

Differential algebraic equations: Mini course 3
Model reduction for switched DAEs

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Switched DAEs

Switched DAE

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, & x(t_0^-) &= \mathcal{X}_0 \subseteq \mathbb{R}^n, \\ y &= C_\sigma x + D_\sigma u, \end{aligned} \quad (\text{swDAE})$$

- ▶ **Switching signal:** $\sigma : [t_0, t_f) \rightarrow \mathbb{M} := \{0, 1, \dots, m\}$
- ▶ **Modes:** $(E_k, A_k, B_k, C_k, D_k)$ for $k \in \mathbb{M}$
- ▶ **Singular system:** $E_k \in \mathbb{R}^{n \times n}$ usually singular

Motivation

- ▶ Electrical **circuits** with switches
- ▶ (Linearized) models of **water distribution networks** with valves
- ▶ Mathematical curiosity

Toy Example

Consider (swDAE) given by:

on $[t_0, s_1)$:

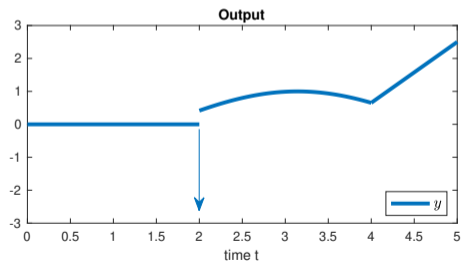
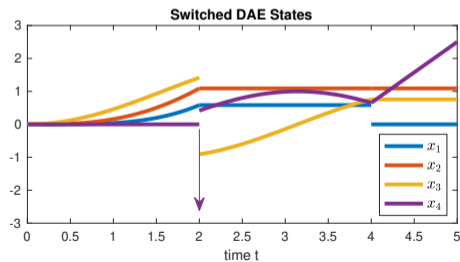
$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = 0$$

on $[s_1, s_2)$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \dot{x} = x + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = [0 \ 0 \ 0 \ 1] x$$

on $[s_2, t_f)$:

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} x$$
$$y = [0 \ 0 \ 0 \ 1] x$$

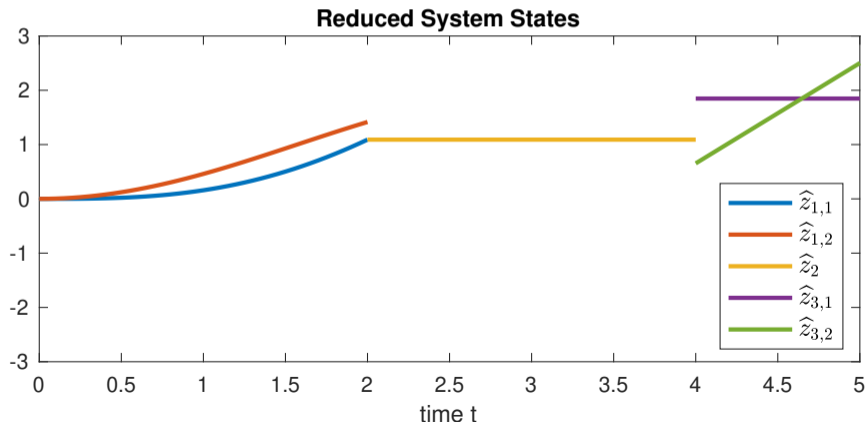


Model reduction

Model reduction task

(Approximately) same input-output behavior with smaller size switched system

For the toy example: possible to reduce to mode-dependent state-dimensions (2, 1, 2):

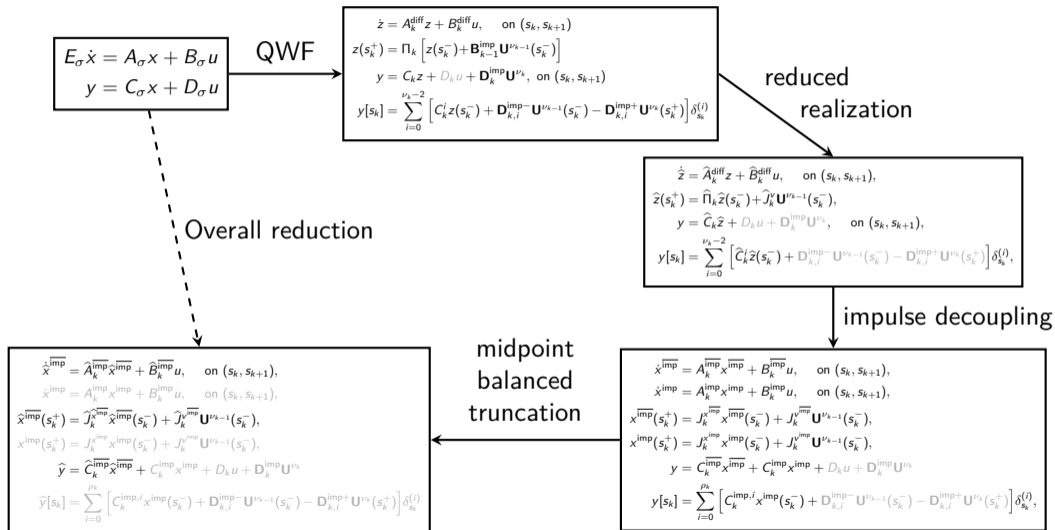


Key challenges and novelties

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, & x(t_0^-) &= \mathcal{X}_0 \subseteq \mathbb{R}^n, \\ y &= C_\sigma x + D_\sigma u, \end{aligned} \quad (\text{swDAE})$$

- ▶ **Fixed switching signal** on fixed **finite** time interval $[t_0, t_f)$
- ▶ **No stability** assumption for individual modes
- ▶ No restriction on **index** of DAE \rightsquigarrow **Dirac impulses** in state and output
- ▶ Allow **non-zero** (possibly inconsistent) **initial values** via subspace \mathcal{X}_0
- ▶ Reduced model should again be a **switched system** (with same switching signal)
- ▶ Allow **mode-dependent reduced state dimension**

