Differential algebraic equations: Mini course 2

Inconsistent initial values and distributional solutions

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Explicit solution formula for regular DAEs

$$E\dot{x} = Ax + Bu \qquad\qquad (E,A) \overset{S,T}{\cong} \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix}\right)$$

Definition (Consistency projector, differential/impulsive selector)

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} \qquad \Pi^{\mathrm{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S \qquad \Pi^{\mathrm{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$$

$$A^{\mathrm{diff}} := \Pi^{\mathrm{diff}}_{(E,A)} A \qquad B^{\mathrm{diff}} := \Pi^{\mathrm{diff}}_{(E,A)} B \qquad E^{\mathrm{imp}} := \Pi^{\mathrm{imp}}_{(E,A)} E \qquad B^{\mathrm{imp}} := \Pi^{\mathrm{imp}}_{(E,A)} B$$

Theorem (Solution formula, cf. Trenn 2012)

(x,u) is a smooth solution of $E\dot{x} = Ax + Bu$ $\iff$

$$x(t) = \mathrm{e}^{A^{\mathrm{diff}} t} \Pi_{(E,A)} x(0) + \int_0^t \mathrm{e}^{A^{\mathrm{diff}}(t-s)} B^{\mathrm{diff}} u(s) \mathrm{d}s - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t)$$

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Decomposition of solution

$$x(t) = \mathrm{e}^{A^{\mathrm{diff}} t}\Pi_{(E,A)}x(0) + \int_0^t \mathrm{e}^{A^{\mathrm{diff}}(t-s)} B^{\mathrm{diff}} u(s)\mathrm{d}s - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t)$$

Corollary

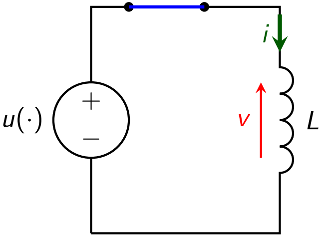
x solves $E\dot{x} = Ax + Bu$ $\iff$ $x = x^{\mathrm{diff}} \oplus x^{\mathrm{imp}}$ *where*

$$\dot{x}^{\mathrm{diff}} = A^{\mathrm{diff}} x + B^{\mathrm{diff}} u, \quad x^{\mathrm{diff}}(0) \in \operatorname{im} \Pi_{(E,A)}$$
$$E^{\mathrm{imp}} \dot{x}^{\mathrm{imp}} = x^{\mathrm{imp}} + B^{\mathrm{imp}} u$$

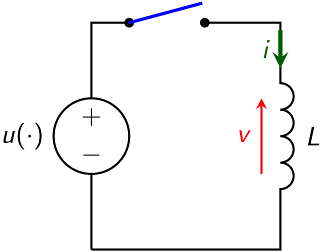
Furthermore $x^{\mathrm{diff}}(t) \in \mathcal{V}^*$ *and* $x^{\mathrm{imp}}(t) \in \operatorname{im}[B^{\mathrm{imp}}, E^{\mathrm{imp}} B^{\mathrm{imp}}, \ldots, (E^{\mathrm{imp}})^{n-1} B^{\mathrm{imp}}] \subseteq \mathcal{W}^*$.

Motivating example

$t < 0$



$t \geq 0$

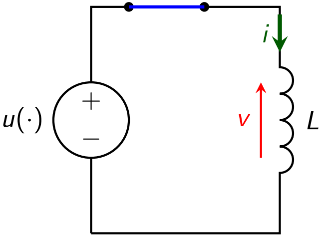


inductivity law: $L\frac{\mathrm{d}}{\mathrm{d}t}i = v$

switch dependent: $0 = v - u$ $\qquad$ $0 = i$

Motivating example

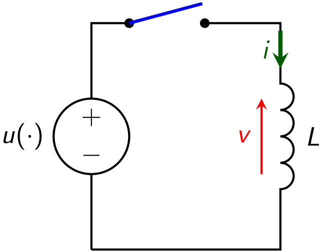
$t < 0$



$x = [i, v]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$t \geq 0$



$x = [i, v]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

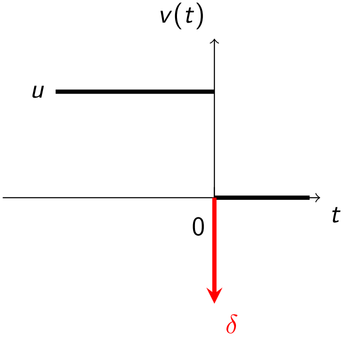
$\rightarrow$ switched differential-algebraic equation

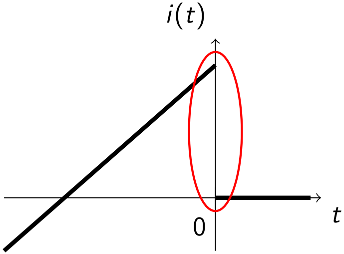
$\rightsquigarrow$ $x(0^-)$ not consistent with open-switch DAE

