**Differential algebraic equations: Mini course 1**

# Motivation, quasi-Kronecker and quasi-Weierstrass form, regularity

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# DAE - defintions

## General nonlinear DAEs

DAE = implicit ODE | in semi-linear form | in semi-explicit form

$$0 = F(t, w, \dot{w})$$

$$E(w)\dot{w} = f(t, w)$$

$$\begin{aligned} \dot{w}_1 &= f(t, w_1, w_2) \\ 0 &= g(t, w_1, w_2) \end{aligned}$$

Note: implicit ODE can always be rewritten as semi-linear DAE:

$$0 = F(t, w, \dot{w}) \quad \overset{w=w_1}{\Longleftrightarrow} \quad \begin{cases} \dot{w}_1 = w_2 \\ 0 = F(t, w_1, w_2) \end{cases}$$

## Linear DAEs

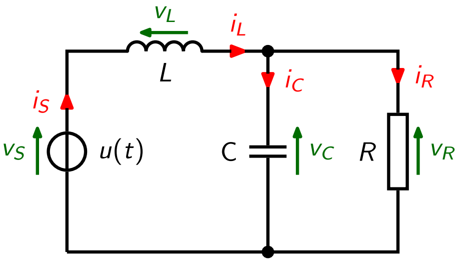
homogeneous | inhomogeneous | with inputs and outputs

$$E\dot{w} = Aw$$

$$E\dot{x} = Ax + v$$

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

# Modeling of electrical circuits




## Basic circuit elements

**Resistor**: $v_R(t) = R\, i_R(t)$

**Capacitor**: $i_C(t) = C \frac{\mathrm{d}}{\mathrm{d}t} v_C(t)$

**Inductor**: $v_L(t) = L \frac{\mathrm{d}}{\mathrm{d}t} i_L(t)$

**Voltage source**: $v_S(t) = u(t)$

## DAEs

All components are given by a differential-algebraic equation (DAE)

$$E\dot{x} = Ax + Bu$$



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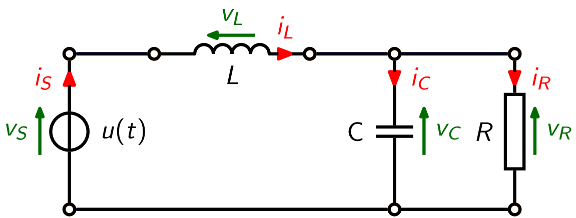
# Hierarchical model building




Overall model
$\Rightarrow$ Again DAE:

$$\boxed{E\dot{x} = Ax + Bu}$$

$$\begin{bmatrix} 0 & 0 & & & & & & \\ & & C & 0 & & & & \\ & & & & 0 & L & & \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \\ & & & & & & & 0 \end{bmatrix} \begin{pmatrix} \dot{v}_R \\ \dot{i}_R \\ \dot{v}_C \\ \dot{i}_C \\ \dot{v}_L \\ \dot{i}_L \\ \dot{v}_S \\ \dot{i}_S \end{pmatrix} = \begin{bmatrix} -1 & R & & & & & & \\ & & 0 & 1 & & & & \\ & & & & 1 & 0 & & \\ & & & & & & -1 & 0 \\ 1 & & -1 & & & & & \\ & 1 & & 1 & & -1 & & \\ & & 1 & & 1 & & -1 & \\ & & & & & 1 & & -1 \end{bmatrix} \begin{pmatrix} v_R \\ i_R \\ v_C \\ i_C \\ v_L \\ i_L \\ v_S \\ i_S \end{pmatrix} + \begin{bmatrix} \\ \\ \\ 1 \\ \\ \\ \\ \\ \end{bmatrix} u$$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# Recall (linear) ODEs