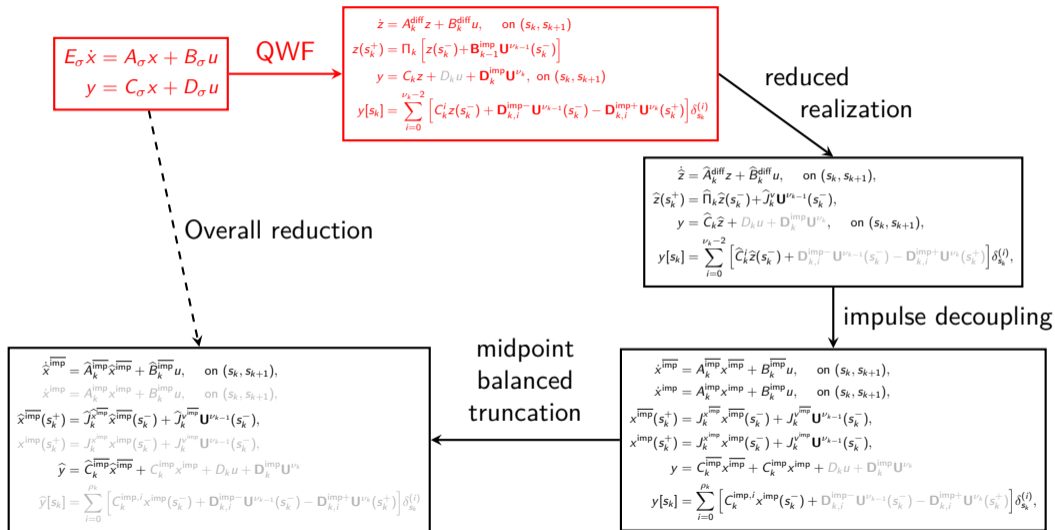
Overview: reduction approach



The three main steps

- 1. Reduced realization** (always possible, depends only on mode sequence)
 - ▶ Via **Wong-sequences** and **Quasi-Weierstrass** form rewrite (swDAE) as **switched ODE** with **jumps** and impulsive output of **same size**
 - ▶ Calculate **extended reachability** and **restricted unobservability subspaces**
 - ▶ Calculate **weak Kalman decomposition** and remove **unreachable/unobservable parts**
 - ▶ Define **reduced jump maps**, **output impulses**, **initial value space** and **initial projector**
- 2. Impulse decoupling** (structural assumption, depends only on mode sequence)
 - ▶ Key observation: Dirac impulse = **infinite peak**
 \leadsto do **not change** states which effect output Diracs
 - ▶ Assumption: States evolve in two disjoint **invariant (mode-dependent) subspaces**
- 3. Midpoint balanced truncation** (invertability assumption on Gramians)
 - ▶ Solution = Solution for **continuous input** + Solution for **discrete input**
 - ▶ Calculate **midpoint reachability Gramians** for continuous and discrete time system
 - ▶ Calculate **midpoint observability Gramians**
 - ▶ Apply mode-wise **balanced truncation** via the midpoint Gramians

From (swDAE) to switched ODE



Reminder: Solution decoupling based on Wong limits

$$E\dot{x} = Ax + Bu \quad \text{with QWF } (SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \quad (\text{DAE})$$

Definition (Some matrix definition based on Wong limits and QWF)

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

$$\Pi_{(E,A)}^{\text{diff}} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$$

$$\Pi_{(E,A)}^{\text{imp}} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

$$A^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} A$$

$$B^{\text{diff}} := \Pi_{(E,A)}^{\text{diff}} B$$

$$E^{\text{imp}} := \Pi_{(E,A)}^{\text{imp}} E$$

$$B^{\text{imp}} := \Pi_{(E,A)}^{\text{imp}} B$$

Theorem (Solution decomposition)

x solves (DAE) with $x(t_0^-) = x_0 \iff x = x^{\text{diff}} + x^{\text{imp}} \in \mathcal{V}^* \oplus \mathcal{W}^*$ where

$$\begin{aligned} \dot{x}^{\text{diff}} &= A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u, & x^{\text{diff}}(t_0^-) &= \Pi_{(E,A)} x_0, \\ E^{\text{imp}} \dot{x}^{\text{imp}} &= x^{\text{imp}} + B^{\text{imp}} u, & x^{\text{imp}}(t_0^-) &= (I - \Pi_{(E,A)}) x_0. \end{aligned}$$

Explicit impulsive solution formula

Lemma

x^{imp} solves $E^{\text{imp}} \dot{x}^{\text{imp}} = x^{\text{imp}} + B^{\text{imp}} u$, $x^{\text{imp}}(t_0^-) = (I - \Pi)x_0 \iff$

$$x^{\text{imp}} = \mathbf{B}^{\text{imp}} \mathbf{U}^\nu \quad \text{on } (t_0, t_f)$$

$$x^{\text{imp}}[t_0] = - \sum_{i=0}^{\nu-2} (E^{\text{imp}})^{i+1} (x_0 - \mathbf{B}^{\text{imp}} \mathbf{U}^\nu(t_0^+)) \delta_{t_0}^{(i)},$$

where $\nu \in \mathbb{N}$ is the nilpotency index of E^{imp} and

$$\mathbf{U}^\nu := \left[u^\top, \dot{u}^\top, \dots, u^{(\nu-1)\top} \right]^\top$$

$$\mathbf{B}^{\text{imp}} := - \left[B^{\text{imp}}, E^{\text{imp}} B^{\text{imp}}, \dots, (E^{\text{imp}})^{\nu-1} B^{\text{imp}} \right].$$