Solution of circuit example

$t < 0$	$t \geq 0$
$v = u$	$i = 0$
$L\frac{\mathrm{d}}{\mathrm{d}t}i = v$	$v = L\frac{\mathrm{d}}{\mathrm{d}t}i$

Solution (assume constant input u):





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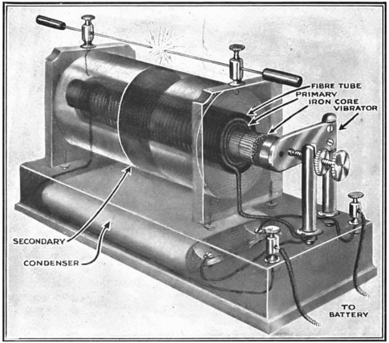
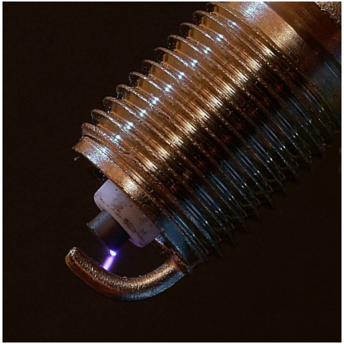
Dirac impulse is “real”

Dirac impulse

Not just a mathematical artifact!



Drawing: Harry Winfield Secor, public domain



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Distribution theory - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ is "derivative" of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1) Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2) Axiomatic: Space of all "derivatives" of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\ \varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is smooth with compact support }\}$

Definition (Distributions)

$\mathbb{D} := \{\ D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous }\}$

Definition (Regular distributions)

$f \in \mathcal{L}_{1,\mathrm{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_\mathbb{R} f(t)\varphi(t)\,\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

$$({\mathbb{1}_{[0,\infty)}}_\mathbb{D})'(\varphi) = -\int_\mathbb{R} \mathbb{1}_{[0,\infty)}\varphi' = -\int_0^\infty \varphi' = -(\varphi(\infty) - \varphi(0)) = \varphi(0)$$

Distributional DAE and ITPs

Distributional solutions

Distributional DAE: $E\dot{X} = AX + BU, \quad X \in \mathbb{D}^n,\ U \in \mathbb{D}^m$

- Classical solution behavior dense in distributional solution behaavior
  $\rightsquigarrow$ essentially no difference between classical and distributional solutions
- No differentiability requirements for U (all distributions are "$\mathcal{C}^\infty$")
- Initial value problems cannot be formulated, $X(0)$ not defined

Initial trajectory problem (ITP)

Given $X^0 \in \mathbb{D}^n$ (initial trajectory) and $U \in \mathbb{D}^m$ find $X \in \mathbb{D}^\ell$ with

$$\begin{aligned} X_{(-\infty,0)} &= X^0_{(-\infty,0)} \\ (E\dot{X})_{[0,\infty)} &= (AX + BU)_{[0,\infty)} \end{aligned} \tag{ITP}$$

Restriction not well defined, cf. Trenn 2021

Restriction of general distributions to interval not well defined (actually not definable)

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Piecewise-smooth distributions

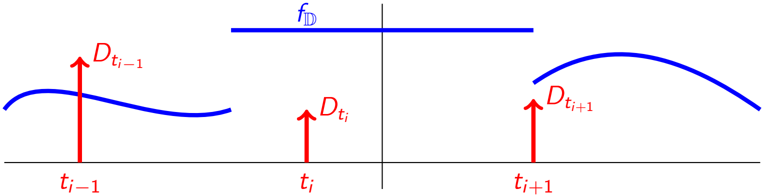
Dilemma

- Examples indicate presence of Dirac impulses in response to inconsistent initial values
- Inconsistent initials cannot be considered for distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$)

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t \in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}},\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\mathrm{pw}}$ "$\subseteq$" $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$
- Closed under differentiation, i.e. $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$
- Well definded restriction $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$

$$D = f_{\mathbb{D}} + \sum_{t\in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t \in T\cap M} D_t$$

- Multiplication well defined (Fuchssteiner multiplication)
- Evaluation at $t \in \mathbb{R}$: $D(t^-) := f(t^-)$, $D(t^+) := f(t^+)$
- Impulses at $t \in \mathbb{R}$: $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$
- Well defined unique antiderivative $G = \int_{0^-} F$, i.e. $G(0^-) = 0$ and $G' = F$

Examples:

$$\delta_{[0,\infty)} = \delta, \qquad \delta[0] = \delta \qquad \delta(t\pm) = 0 \;\forall t$$

$$\delta_{(0,\infty)} = 0, \qquad \delta[t] = 0 \;\forall t \neq 0 \qquad \delta^2 = 0$$

$$\int_{0^-} \delta = (\mathbb{1}_{[0,\infty)})_{\mathbb{D}}$$

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ITP-solutions

Theorem (cf. Trenn 2012)

Let (E, A)*, then* $\forall X^0 \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *and* $\forall U \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$ *there is a unique* $X \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$ *satisfying*

$$\begin{aligned} X_{(-\infty,0)} &= X^0_{(-\infty,0)} \\ (E\dot{X})_{[0,\infty)} &= (AX + BU)_{[0,\infty)} \end{aligned} \tag{ITP}$$