Ordinary differential equations (ODEs):

$$\dot{x} = Ax + Bu$$

- Initial values: arbitrary
- Solution uniquely determined by $u$ and $x(0)$
- No constraints on $B$ and $u$
- Solution formula (variation of constant formula):

$$x(t) = \mathrm{e}^{At}x_0 + \int_0^t \mathrm{e}^{A(t-s)}Bu(s)\,\mathrm{d}s$$


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# DAEs are not ODEs

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\begin{array}{lcl} \dot{x}_2 = x_1 + v_1 & \longrightarrow & x_1 = -v_1 - \dot{v}_2 \\ 0 = x_2 + v_2 & \longrightarrow & x_2 = -v_2 \nearrow \\ 0 = v_3 & & \end{array}$$

no restriction on $x_3$

## Key differences to ODEs

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -v_1(0) - \dot{v}_2(0)$, $x_2(0) = v_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value ($x_3$ free)
- Inhomogeneity not arbitrary
  - structural restrictions ($v_3 = 0$)
  - differentiability restrictions ($\dot{v}_2$ must be well defined)

# Solution behavior

$$\begin{aligned} E\dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \qquad (*)$$

## Solution behaviors

full solution behavior: $\mathfrak{B}_{\text{full}} := \left\{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \;\middle|\; (*) \text{ holds } \right\}$

external solution behavior (i/o-behavior): $\mathfrak{B}_{\text{i/o}} := \left\{ \begin{pmatrix} u \\ y \end{pmatrix} \;\middle|\; \exists x \text{ s.t. } (*) \text{ holds } \right\}$

## Theorem

*For given $(E, A, B, C, D)$ and $(\bar{E}, \bar{A}, \bar{B}, \bar{C}, \bar{D})$ with corresponding i/o-behaviors $\mathfrak{B}_{i/o}$ and $\bar{\mathfrak{B}}_{i/o}$ assume $\exists\, S, T$ invertible such that*

$$\bar{E} = SET, \qquad \bar{A} = SAT, \qquad \bar{B} = SB, \qquad \bar{C} = CT, \qquad \bar{D} = D.$$

*Then $\mathfrak{B}_{i/o} = \bar{\mathfrak{B}}_{i/o}$.*

Proof $\hookrightarrow$ handwritten notes



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# Equivalence and four types

## Definition (Equivalence of matrix pairs)

$(E,A)$, $(\bar{E},\bar{A})$ are called equivalent $:\Longleftrightarrow$ $(\bar{E},\bar{A}) = (SET, SAT)$

short: $(E,A) \cong (\bar{E},\bar{A})$ or $(E,A) \overset{S,T}{\cong} (\bar{E},\bar{A})$

## Definition

- $(E,A)$ is of type ODE $:\Longleftrightarrow$ $(E,A) \cong (I,J)$
- $(E,A)$ is of type nDAE $:\Longleftrightarrow$ $(E,A) \cong (N,I)$, $N$ nilpotent
- $(E,A)$ is of type uDAE $:\Longleftrightarrow$ $(E,A) \cong (\operatorname{diag}(E_1,\ldots,E_k), \operatorname{diag}(A_1,\ldots,A_k))$, where

$$(E_i,A_i) = \left( \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \right) \quad \text{underdetermined prototypes}$$

- $(E,A)$ is of type oDAE $:\Longleftrightarrow$ $(E,A) \cong (\operatorname{diag}(E_1,\ldots,E_k), \operatorname{diag}(A_1,\ldots,A_k))$, where

$$(E_i,A_i) = \left( \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \right) \quad \text{overdetermined prototypes}$$


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# Solution properties of four types

$$E\dot{x} = Ax + v$$

- **Type ODE** $(E, A) \cong (I, J)$: $E\dot{x} = Ax + v \iff \dot{x} = E^{-1}Ax + E^{-1}v$
  $\rightsquigarrow$ existence and uniqueness of solutions for all $x_0$ and all $v$
- **Type nDAE** $(E, A) \cong (N, I)$: $E\dot{x} = Ax + v \iff A^{-1}E\dot{x} = x + A^{-1}v$, $A^{-1}E$ nilpotent
  solutions $\hookrightarrow$ handwritten notes
  $\rightsquigarrow$ existence and uniqueness of solutions for all smooth $v$, $x(0)$ fully fixed by $v$
- **Type uDAE**: Structure and solutions $\hookrightarrow$ handwritten notes
  $\rightsquigarrow$ existence of solutions for all $v$ and all $x_0$, but non-unique
- **Type oDAE**: Structure and solutions $\hookrightarrow$ handwritten notes
  $\rightsquigarrow$ non-existence of solutions for general $v$, but if existent, solutions are unique, $x(0)$ fully fixed by $v$