Equivalent switched ODE formulation

Corollary

For each $x_0 \in \mathbb{R}^n$ the *input-output behavior* of (swDAE) is *equal* to the one of

$$\begin{aligned} \dot{z} &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}), \quad z(t_0^-) = x_0 \\ z(s_k^+) &= \Pi_k \left[z(s_k^-) + \mathbf{B}_{k-1}^{\text{imp}} \mathbf{U}^{\nu_{k-1}}(s_k^-) \right], \quad k \geq 0 \\ y &= C_k z + D_k u + \mathbf{D}_k^{\text{imp}} \mathbf{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1}) \end{aligned}$$

$$y[s_k] = \sum_{i=0}^{\nu_k-2} \left[C_k^i z(s_k^-) + \mathbf{D}_{k-1,i}^{\text{imp}-} \mathbf{U}^{\nu_{k-1,i}}(s_k^-) - \mathbf{D}_k^{\text{imp}+} \mathbf{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}$$

where $\mathbf{B}_{-1}^{\text{imp}} := 0$, $\mathbf{D}_k^{\text{imp}} := C_k \mathbf{B}_k^{\text{imp}}$, $C_k^i := -C_k (E_k^{\text{imp}})^{i+1}$,
 $\mathbf{D}_{k,i}^{\text{imp}-} := -C_k (E_k^{\text{imp}})^{i+1} \mathbf{B}_{k-1}^{\text{imp}}$ and $\mathbf{D}_{k,i}^{\text{imp}+} := -C_k (E_k^{\text{imp}})^{i+1} \mathbf{B}_k^{\text{imp}}$.

Toy example - Wong matrices

The matrices $(\Pi_k, A_k^{\text{diff}}, B_k^{\text{diff}}, E_k^{\text{imp}}, B_k^{\text{imp}})$ are given by

$$\begin{aligned}k = 0 : & \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right), \\k = 1 : & \left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right), \\k = 2 : & \left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right).\end{aligned}$$

The corresponding feedthrough terms are then

$$\mathbf{D}_0^{\text{imp}} = 0_{1 \times 0}, \quad \mathbf{D}_1^{\text{imp}} = [0 \ -1], \quad \mathbf{D}_2^{\text{imp}} = 0_{1 \times 1}, \quad \mathbf{D}_{1,0}^{\text{imp}+} = [1 \ 0], \quad \mathbf{D}_{1,0}^{\text{imp}-} = 0_{1 \times 0}.$$

Toy example - switched ODE representation

on (s_0, s_1) :

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$z(s_0^+) = x_0$$

$$y = 0$$

$$y[s_0] = 0$$

on (s_1, s_2) :

$$\dot{z} = 0$$

$$z(s_1^+) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z(s_1^-)$$

$$y = [0 \ 0 \ 0 \ 1] z + [0 \ -1] \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

$$y[s_1] = \left([0 \ 0 \ -1 \ 0] z(s_1^-) - [1 \ 0] \begin{pmatrix} u(s_1^+) \\ \dot{u}(s_1^+) \end{pmatrix} \right) \delta_{s_1}$$

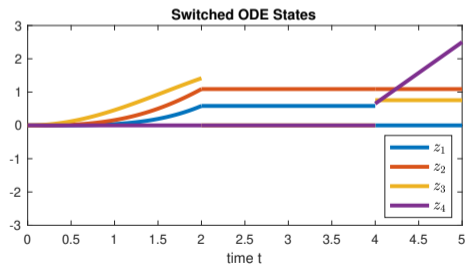
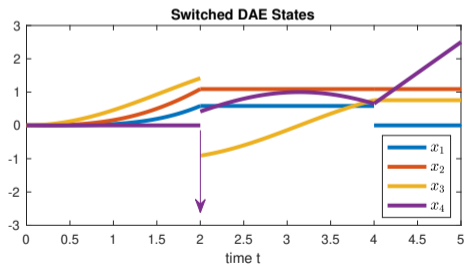
on (s_2, s_3) :

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} z$$

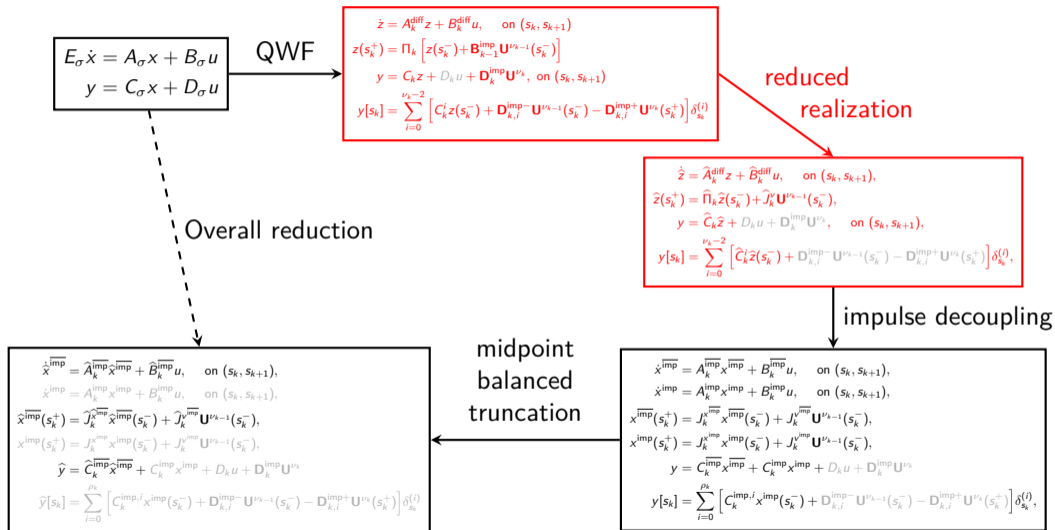
$$z(s_2^+) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} z(s_2^-) - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u(s_1^-) \\ \dot{u}(s_1^-) \end{pmatrix}$$

$$y = [0 \ 0 \ 0 \ 1] z$$

$$y[s_2] = 0$$



Reduced realization of switched ODE



Reduced realization - notation reset

$$\begin{aligned} \dot{z} &= A_k z + B_k u, & \text{on } (s_k, s_{k+1}), & \quad z(t_0^-) = x_0 \in \mathcal{X}_0, \\ z(s_k^+) &= J_k^z z(s_k^-) + J_k^v v_k, & k \geq 0, \\ y &= C_k z, & \text{on } (s_k, s_{k+1}), \\ y[s_k] &= \sum_{i=0}^{\rho_k} C_k^i z(s_k^-) \delta_{s_k}^{(i)}, & k \geq 0, \end{aligned}$$

