Proof $\begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{B}_{i/o} \iff \exists x: \quad \begin{array}{l} E\dot{x} = Ax + Bu \\ y = Cx \end{array}$

$\iff \exists x: \quad \begin{array}{l} SE\dot{x} = SAx + SBu \\ y = Cx \end{array}$

$\overset{x = Tz}{\iff} \exists z: \quad \begin{array}{l} SET\dot{z} = SATz + SBu \\ y = CTz \end{array}$

$\iff \begin{pmatrix} u \\ y \end{pmatrix} \in \bar{\mathcal{B}}_{i/o}$

In fact, $\begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathcal{B}_{full} \iff \begin{pmatrix} T^{-1}x \\ u \\ y \end{pmatrix} \in \bar{\mathcal{B}}_{full}$

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nilpotent DAE $\boxed{N\dot{x} = x + v}$

N nilpotent $:\iff N^{\nu} = 0$ for some $\nu \in \mathbb{N}$

$\left(\iff N^{n} = 0 \quad \text{where } N \in \mathbb{R}^{n \times n} \right)$

Assume x solves $N\dot{x} = x + v$

$\overset{N\frac{d}{dt}}{\implies} N^2 \ddot{x} = N\dot{x} + N\dot{v} = x + v + N\dot{v}$

$\overset{N\frac{d}{dt}}{\implies} N^3 \dddot{x} = N\dot{x} + N\dot{v} + N^2\ddot{v} = x + v + N\dot{v} + N^2\ddot{v}$

$\vdots$

$\overset{N\frac{d}{dt}}{\implies} \underbrace{N^{\nu}}_{=0} x^{(\nu)} = x + \sum_{i=0}^{\nu-1} N^i v^{(i)}$

Hence $x = -\sum_{i=0}^{\nu-1} N^i v^{(i)}$

In fact, the differential operator $(N\frac{d}{dt} - I) : (C^{\infty})^n \to (C^{\infty})^n$ is invertible with

$$\left(N\frac{d}{dt}-I\right)^{-1} = -\sum_{i=0}^{\nu-1} N^i \left(\frac{d}{dt}\right)^i$$

because

$$\left(N\frac{d}{dt}-I\right)\cdot\left(-\sum_{i=0}^{\nu-1} N^i\left(\frac{d}{dt}\right)^i\right)$$

$$= -\sum_{i=0}^{\nu-1} N^{i+1}\left(\frac{d}{dt}\right)^{i+1} + \sum_{i=0}^{\nu-1} N^i\left(\frac{d}{dt}\right)^i$$

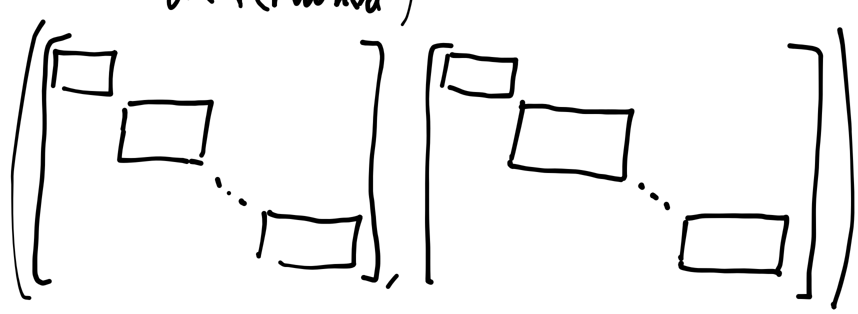
$$= -N^\nu\left(\frac{d}{dt}\right)^\nu + I = I$$

$$\rightsquigarrow \quad N\dot{x} = x + v \iff \left(N\frac{d}{dt}-I\right)x = v$$

$$\iff x = \left(N\frac{d}{dt}-I\right)^{-1} v$$

$$= -\sum_{i=0}^{\nu-1} N^i v^{(i)}$$

Type uDAE (underdetermined)

$(E,A) \cong$ 

where each diagonal block pair has the form

$$\left(\underbrace{\begin{bmatrix} 1 & 0 & & \\ & 1 & 0 & \\ & & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix}}_{k+1} k, \quad \underbrace{\begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \ddots \\ & & 0 & 1 \end{bmatrix}}_{k+1} k \right)$$

$\rightsquigarrow$ more columns than rows

Example: $\begin{bmatrix}1&0&0\\0&1&0\end{bmatrix}\dot{x} = \begin{bmatrix}0&1&0\\0&0&1\end{bmatrix}x + \begin{pmatrix}v_1\\v_2\end{pmatrix}$ $\qquad x = \begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$

$$\Leftrightarrow \begin{pmatrix}\dot{x}_1\\\dot{x}_2\end{pmatrix} = \begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} + \begin{bmatrix}0\\1\end{bmatrix}x_3 + \begin{pmatrix}v_1\\v_2\end{pmatrix}$$