Explicit solution formula

$$X(t^+) = \mathrm{e}^{A^{\mathrm{diff}} t}\Pi_{(E,A)} X^0(0^-) + \mathrm{e}^{A^{\mathrm{diff}} t} \int_{0^-}^{t+} \mathrm{e}^{-A^{\mathrm{diff}} \cdot} B^{\mathrm{diff}} U - \sum_{i=0}^{\nu-1} (E^{\mathrm{imp}})^i B^{\mathrm{imp}} U^{(i)}(t^+)$$

and for $U = 0$ (or $B = 0$)

$$X(0^+) = \Pi_{(E,A)} X^0(0^-) \qquad X[0] = \sum_{i=0}^{\nu-2} (E^{\mathrm{imp}})^{i+1} X^0(0^-) \delta^{(i)}$$

Unique jump

$$E\dot{x} = Ax, \quad x(t_0^-) = x_0 \in \mathbb{R}^n$$

Unique jump

$x(t_0^-) \mapsto x(t_0^+) = \Pi_{(E,A)} x(t_0^-)$

Why no other jump rule? $\hookrightarrow$ handwritten notes ...

Equivalent jump rules, cf. COSTANTINI, TRENN & VASCA 2013; FRASCA ET AL. 2010

The following jump rules are equivalent:

- Consistency projector based on Wong sequences and QWF (TRENN 2012)
- Passivity based energy minimization (FRASCA ET AL. 2010)
- Conservation of charge/flux (SESHU & BALABANIAN 1964)
- Laplace transform approach (OPAL & VLACH 1990)

Index of a regular DAE and solution properties

$$E\dot{x} = Ax + Bu, \qquad (E, A) \simeq \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right)$$

Definition (Index)

Index of regular (E, A) := nilpotency index of N in QWF

Theorem (Index and Diracs)

$\exists x_0 \in \mathbb{R}^n$ such that $x[t_0] \neq 0 \iff$ index of $(E, A) > 1$

Reminder: $x[t_0] = -\sum_{i=0}^{\nu-2} (E^{\mathrm{imp}})^{i+1} x_0 \delta_{t_0}^{(i)}, \qquad E^{\mathrm{imp}} = T \begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix} T^{-1}$

Remark (Index and input derivatives)

Solution x depends on derivatives of u $\implies$ index > 1

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Index 1

Conclusion

Index 1 $\implies$ no Diracs in response to inconsistent initial values and discontinuous inputs

Index 1: $\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad \iff \quad \begin{aligned} \dot{x} &= A^{\mathrm{diff}}x + B^{\mathrm{diff}}u \\ y &= Cx + (D - CB^{\mathrm{imp}})u \end{aligned} \quad x(0^+) \in \operatorname{im}\Pi_{(E,A)}$

Theorem (Index 1 characterization)

(E, A) *with singular and square* E *is index 1 (i.e.* $N = 0$ *in QWF)*

$\iff \mathcal{V}_1 \cap \mathcal{W}_1 = A^{-1}(\operatorname{im} E) \cap \ker E = \{0\}$

$\iff \mathcal{V}_1 \oplus \mathcal{W}_1 = \mathbb{R}^n$

$\iff \deg\det(sE - A) = \operatorname{rank} E$

$\iff A_{22}$ *is invertible where* $(E, A) \simeq \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right)$

$\iff \begin{bmatrix} E_1 \\ A_2 \end{bmatrix}$ *is invertible, where* $(E, A) \simeq \left(\begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \right)$ *with* E_1 *full row rank*

Summary

$$E\dot{x} = Ax + Bu, \quad x(t_0^-) = x_0$$

- Smooth solutions
  - Explicit solution formula:
    $x(t) = \mathrm{e}^{A^{\mathrm{diff}}(t-t_0)}\Pi_{(E,A)}x_0 + \int_{t_0}^t \mathrm{e}^{A^{\mathrm{diff}}(t-s)}B^{\mathrm{diff}}u(s)\mathrm{d}s - \sum_{i=0}^{\nu-1}(E^{\mathrm{imp}})^i B^{\mathrm{imp}} u^{(i)}(t)$
  - Solution decomposition according to $\mathcal{V}_* \oplus \mathcal{W}_*$: $x = x^{\mathrm{diff}} \oplus x^{\mathrm{imp}}$ with $\dot{x}^{\mathrm{diff}} = A^{\mathrm{diff}}x^{\mathrm{diff}} + B^{\mathrm{diff}}u$ and $E^{\mathrm{imp}}\dot{x}^{\mathrm{imp}} = x^{\mathrm{imp}} + B^{\mathrm{imp}}u$
- Inconsistent initial values
  - Real world applications motivate presence of Dirac delta
  - Standard distributional solutions not suitable for ITP
  - Piecewise-smooth distributions are suitable
- Distributional solution theory
  - Existence and uniqueness for ITP
  - Unique jump rule for inconsistent initial values
  - Index and solution properties