↓ reduction

$$\begin{aligned} \dot{\hat{z}} &= \hat{A}_k \hat{z} + \hat{B}_k u, & \text{on } (s_k, s_{k+1}), & \quad \hat{z}(t_0^-) = \hat{z}_0(x_0), \\ \hat{z}(s_k^+) &= \hat{J}_k^z \hat{z}(s_k^-) + \hat{J}_k^v v_k, & k \geq 0, \\ y &= \hat{C}_k \hat{z}, & \text{on } (s_k, s_{k+1}), \\ y[s_k] &= \sum_{i=0}^{\rho_k} \hat{C}_k^i \hat{z}(s_k^-) \delta_{s_k}^{(i)}, & k \geq 0, \end{aligned}$$

Recall: Kalman decomposition

Reachable subspace for $\dot{x} = Ax + Bu$

$\mathcal{R} := \langle A \mid \text{im } B \rangle := \text{im}[B, AB, \dots, A^{n-1}B] \rightsquigarrow$ smallest A -inv. subspace containing $\text{im } B$

Unobservable subspace for $\dot{x} = Ax, y = Cx$

$\mathcal{U} := \langle \ker C \mid A \rangle := \ker[C/CA/\dots/CA^{n-1}] \rightsquigarrow$ largest A -inv. subspace contained in $\ker C$

Kalman decomposition

Choose coordinate transformation $P = [P^1, P^2, P^3, P^4]$ such that

$$\text{im } P^1 = \mathcal{R} \cap \mathcal{U}, \quad \text{im}[P^1, P^2] = \mathcal{R}, \quad \text{im}[P^1, P^3] = \mathcal{U}$$

then $(P^{-1}AP, P^{-1}B, CP)$ is a **Kalman decomposition**:

$$\left(\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, [0 \ C_2 \ 0 \ C_4] \right)$$

$\rightsquigarrow (A_{22}, B_2, C_2)$ has **same input-output behavior** as (A, B, C) for $x_0 \in \mathcal{R}$

Removing unreachable/unobservable states

Reduced realization: Basic idea

Remove unreachable/unobservable states

↪ reduced system with same input-output behavior

Challenges for switched DAE

- ▶ **Structurally unreachable**: States evolve within consistency subspace
- ▶ **Initial value** before switch structurally **unreachable** for current mode
- ▶ Reachable and unobservable subspaces **fully time-varying** for switched systems

Example to illustrate time-varying nature of reachable space:

$$\dot{x} = 0 \cdot x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \text{ on } [t_0, s_1), \quad \dot{x} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \text{ on } [s_1, t_f)$$

$$\mathcal{R}_{[t_0, t)} = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ for } t \in (t_0, s_1], \quad \mathcal{R}_{[t_0, t)} = \text{im} \begin{bmatrix} \cos(t-s_1) & 0 \\ \sin(t-s_1) & 0 \\ 0 & 1 \end{bmatrix} \text{ for } t \in (s_1, t_f)$$

Weak Kalman decomposition

Definition

- ▶ $\overline{\mathcal{R}} \subseteq \mathbb{R}^n$ is called **extended reachable subspace**
: $\iff \overline{\mathcal{R}}$ is A -invariant and contains $\text{im } B$ (and hence \mathcal{R})
- ▶ $\underline{\mathcal{U}} \subseteq \mathbb{R}^n$ is called **restricted unobservable subspace**
: $\iff \underline{\mathcal{U}}$ is A -invariant and is contained in $\ker C$ (and hence in \mathcal{U})

Weak Kalman decomposition

Choose coordinate transformation $\overline{P} = [\overline{P}^1, \overline{P}^2, \overline{P}^3, \overline{P}^4]$ such that

$$\text{im } \overline{P}^1 = \overline{\mathcal{R}} \cap \underline{\mathcal{U}}, \quad \text{im}[\overline{P}^1, \overline{P}^2] = \overline{\mathcal{R}}, \quad \text{im}[\overline{P}^1, \overline{P}^3] = \underline{\mathcal{U}}$$

then $(\overline{P}^{-1}A\overline{P}, \overline{P}^{-1}B, C\overline{P})$ is a **Weak Kalman decomposition**:

$$\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, [0 \ C_2 \ 0 \ C_4] \right)$$

$\rightsquigarrow (A_{22}, B_2, C_2)$ has **same input-output behavior** as (A, B, C) for $x_0 \in \overline{\mathcal{R}}$

Sequence of extended reachable / restricted unobservable subspaces

$$\dot{z} = A_k z + B_k u, \quad z(t_0^-) = x_0 \in \mathcal{X}_0,$$

$$z(s_k^+) = J_k^z z(s_k^-) + J_k^v v_k$$

Back to switched ODE with jumps and Diracs:

$$y = C_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} C_k^i z(s_k^-) \delta_{s_k}^{(i)}$$

Lemma (Exact reachable / unobservable subspaces)

$\mathcal{M}_k^\sigma := \mathcal{R}_{[t_0, s_{k+1})}^\sigma$ and $\mathcal{N}_k^\sigma := \mathcal{U}_{(s_k, t_f]}^\sigma$ are recursively given by:

$$\mathcal{M}_{-1}^\sigma = \mathcal{X}_0, \quad \mathcal{M}_k^\sigma := \mathcal{R}_k + e^{A_k \tau_k} (J_k^x \mathcal{M}_{k-1}^\sigma + \text{im } J_k^v), \quad k = 0, 1, \dots, m,$$

$$\mathcal{N}_m^\sigma = \mathcal{U}_m, \quad \mathcal{N}_k^\sigma = \mathcal{U}_k \cap e^{-A_k \tau_k} (((J_k^x)^{-1} \mathcal{N}_{k+1}^\sigma) \cap \mathcal{U}_{k+1}^{\text{imp}}), \quad k = m-1, \dots, 0,$$

