dynamics of DAE, Tanwani & Trenn 2010)

x *solves* $E\dot{x} = Ax \quad \iff \quad \dot{x} = \underbrace{\Pi^{\text{diff}}_{(E,A)} A}_{=: A^{\text{diff}}} x,\ x(0) \in \operatorname{im} \Pi_{(E,A)}$

Note: $A^{\text{diff}} = T \left[\begin{smallmatrix} J & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1}$, hence $\Pi_{(E,A)} A^{\text{diff}} = A^{\text{diff}} = A^{\text{diff}} \Pi_{(E,A)}$

Commutativity condition

(swDAE) $E_\sigma \dot{x} = A_\sigma x$

Theorem (Liberzon, Trenn & Wirth 2011)

(IFC) $\wedge$ **(**$\exists \boldsymbol{V_p}$**)** $\wedge$

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \ldots, \mathtt{m}\} \qquad \text{(C)}$$

$\implies$ **(swDAE)** *is asymptotically stable* $\forall \sigma$.

(IFC) $\wedge$ **(**$\exists \boldsymbol{V_p}$**)** $\wedge$ (C) $\implies$ $\exists$ *common quadratic Lyapunov function with*

$$V(\Pi_p x) \leq V(x) \quad \forall x \ \forall p$$

Remarkable: No explicit condition on jumps!

Proof idea and extenstions

Key property:

$$[A_p^{\text{diff}}, A_q^{\text{diff}}] = 0 \quad \forall p, q \in \{1, 2, \dots, \mathtt{m}\} \tag{C}$$

implies

$$[\Pi_p, A_q^{\text{diff}}] = 0 \quad \wedge \quad [\Pi_p, \Pi_q] = 0.$$

Consider switching times $t_0 < t_1 < \dots < t_k < t$ and $p_i := \sigma(t_i+)$, then

$$\begin{aligned} x(t) &= e^{A_{p_k}^{\text{diff}}(t-t_k)} \Pi_{p_k} e^{A_{p_{k-1}}^{\text{diff}}(t_k-t_{k-1})} \Pi_{p_{k-1}} \cdots e^{A_{p_1}^{\text{diff}}(t_2-t_1)} \Pi_{p_1} e^{A_{p_0}^{\text{diff}}(t_1-t_0)} \Pi_{p_0} x_0 \\ &\overset{(C)}{=} e^{A_1^{\text{diff}}\Delta t_1} \Pi_1 \; e^{A_2^{\text{diff}}\Delta t_2} \Pi_2 \cdots e^{A_{\mathtt{m}}^{\text{diff}}\Delta t_{\mathtt{m}}} \Pi_{\mathtt{m}} x_0 \end{aligned}$$

and $\Delta t_p \to \infty$ for at least one p and $t \to \infty$.

Extension to Lie-algebraic conditions

Commutativity $\Longrightarrow$ jointly diagonalizable $\Longleftrightarrow$ matrices form solvable Lie-algebra
$\rightsquigarrow$ recent results available [Raj & Pal 2021,2024]

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Evolution operator

$$x(t) = \underbrace{e^{A_k^{\text{diff}}(t-t_k)}\Pi_k e^{A_{k-1}^{\text{diff}}(t_k-t_{k-1})}\Pi_{k-1}\cdots e^{A_1^{\text{diff}}(t_2-t_1)}\Pi_1 e^{A_0^{\text{diff}}(t_1-t_0)}\Pi_0}_{=:\,\Phi^\sigma(t,t_0)} x(t_0-)$$

Let $\mathcal{M} := \left\{(A_p^{\text{diff}},\Pi_p)\ \middle|\ \text{corresponding to } (E_p,A_p), p=1,\ldots,\mathtt{m}\right\}$.