$\rightsquigarrow$ ODE with free additional „input" x_3

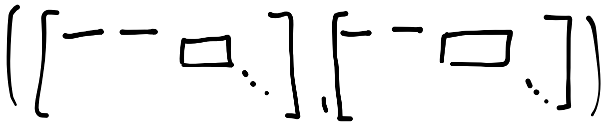
$\rightsquigarrow$ in general: $E\dot{x} = Ax + v \iff \dot{z} = \tilde{A}z + \tilde{B}z^f + v$

$$x \cong \begin{pmatrix}z\\z^f\end{pmatrix}$$

Attention: 0×1 blocks allowed

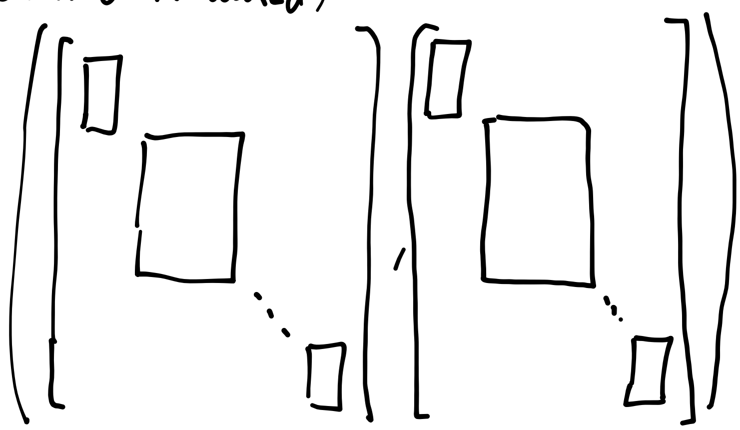
$$\underbrace{[\;]}_{=0\in\mathbb{R}^0}\dot{x} = \underbrace{[\;]}_{=0\in\mathbb{R}^0}x + \underbrace{[\;]}_{=0\in\mathbb{R}^0}$$ Satisfied $\forall x$ and $\dot{x}$

$\rightsquigarrow$ corresponds to common zero columns in E and A



Type oDAE (overdetermined)

$(E,A) \cong$



where each diagonal block pair has the form

$$\left(\begin{array}{c}\begin{bmatrix}0&&&\\1&0&&\\&1&\ddots&\\&&\ddots&0\\&&&1\end{bmatrix}_{k+1}\\ k\end{array},\ \begin{array}{c}\begin{bmatrix}1&&&\\0&1&&\\&0&\ddots&\\&&\ddots&1\\&&&0\end{bmatrix}_{k+1}\\ k\end{array}\right)$$

$k=0$ allowed

$\rightsquigarrow$ common zero row

Example: $\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} x + v \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad v(t) = \begin{pmatrix} 1 \\ t \\ t^2 \end{pmatrix}$

$$
\begin{aligned}
0 &= x_1(t) + v_1(t) \rightsquigarrow x_1(t) = -1 \\
\dot{x}_1(t) &= x_2(t) + v_2(t) \rightsquigarrow x_2(t) = -t \\
\dot{x}_2(t) &= 0 + v_3(t) \rightsquigarrow -1 = t^2 \qquad ⚡
\end{aligned}
$$

In general: - subset of equations uniquely determine x in terms of v
- remaining eq. constrain v

$$E\dot{x} = Ax + v \iff N\dot{x} = x + \tilde{v}$$
$$+ \text{ constraints on } v$$

Why no other jump rule?

- Consider QWF and $x = T\begin{pmatrix} v \\ w \end{pmatrix}$:

$$
\begin{aligned}
\dot{v} &= Jv \qquad v(t_0^-) = v_0 \\
N\dot{w} &= w \qquad w(t_0^-) = w_0
\end{aligned}
\qquad \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = T^{-1} x_0
$$

- On open interval (t_0, ∞) we have:

$$N\dot{w} = w \underset{N\frac{d}{dt}}{\Rightarrow} N^2 \ddot{w} = N\dot{w} = w \underset{N\frac{d}{dt}}{\Rightarrow} N^3 \dddot{w} = N\dot{w} = w \underset{N\frac{d}{dt}}{\Rightarrow} \ldots \Rightarrow \underbrace{N^n}_{=0} w^{(n)} = w$$

$\rightsquigarrow w(t_0^+) = 0$ (independent of w_0)

$\rightsquigarrow$ unique jump $w_0 = w(t_0^-) \mapsto w(t_0^+) = 0$

- We will show that $v(t_0^+) = v(t_0^-)$ for all distributional solutions of $\dot{v} = Jv$
  - assume otherwise: $v(t_0^+) \neq v(t_0^-)$ and $v[t_0] = \sum_{i=0}^{d} \alpha_i \delta_{t_0}^{(i)}$ for some finite $d \in \mathbb{N}$
  - then $\dot{v}[t_0] = J v[t_0] = \sum_{i=0}^{d} J \alpha_i \delta_{t_0}^{(i)}$

$$\overset{\shortparallel}{\sum_{i=0}^{d}} \alpha_i \delta_{t_0}^{(i+1)} + \underbrace{(v(t_0^+) - v(t_0^-))}_{=: \alpha_{-1} \neq 0} \delta_{t_0}$$

$$
\begin{aligned}
\rightsquigarrow \quad 0 \neq \alpha_{-1} = J\alpha_0 &\Rightarrow 0 \neq \alpha_0 = J\alpha_1 \\
&\Rightarrow 0 \neq \alpha_1 = J\alpha_2
\end{aligned}
$$

$$\Rightarrow \dots$$

$$\Rightarrow 0 \neq \alpha_{d-1} = \gamma \alpha_d$$

$$\Rightarrow 0 \neq \alpha_d = 0 \quad ⚡$$

- QWF $\rightsquigarrow$ unique jump $\begin{pmatrix} v_0 \\ w_0 \end{pmatrix} \mapsto \begin{pmatrix} v_0 \\ 0 \end{pmatrix} = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix}$

$$\rightsquigarrow x(t_0^+) = T \begin{pmatrix} v(t_0^+) \\ w(t_0^+) \end{pmatrix} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \underbrace{T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}}_{= \Pi_{(E,A)}} x_0$$