Key fact

For any subspace $\mathcal{V} \subseteq \mathbb{R}^n$ and any $A \in \mathbb{R}^{n \times n}$: $\langle \mathcal{V} \mid A \rangle \subseteq e^{At} \mathcal{V} \subseteq \langle A \mid \mathcal{V} \rangle$

Sequence of extended reachable / restricted unobservable subspaces

$$\dot{z} = A_k z + B_k u, \quad z(t_0^-) = x_0 \in \mathcal{X}_0,$$

$$z(s_k^+) = J_k^z z(s_k^-) + J_k^v v_k$$

Back to switched ODE with jumps and Diracs:

$$y = C_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} C_k^i z(s_k^-) \delta_{s_k}^{(i)}$$

Definition (extended reachable / restricted unobservable subspaces)

$\bar{\mathcal{R}}_k \subseteq \mathcal{R}_{[t_0, s_{k+1}]^\sigma}$ and $\underline{\mathcal{U}}_k \subseteq \mathcal{U}_{(s_k, t_f]^\sigma}$ are recursively given by:

$$\bar{\mathcal{R}}_{-1} := \mathcal{X}_0, \quad \bar{\mathcal{R}}_k := \mathcal{R}_k + \langle A_k \mid J_k^x \bar{\mathcal{R}}_{k-1} + \text{im } J_k^v \rangle, \quad k = 0, 1, \dots, m,$$

$$\underline{\mathcal{U}}_m := \mathcal{U}_m, \quad \underline{\mathcal{U}}_k := \mathcal{U}_k \cap \langle ((J_k^x)^{-1} \underline{\mathcal{U}}_{k+1}) \cap \mathcal{U}_{k+1}^{\text{imp}} \mid A_k \rangle, \quad k = m-1, \dots, 0,$$

Key fact

For any subspace $\mathcal{V} \subseteq \mathbb{R}^n$ and any $A \in \mathbb{R}^{n \times n}$: $\langle \mathcal{V} \mid A \rangle \subseteq e^{At} \mathcal{V} \subseteq \langle A \mid \mathcal{V} \rangle$

Reduced realization via weak Kalman decomposition

For each mode k : $\overline{\mathcal{R}}_k, \underline{\mathcal{U}}_k \rightsquigarrow$ weak Kalman decomposition:

$$\begin{bmatrix} * \\ W_k \\ * \\ * \end{bmatrix} A_k \begin{bmatrix} * & V_k & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ 0 & \widehat{A}_k & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \quad \begin{bmatrix} * \\ W_k \\ * \\ * \end{bmatrix} B_k = \begin{bmatrix} * \\ \widehat{B}_k \\ 0 \\ 0 \end{bmatrix}$$

$$C_k \begin{bmatrix} * & V_k & * & * \end{bmatrix} = \begin{bmatrix} 0 & \widehat{C}_k & 0 & * \end{bmatrix}$$

$$\widehat{C}_k^i := C_k^i V_{k-1},$$

$$\widehat{J}_k^z := W_k J_k^z V_{k-1},$$

$$\widehat{J}_k^v := W_k J_k^v$$

$$\dot{\widehat{z}} = \widehat{A}_k \widehat{z} + \widehat{B}_k u, \quad \widehat{z}(t_0^-) = \Pi^{x_0} x_0 \in \widehat{\mathcal{X}}_0,$$

$$\widehat{z}(s_k^+) = \widehat{J}_k^z \widehat{z}(s_k^-) + \widehat{J}_k^v v_k$$

Reduced sw. ODE with jumps and Diracs:

$$y = \widehat{C}_k z, \quad y[s_k] = \sum_{i=0}^{\rho_k} \widehat{C}_k^i \widehat{z}(s_k^-) \delta_{s_k}^{(i)}$$

Toy example - reduced realization

on (s_0, s_1) :

$$\dot{z} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$z(s_0^+) = x_0$$

$$y = 0$$

$$y[s_0] = 0$$

on (s_1, s_2) :

$$\dot{z} = 0$$

$$z(s_1^+) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} z(s_1^-)$$

$$y = [0 \ 0 \ 0 \ 1] z + [0 \ -1] \begin{pmatrix} u \\ \dot{u} \end{pmatrix}$$

$$y[s_1] = \left([0 \ 0 \ -1 \ 0] z(s_1^-) - [1 \ 0] \begin{pmatrix} u(s_1^+) \\ \dot{u}(s_1^+) \end{pmatrix} \right) \delta_{s_1}$$

on (s_2, s_3) :

$$\dot{z} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} z$$

$$z(s_2^+) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} z(s_2^-) - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u(s_1^-) \\ \dot{u}(s_1^-) \end{pmatrix}$$

$$y = [0 \ 0 \ 0 \ 1] z$$

$$y[s_2] = 0$$

$$\bar{\mathcal{R}}_0 = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathcal{R}}_1 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\bar{\mathcal{R}}_2 = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathcal{U}}_0 = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\mathcal{U}}_1 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathcal{U}}_2 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Toy example - reduced realization

on (s_0, s_1) :

$$\begin{aligned}\hat{z} &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \hat{z} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= 0 \\ y[s_0] &= 0\end{aligned}$$

on (s_1, s_2) :

$$\begin{aligned}\hat{z} &= 0 \\ \hat{z}(s_1^+) &= [1 \ 0] \hat{z}(s_1^-) \\ y &= [0 \ -1] \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \\ y[s_1] &= \left([0 \ -1] \hat{z}(s_1^-) - [1 \ 0] \begin{pmatrix} u(s_1^+) \\ \dot{u}(s_1^+) \end{pmatrix} \right) \delta_{s_1}\end{aligned}$$

on (s_2, s_3) :

$$\begin{aligned}\hat{z} &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \hat{z} \\ \hat{z}(s_2^+) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} z(s_2^-) - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} u(s_2^-) \\ \dot{u}(s_2^-) \end{pmatrix} \\ y &= [0 \ 1] \hat{z} \\ y[s_2] &= 0\end{aligned}$$

$$\bar{\mathcal{R}}_0 = \text{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\bar{\mathcal{R}}_1 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