Definition (Set of all evolution matrices with fixed time span $t>0$)

$$\mathcal{S}_t := \{\Phi^\sigma(t,0)\ |\ \sigma \text{ arbitrary switching signal}\}$$
$$= \left\{\prod_{i=0}^{k} e^{A_i^{\text{diff}}\tau_i}\Pi_i\ \middle|\ (A_i^{\text{diff}},\Pi_i)\in\mathcal{M},\ \sum_{i=0}^{k}\tau_i = \Delta t,\ \tau_i>0\right\}$$

Lemma (Semi group, Trenn & Wirth 2012)

The set $\mathcal{S} := \bigcup_{t>0}\mathcal{S}_t$ is a semi group with $\mathcal{S}_{s+t} = \mathcal{S}_s\mathcal{S}_t := \{\Phi_s\Phi_t\ |\ \Phi_s\in\mathcal{S}_s, \Phi_t\in\mathcal{S}_t\}$

Exponential growth bound

Definition (Exponential growth bound)

For $t > 0$ the *exponential growth bound* of $E_\sigma \dot{x} = A_\sigma x$ is

$$\lambda_t(\mathcal{S}_t) := \sup_{\Phi_t \in \mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t} \in \mathbb{R} \cup \{-\infty, \infty\}$$

Definition implies for all solutions x of $E_\sigma \dot{x} = A_\sigma x$:

$$\|x(t)\| = \|\Phi_t x(0-)\| \leq \|\Phi_t\| \, \|x(0-)\| \leq e^{\lambda_t(\mathcal{S}_t)\, t} \|x(0-)\|$$

Difference to switched ODEs without jumps

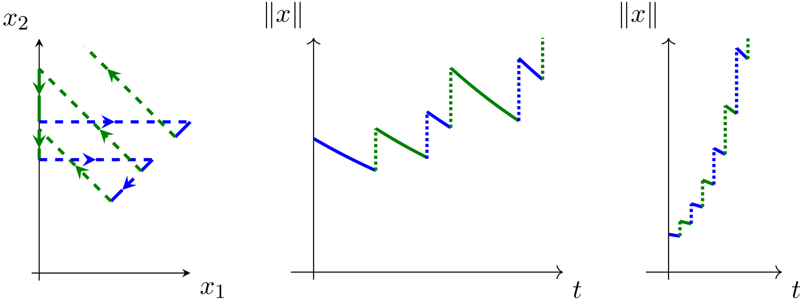
$\lambda_t(\mathcal{S}_t) = \pm\infty$ is possible!

All jumps are trivial, i.e. $\Pi_p = 0 \quad \Longleftrightarrow \quad \lambda_t(\mathcal{S}_t) = -\infty$

Infinite exponential growth bound

Example 1a revisited:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right) \quad (E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



For small dwell times: $\Phi_t \approx (\Pi_1 \Pi_2)^k = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^k = 2^{k-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

Lyapunov exponent of a switched DAE

Theorem (Boundedness of $\mathcal{S}_t$, Trenn & Wirth 2012)

$\mathcal{S}_t$ *is bounded* $\iff$ *the set of consistency projectors is product bounded*

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x$

Theorem (Lyapunov exponent well defined, Trenn & Wirth 2012)

Let the consistency projectors be product bounded and not all be trivial, then the (upper) Lyapunov exponent

$$\lambda(\mathcal{S}) := \lim_{t\to\infty} \lambda_t(\mathcal{S}_t) = \lim_{t\to\infty} \sup_{\Phi_t\in\mathcal{S}_t} \frac{\ln \|\Phi_t\|}{t}$$

of **(swDAE)** *is well defined and finite.*

Note that: **(swDAE)** uniformly exponentially stable

$:\Leftrightarrow \quad \exists M \geq 1, \mu > 0 : \|x(t)\| \leq M e^{-\mu t}\|x(0-)\| \quad \forall t \geq 0$

$\iff \quad \lambda(\mathcal{S}) \leq -\mu < 0$

Converse Lyapunov theorem for switched DAEs

For $\varepsilon > 0$ define "Lyapunov norm"

$$|||x|||_\varepsilon := \sup_{t>0} \sup_{\Phi_t \in \mathcal{S}_t} e^{-(\lambda(\mathcal{S})+\varepsilon)t} \|\Phi_t x\|$$

(swDAE) $\quad E_\sigma \dot{x} = A_\sigma x$

Theorem (Converse Lyapunov theorem, Trenn & Wirth 2012)

(swDAE) *is uniformly exponentially stable* $\forall \sigma$
$\Rightarrow V = |||\cdot|||_\varepsilon$ *is Lyapunov function for sufficiently small* $\varepsilon > 0$

In particular: $V(\Pi x) \leq V(x)$ for all consistency projectors Π

Non-smooth Lyapunov function

$|||\cdot|||_\varepsilon$ in general non-smooth.
"Smoothification" as in [Yin, Sontag & Wang 1996] might violate jump condition!