General DAE can contain **arbitrary combination of above four types** ... and maybe more?

# Simple check for types

## Theorem

- $(E, A)$ *is of type ODE* $\iff$ $\lambda = \infty$ : $\operatorname{rank}(\lambda E - A) = n = \ell$
- $(E, A)$ *is of type nDAE* $\iff$ $\forall \lambda \in \mathbb{C}$ : $\operatorname{rank}(\lambda E - A) = n = \ell$
- $(E, A)$ *is of type uDAE* $\iff$ $\forall \lambda \in \mathbb{C} \cup \{\infty\}$ : $\operatorname{rank}(\lambda E - A) = \ell$
- $(E, A)$ *is of type oDAE* $\iff$ $\forall \lambda \in \mathbb{C} \cup \{\infty\}$ : $\operatorname{rank}(\lambda E - A) = n$

$$\operatorname{rank}(\lambda E - A) = \operatorname{rank}(E - \tfrac{1}{\lambda}A) \quad \rightsquigarrow \quad \operatorname{rank}(\infty E - A) := \operatorname{rank}(E - \tfrac{1}{\infty}A) = \operatorname{rank}(E)$$

## Example revisited

Consider again the DAE

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

Is this DAE of any of the above four types? NO, neither $E$ nor $A$ have full rank


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# The quasi-Kronecker form

**Theorem (Quasi-Kronecker Form , Berger & Trenn 2012,2013)**

*For any $E, A \in \mathbb{R}^{\ell \times m}$, $\exists$ invertible $S \in \mathbb{R}^{\ell \times \ell}$ and invertible $T \in \mathbb{R}^{n \times n}$:*

$$(E, A) \overset{S,T}{\cong} \left( \begin{bmatrix} E_U & & & \\ & E_J & & \\ & & E_N & \\ & & & E_O \end{bmatrix}, \begin{bmatrix} A_U & & & \\ & A_J & & \\ & & A_N & \\ & & & A_O \end{bmatrix} \right)$$

*where*

- $(E_U, A_U)$ *is of type uDAE (underdetermined part)*
- $(E_J, A_J)$ *is of type ODE (ODE part)*
- $(E_N, A_N)$ *is of type nDAE (nilpotent part)*
- $(E_O, A_O)$ *is of type oDAE (overdetermined part)*

# QKF for simple example

## Example revisited

Consider again the DAE

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

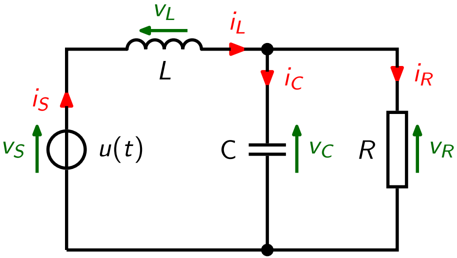
What is the QKF for this DAE?

simple column permutation gives: $$(E, A) \simeq \left( \begin{bmatrix} \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} \end{bmatrix}, \begin{bmatrix} \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{bmatrix} \right)$$

QKF consists of: 1 uDAE ($0 \times 1$), no ODE, 1 nDAE ($2 \times 2$), 1 oDAE ($1 \times 0$)

Solution properties: one free variable, one differentiability and one structural constraint on $v$