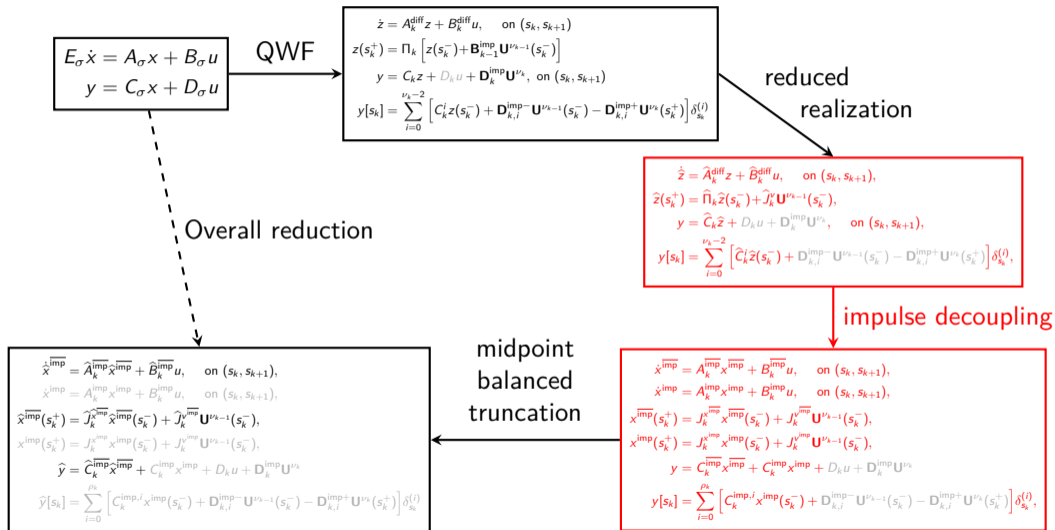
$$\bar{\mathcal{R}}_2 = \text{im} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underline{\mathcal{U}}_0 = \text{im} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\mathcal{U}}_1 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\underline{\mathcal{U}}_2 = \text{im} \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Impulse decoupling



Approximation of Dirac impulses?

Assume output Dirac is given by $y[s_k] = C_k^0 z(s_k^-) \delta_{s_k}$

\rightsquigarrow model reduction $\hat{y}[s_k] = \hat{C}_k^0 \hat{z}(s_k^-) \delta_{s_k}$

\rightsquigarrow error $\varepsilon := C_k^0 z(s_k^-) - \hat{C}_k^0 \hat{z}(s_k^-)$ leads to output error $y[s_0] - \hat{y}[s_0] = \varepsilon \delta_{s_0}$

\rightsquigarrow arbitrarily small approximation error leads to **infinite error peak**

Conclusion for model reduction

Unclear how to **quantify** error in Dirac impulses (especially for higher order Diracs)

\rightsquigarrow do **not reduce** parts of states which effect output Diracs

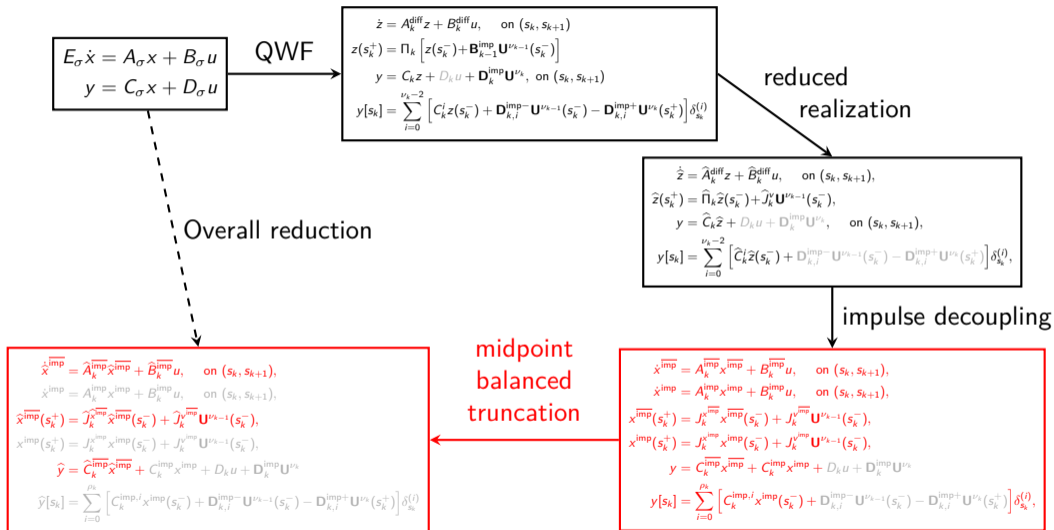
\rightsquigarrow apply further model reduction only on the **impulse-unobservable part** of the state

Impulse decoupling assumption

For each mode there exists a state **decomposition** $\mathbb{R}^{n_k} = \mathcal{X}_k^{\text{imp}} \oplus \overline{\mathcal{X}_k^{\text{imp}}}$ s.t.:

1. $\overline{\mathcal{X}_{k-1}^{\text{imp}}} \subseteq \ker[C_k^0 / C_k^1 / \dots / C_k^{\nu_k-2}]$
2. $\mathcal{X}_k^{\text{imp}}$ and $\overline{\mathcal{X}_k^{\text{imp}}}$ are A_k -invariant
3. $J_k^z \mathcal{X}_{k-1}^{\text{imp}} \subseteq \mathcal{X}_k^{\text{imp}}$ and $J_k^z \overline{\mathcal{X}_{k-1}^{\text{imp}}} \subseteq \overline{\mathcal{X}_k^{\text{imp}}}$

Midpoint balanced truncation



Notation reset

$$\begin{aligned} \dot{x} &= A_k x + B_k u, & \text{on } (s_k, s_{k+1}), & & x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^v v_k, & k \geq 0, & & \\ y &= C_k x, & \text{on } (s_k, s_{k+1}), & & \end{aligned}$$