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Control task

Goal: Stabilization

Find σ such that $x(t) \to 0$ as $t \to \infty$.

Usual approach

State-depending switching $x \mapsto \sigma(x)$

Problems

- No solution theory available for state-dependent switched DAEs!
- State x may not be available for feedback control
  $\to$ observer with estimation $\widehat{x}$
  $\to$ non-matching switching signals $\sigma(x) \neq \sigma(\widehat{x})$, NO seperation principle

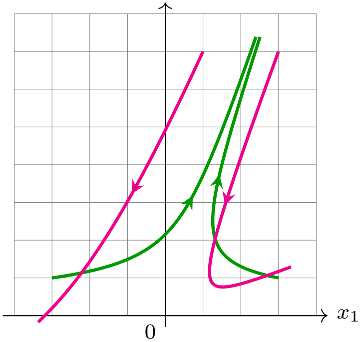
Alternative approach

Time-dependent switching $t \mapsto \sigma(t)$

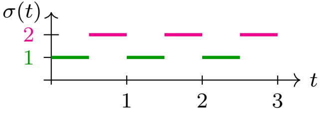
Example: Stabilization of switched ODEs

$$\dot{x} = A_\sigma x, \quad A_1 = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$$

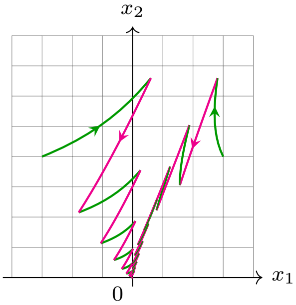
Unstable modes



Periodic switching signal:



$\Longrightarrow$ Stability:



Why does the example work?

Convex combination

$$\tfrac{1}{2}A_1 + \tfrac{1}{2}A_2 = \tfrac{1}{2}\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{Hurwitz!}$$

Classical averaging result

For switched ODE $\dot{x} = A_\sigma x$ any convex combination

$$\dot{x} = A_{\mathsf{av}} x, \quad A_{\mathsf{av}} := \sum_{k=1}^{\mathtt{m}} d_k A_k, \quad d_1, d_2, \ldots, d_{\mathtt{m}} \in [0,1], \ \sum_{k=1}^{\mathtt{m}} d_k = 1,$$

can be approximated arbitrarily well by sufficiently fast (periodic) switching.

Corollary

$\exists$ *Hurwitz convex combination* $\implies$ *Stabilizable by fast (time-dependent) switching*

The Mironchenko-Wirth-Wulff Approach

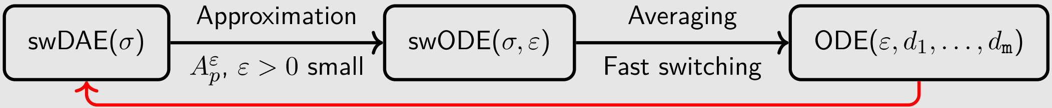
Key observation

$$e^{A^{\text{diff}}t}\Pi \approx e^{A^\varepsilon t} \quad \text{with } A^\varepsilon := T \begin{bmatrix} J & 0 \\ 0 & -\frac{1}{\varepsilon} I \end{bmatrix} T^{-1} \quad \text{hence} \quad E_\sigma \dot{x} = A_\sigma x \quad \approx \quad \dot{x} = A^\varepsilon_\sigma x$$

Theorem (Mironchenko, Wirth & Wulff 2013)

σ *stabilizes* $\dot{x} = A^\varepsilon_\sigma x \quad \forall \varepsilon \in (0, \varepsilon_0) \quad \Longrightarrow \quad \sigma$ *stabilizes* $E_\sigma \dot{x} = A_\sigma x$

Overall stabilization strategy



stabilization with suitable $d_1, \dots, d_{\mathtt{m}}$ and sufficiently high switching frequency