↓
reduction

$$\begin{aligned} \dot{\hat{x}} &= \hat{A}_k \hat{x} + \hat{B}_k u, & \text{on } (s_k, s_{k+1}), & & \hat{x}(t_0^-) = \hat{x}_0(x_0), \\ \hat{x}(s_k^+) &= \hat{J}_k^x \hat{x}(s_k^-) + \hat{J}_k^v v_k, & k \geq 0, & & \\ y &= \hat{C}_k \hat{x}, & \text{on } (s_k, s_{k+1}), & & \end{aligned}$$

Challenge: Two types of inputs

$$\begin{aligned} \dot{x} &= A_k x + B_k u, & \text{on } (s_k, s_{k+1}), & & x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^y v_k, & k \geq 0, & & \\ y &= C_k x, & \text{on } (s_k, s_{k+1}), & & \end{aligned} \quad (\text{swODE})$$

Two types of input

- ▶ **Continuous input u** : Effects $\dot{x} = A_k x + B_k u$ on (s_k, s_{k+1})
- ▶ **Discrete input v_k** : Effects $x(s_k^+) = J_k^x x(s_k^-) + J_k^y v_k$ at switching times s_k

Lemma (Input decoupling)

x solves (swODE) : $\iff x = x_u + x_v$ where

- ▶ x_u solves (swODE) with $v_k = 0$ and $x_u(t_0^-) = 0$
- ▶ x_v solves (swODE) with $u = 0$ and $x_v(t_0^-) = x_0$

Continuous-time Gramians

Definition (Local time-dependent Gramians)

Local reachability Gramian: $P_k(t) := \int_{s_k}^t e^{A_k(\tau-s_k)} B_k B_k^\top e^{A_k^\top(\tau-s_k)} d\tau$

Local observability Gramian: $Q_k(t) := \int_t^{s_{k+1}} e^{A_k^\top(s_{k+1}-\tau)} C_k^\top C_k e^{A_k(s_{k+1}-\tau)} d\tau$

Definition (Global time-varying Gramians)

- ▶ Global reachability Gramian:

$$P^\sigma(t) := P_0(t) \text{ for } t \in (t_0, s_1)$$

$$P^\sigma(t) := e^{A_k(t-s_k)} J_k^x P^\sigma(s_k^-) (J_k^x)^\top e^{A_k^\top(t-s_k)} + P_k(t) \text{ for } t \in (s_k, s_{k+1})$$

- ▶ Global observability Gramian:

$$Q^\sigma(t) := Q_m(t) \text{ for } t \in (s_m, t_f)$$

$$Q^\sigma(t) := e^{A_k^\top(s_{k+1}-t)} (J_k^x)^\top Q^\sigma(s_{k+1}^+) J_k^x e^{A_k^\top(s_{k+1}-t)} + Q_k(t) \text{ for } t \in (s_k, s_{k+1})$$

Energy interpretation Gramians

Theorem (Reachability Gramian and input energy)

Consider (swODE) with $v_k = 0$ and $x_0 = 0$ and assume that $\mathbf{P}_k^\sigma(t^-)$ and $P_k(t)$ are **positive definite** for all $t \in (t_0, t_f)$. Then for all $x_t \in \mathbb{R}^{n_k}$:

$$\min_{\substack{u \text{ s.t.} \\ 0 \xrightarrow{u} x_t}} \int_{t_0}^t u(\tau)^\top u(\tau) d\tau = x_t^\top (\mathbf{P}_k^\sigma(t^-))^{-1} x_t$$

Theorem (Observability Gramian)

Consider (swODE) with zero input. Then for all $t \in (t_0, t_f)$

$$\int_t^{t_f} y(\tau)^\top y(\tau) d\tau = x(t^+)^\top \mathbf{Q}_k^\sigma(t^+) x(t^+)$$

Midpoint Gramians

Definition

- ▶ Midpoint reachability Gramian: $\bar{\mathbf{P}}_k^\sigma := \mathbf{P}^\sigma\left(\frac{s_k + s_{k+1}}{2}\right)$
- ▶ Midpoint observability Gramian: $\bar{\mathbf{Q}}_k^\sigma := \mathbf{Q}^\sigma\left(\frac{s_k + s_{k+1}}{2}\right)$

Intuition/Assumption

States which are **difficult to reach and observe at midpoint** of interval (s_k, s_{k+1}) (quantified by $\bar{\mathbf{P}}_k^\sigma$ and $\bar{\mathbf{Q}}_k^\sigma$) are also difficult to reach and observe **on the whole (finite) time interval**.

Midpoint balanced truncation

Use classical **balanced truncation** for each mode w.r.t. **midpoint Gramians**

Problem

Effect of **discrete input** v_k not yet considered!

Discrete time midpoint dynamics

$$\begin{aligned} \dot{x} &= A_k x, & \text{on } (s_k, s_{k+1}), & & x(t_0^-) &= x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^v v_k, & k \geq 0, & & & \end{aligned} \quad (\text{swODE})$$

Lemma (Solutions at midpoints)

The sequence $x_k^m := x\left(\frac{s_k + s_{k+1}}{2}\right)$ of solution midpoints of (swODE) satisfy the linear (rectangular) discrete-time system:

$$x_{k+1}^m = A_k^m x_k^m + B_k^m v_k$$

where

$$A_k^m := e^{A_k \tau_k / 2} J_k^x e^{A_{k-1} \tau_{k-1} / 2} \in \mathbb{R}^{n_k \times n_{k-1}} \quad \text{and} \quad B_k^m := e^{A_k \tau_k / 2} J_k^v$$

Overall midpoint reachability Gramians

Definition (Discrete-time reachability Gramians)

$$\mathbf{P}_{-1}^m := \gamma \mathbf{X}_0 \mathbf{X}_0^\top \quad \text{and} \quad \mathbf{P}_k^m = \mathbf{A}_k^m \mathbf{P}_{k-1}^m \mathbf{A}_k^{m\top} + \mathbf{B}_k^m \mathbf{B}_k^{m\top}$$

where \mathbf{X}_0 is an orthogonal basis matrix of \mathcal{X}_0 .