Discussion of the MWW-approach

No further assumptions needed for individual approximations

swDAE(σ) → swODE(σ, ε) $x_{\sigma,\mathfrak{p}}(t^-) - x^\varepsilon_{\sigma,\mathfrak{p}}(t) \to 0$ as $\varepsilon \to 0$

swODE(σ, ε) → ODE($\varepsilon, d_1, \ldots, d_{\mathfrak{m}}$) $x^\varepsilon_{\sigma,\mathfrak{p}}(t) - x^\varepsilon_{\mathsf{av}}(t) \to 0$ as $\mathfrak{p} \to 0$

Problem

For fixed $\varepsilon > 0$ it is possible that $x_{\sigma,\mathfrak{p}}(t^-) - x^\varepsilon_{\sigma,\mathfrak{p}}(t) \to \infty$ as $\mathfrak{p} \to 0$

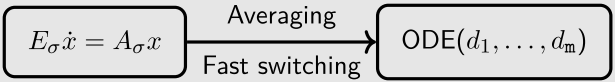
Underlying problem

Consistency projectors not explicitly considered:

- Destabilizing effect for fast switching
- Non-existence of averaged model

Direct approach

Directly utilize averaging approach for switched DAEs



Assumptions

- (E_k, A_k) regular and index-1 with Π_k, A_k^{diff}
- $\sigma : \mathbb{R} \to \{1, 2, \ldots, \mathtt{m}\}$ periodic with
  - period $\mathfrak{p} > 0$
  - duty cycles $d_1, \ldots, d_\mathtt{m} \in (0, 1)$ for fixed (periodic) mode sequence $(1, 2, \ldots, \mathtt{m})$

Existence of averaged ODE

When can **(swDAE)** be approximated by averaged ODE $\dot{x}_{\text{av}} = A_{\text{av}} x_{\text{av}}$?

Existence of an averaged model

Definition (Averaged model)

We call $\dot{x}_{\mathsf{av}} = A_{\mathsf{av}} x_{\mathsf{av}}$ an averaged model of $E_\sigma \dot{x} = A_\sigma x$
$:\Leftrightarrow \forall T > 0\ \forall x_0\ \forall \varepsilon > 0\ \exists x_0^{\mathsf{av}}\ \exists \overline{\mathfrak{p}} > 0\ \exists C > 0$

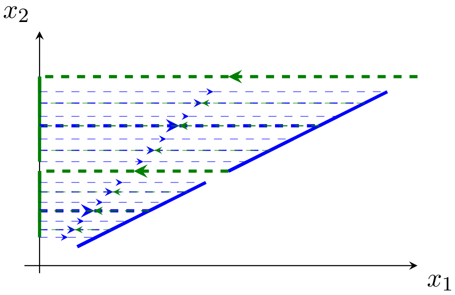
$$\forall \mathfrak{p} \in (0, \overline{\mathfrak{p}}): \quad \|x_{\sigma,\mathfrak{p}}(t^\pm) - x_{\mathsf{av}}(t)\| \leq C\mathfrak{p} \quad \forall t \in [\mathfrak{p}, T]$$

Problem

Averaged model does not always exist
(even for exponentially stable (swDAE)!

Example 1b revisited:

$$(E_1, A_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix} \right)$$
$$(E_2, A_2) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \right)$$



Underlying problem

Although $\{\Pi_1, \Pi_2\}$ is product bounded, jumps in $x_{\sigma,\mathfrak{p}}$ do not converge to zero as $\mathfrak{p} \to 0$

BUT: $x_{\sigma,\mathfrak{p}} \to x_{\text{av}}$ requires vanishing jumps because x_{av} is always continuous!

In which space would x_{av} evolve?

$x_{\sigma,\mathfrak{p}}((t+\mathfrak{p})^-) = \mathrm{e}^{A_{\mathtt{m}}^{\text{diff}} d_{\mathtt{m}} \mathfrak{p}} \Pi_{\mathtt{m}} \cdots \mathrm{e}^{A_2^{\text{diff}} d_2 \mathfrak{p}} \Pi_2 \cdot \mathrm{e}^{A_1^{\text{diff}} d_1 \mathfrak{p}} \Pi_1 x_{\sigma,\mathfrak{p}}(t^-)$ for $t = k\mathfrak{p}$

If averaged model exists then

$$x_{\text{av}}(t) = \lim_{\mathfrak{p}\to 0} x_{\sigma,\mathfrak{p}}((t+\mathfrak{p})^-) = \underbrace{\Pi_{\mathtt{m}} \cdots \Pi_2 \Pi_1}_{=: \Pi_\cap} x_{\text{av}}(t)$$