Definition (Overall midpoint reachability Gramian)

$$\mathbf{P}_k^\lambda := \bar{\mathbf{P}}_k^\sigma + \lambda \mathbf{P}_k^m$$

Role of parameters γ and λ

- ▶ γ : How difficult is it to reach the **initial value**?
- ▶ λ : Cost **relation** between **discrete** input v_k and **continuous** input v_k

Model reduction based on midpoint Gramians

$$\begin{aligned} \dot{x} &= A_k x + B_k u, & \text{on } (s_k, s_{k+1}), & \quad x(t_0^-) = x_0 \in \mathcal{X}_0, \\ x(s_k^+) &= J_k^x x(s_k^-) + J_k^y v_k, & k \geq 0, & \\ y &= C_k x, & \text{on } (s_k, s_{k+1}), & \end{aligned} \quad (\text{swODE})$$

Midpoint Gramians for each mode interval (s_k, s_{k+1}) : \mathbf{P}_k^λ and $\bar{\mathbf{Q}}_k$

Balanced truncation w.r.t. midpoint Gramians

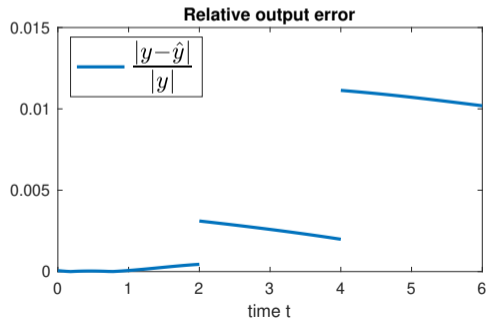
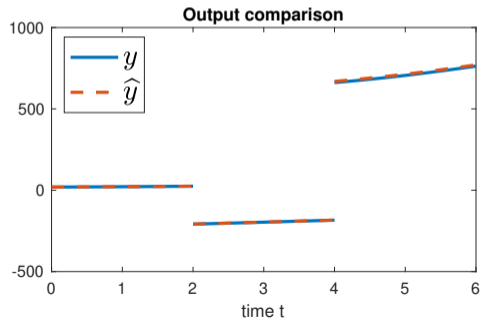
1. Find modewise **coordinate transformation** T_k such that Gramians are **equal and diagonal**:

$$T_k^{-1} \mathbf{P}_k^\lambda T_k^{-\top} = \begin{bmatrix} \sigma_k^1 & & \\ & \ddots & \\ & & \sigma_k^n \end{bmatrix} = T_k^\top \bar{\mathbf{Q}}_k T_k \quad (\text{Hankel singular values})$$

2. Choose **reduction size** r_k such that $\sigma_k^{r_k}$ sufficiently small
3. **Left and right projector matrices** $T_k = [V_k, *]$ and $T_k^{-1} = [W_k^\top, *]^\top$
4. Reduced system: $\hat{A}_k = W_k A_k V_k$, $\hat{B}_k = W_k B_k$, $\hat{C}_k = W_k C_k$, $\hat{J}_k^x = W_k J_k^x V_{k-1}$, $\hat{J}_k^y = W_k J_k^y$

Medium size academic example

- ▶ (swODE) state dimensions: $n_0 = 50$, $n_1 = 60$, $n_2 = 40$
- ▶ Coefficient matrices randomly chosen, single input and single output
- ▶ Discrete input $v_k = (u(s_k), \dot{u}(s_k))$
- ▶ Initial values subspace: $\mathcal{X}_0 = \mathbb{R}^5$
- ▶ Reachability Gramian parameters: $\gamma = 0.1$ and $\lambda = 1$
- ▶ Hankel singular values threshold: $\varepsilon_0 = \varepsilon_1 = \varepsilon_2 = 0.001$
- ▶ Reduced system state dimensions: $\hat{n}_0 = 8$, $\hat{n}_1 = 10$, $\hat{n}_2 = 6$



Summary: Model reduction for switched DAEs

- 1. Reduced realization** (always possible, depends only on mode sequence)
 - ▶ Via **Wong-sequences** and **Quasi-Weierstrass** form rewrite (swDAE) as **switched ODE** with **jumps** and impulsive output of **same size**
 - ▶ Calculate **extended reachability** and **restricted unobservability subspaces**
 - ▶ Calculate **weak Kalman decomposition** and remove **unreachable/unobservable parts**
 - ▶ Define **reduced jump maps**, **output impulses**, **initial value space** and **initial projector**
- 2. Impulse decoupling** (structural assumption, depends only on mode sequence)
 - ▶ Key observation: Dirac impulse = **infinite peak**
 \leadsto do **not change** states which effect output Diracs
 - ▶ Assumption: States evolve in two disjoint **invariant (mode-dependent) subspaces**
- 3. Midpoint balanced truncation** (invertability assumption on Gramians)
 - ▶ Solution = Solution for **continuous input** + Solution for **discrete input**
 - ▶ Calculate **midpoint reachability Gramians** for continuous and discrete time system
 - ▶ Calculate **midpoint observability Gramians**
 - ▶ Apply mode-wise **balanced truncation** via the midpoint Gramians

Remaining challenges and literature

Remaining challenges

- ▶ Precise **rank decisions** required for reduced realization
- ▶ Impulse decoupling assumption **not constructive**
- ▶ Large-scale **matrix-exponentials** are required for midpoint balanced truncation
- ▶ **Switching signal** must be known a-priori
- ▶ **Error bounds** for midpoint balanced truncation

References:

- ▶ Hossain & T. (2024): Model reduction for switched differential-algebraic equations with known switching signal, submitted to DAE-Panel
- ▶ Hossain & T. (2023): Reduced realization for switched linear systems with known mode sequence, Automatica
- ▶ Hossain & T. (2024): Midpoint based balanced truncation for switched linear systems with known switching signal, IEEE TAC