$\rightsquigarrow$ $x_{\text{av}}(t) \in \operatorname{im} \Pi_\cap$

Condition for vanishing jumps

$$\forall k \in \{1, 2, \ldots, \mathtt{m}\}: \quad \Pi_k \cdots \Pi_2 \Pi_1 \Pi_\cap = \Pi_\cap \quad \text{and} \quad \Pi_\cap \Pi_{\mathtt{m}} \cdots \Pi_{k+1} \Pi_k = \Pi_\cap$$

$$\iff \forall k \in \{1, 2, \ldots, \mathtt{m}\}: \quad \operatorname{im} \Pi_k \supseteq \operatorname{im} \Pi_\cap \quad \text{and} \quad \ker \Pi_k \subseteq \ker \Pi_\cap \qquad \text{(PA)}$$

Averaged model

For $E_\sigma \dot{x} = A_\sigma x$, $x(0^-) = x_0$, define averaged system:

$$\dot{x}_{\mathsf{av}} = \Pi_\cap A_{\mathsf{av}}^{\mathsf{diff}} \Pi_\cap x_{\mathsf{av}}, \quad x_{\mathsf{av}}(0) = \Pi_\cap x_0 \qquad (\Sigma_{\mathsf{av}})$$

where $A_{\mathsf{av}}^{\mathsf{diff}} := \sum_{k=1}^{\mathtt{m}} d_k A_k^{\mathsf{diff}}$ and $\Pi_\cap = \Pi_{\mathtt{m}} \ldots \Pi_2 \Pi_1$ with projector assumption

$$\forall k \in \{1, 2, \ldots, \mathtt{m}\}: \quad \operatorname{im} \Pi_k \supseteq \operatorname{im} \Pi_\cap \quad \text{and} \quad \ker \Pi_k \subseteq \ker \Pi_\cap \qquad \text{(PA)}$$

Theorem (Mostacciuolo, Trenn & Vasca 2017)

If (PA) *then* (Σ_{av}) is an averaged system for (swDAE), i.e.

$$\|x_{\sigma,\mathfrak{p}} - x_{\mathsf{av}}\|_\infty = O(p)$$

on every compact interval in $(0, \infty)$

Remarks on (PA) condition

- (PA) $\Longrightarrow$ $\Pi_\cap$ is a projector (converse is not true in general)
- (PA) depends on order of modes $\rightsquigarrow$ existence of averaged system depends on mode sequence
- $\Pi_i \Pi_j = \Pi_j \Pi_i \Longrightarrow$ (PA) (converse not true in general)

Stabilization via fast switching

Corollary

$\exists$ mode sequence satisfying (PA) and $d_1, \ldots, d_\mathfrak{m}$ such that (Σ_{av}) is exponentially stable
$\implies$ $\exists \mathfrak{p} > 0$ sufficiently small: $E_\sigma \dot{x} = A_\sigma x$ exponentially stable

Key steps of proof:

1. Chose $T > 0$ and $c < 1$ such that

$$\|x_{\mathsf{av}}(T)\| < c\|x_{\mathsf{av}}(T/2)\|$$

2. Chose $\mathfrak{p} > 0$ sufficiently small such that

$$x_{\sigma,\mathfrak{p}}(T^-) \approx x_{\mathsf{av}}(T) \quad \text{and} \quad x_{\sigma,\mathfrak{p}}(T/2^-) \approx x_{\mathsf{av}}(T/2)$$

so that we can conclude for some $\tilde{c} \in (c, 1)$

$$\|x_{\sigma,\mathfrak{p}}(T^-)\| < \tilde{c}\|x_{\sigma,\mathfrak{p}}(T/2^-)\|$$

3. Conclude exponential stability.

Summary

$$E_\sigma \dot{x} = A_\sigma x \quad \textbf{(swDAE)}$$

- **Stability**
  - Impulse freeness
  - Lyapunov jump condition for arbitrary switching
  - Stability under slow switching
  - Generalization to nonlinear case
  - Commutativity and stability
  - Converse Lyapuynov Theorem
- **Stabilization by fast switching**
  - MWW approach
  - Averaging approach