# Circuit example revisited




$$(E,A) \simeq \left( \begin{bmatrix} I_{2\times 2} & \\ & 0_{6\times 6} \end{bmatrix}, \begin{bmatrix} \begin{matrix} 0 & 1/C \\ -1/L & -1/RC \end{matrix} & \\ & I_{6\times 6} \end{bmatrix} \right)$$

# One more example

$$(E, A) = \left( \begin{bmatrix} 0 & 0 & -2 & 1 & 3 & -4 & 2 & -5 \\ -1 & -2 & -5 & 2 & 6 & -5 & 3 & -8 \\ -2 & -3 & -3 & 0 & 1 & 0 & 0 & -3 \\ 0 & -2 & -4 & 4 & 7 & -3 & 1 & -6 \\ -1 & -1 & -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -2 & -1 & -3 & -2 & 4 & 1 & 4 \\ 0 & 1 & 5 & -4 & -7 & 8 & -2 & 11 \\ -1 & -1 & -1 & -1 & 0 & 0 & 1 & -1 \\ -2 & -3 & -10 & 5 & 10 & -17 & 4 & -25 \end{bmatrix}, \begin{bmatrix} 0 & -1 & 1 & 2 & 0 & -1 & -3 & -4 \\ -1 & -3 & -1 & 3 & 1 & -2 & -4 & -9 \\ 1 & 0 & -1 & 4 & 6 & -2 & 0 & -5 \\ -5 & -4 & -5 & -8 & -10 & 0 & 1 & -2 \\ 3 & 1 & 4 & 7 & 8 & 2 & -3 & -1 \\ 3 & 1 & 3 & 7 & 9 & 2 & -2 & -1 \\ 4 & 5 & 8 & 0 & 3 & 11 & 2 & 19 \\ 3 & 1 & 4 & 7 & 8 & 2 & -3 & -1 \\ 0 & -3 & -4 & 10 & 7 & -15 & -6 & -27 \end{bmatrix} \right)$$

$$\text{QKF: } (SET, SAT) = \left( \begin{bmatrix} -\frac{1}{2} & -\frac{7}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{3}{2} & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{65}{2} & 0 & \frac{15}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{13}{2} & 0 & -\frac{3}{4} & 0 \\ 0 & 0 & 0 & 0 & -\frac{117}{2} & 0 & -\frac{27}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{52}{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{80}{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{92}{37} \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & -12 & -12 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{141}{2} & 10 & -\frac{9}{4} & 0 \\ 0 & 0 & 0 & 0 & 19 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{519}{4} & -18 & \frac{33}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{12}{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{64}{37} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{44}{37} \end{bmatrix} \right)$$

# QKF $\rightarrow$ Quasi-Weierstrass form

$$(E, A) \cong \left( \begin{bmatrix} \xcancel{\boxed{E_U}} & & & \\ & \boxed{I} & & \\ & & \boxed{N} & \\ & & & \xcancel{\boxed{E_O}} \end{bmatrix}, \begin{bmatrix} \xcancel{\boxed{A_U}} & & & \\ & \boxed{J} & & \\ & & \boxed{I} & \\ & & & \xcancel{\boxed{A_O}} \end{bmatrix} \right)$$

## Corollary (Quasi-Weierstrass-Form (QWF))

$E\dot{x} = Ax + f$ *has solution* $x$ *for any sufficiently smooth* $f$ *and each solution* $x$ *is uniquely determined by* $x(0)$ *and* $f$

$$\Leftrightarrow \quad (E, A) \cong \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

*quasi-Weierstrass form*

# Regularity

**Theorem (cf. Kunkel & Mehrmann 2006; Berger, Ilchmann & Trenn 2012)**

$(E, A)$ *is regular, i.e.* $\det(sE - A) \not\equiv 0$

$\iff$ *QWF:* $(E, A) \simeq \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)$

$\iff$ $\exists!$ *subspaces* $\mathcal{V}, \mathcal{W} \in \mathbb{R}^n$ : $A\mathcal{V} \subseteq E\mathcal{V}, E\mathcal{W} \subseteq A\mathcal{W}, \mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n, E\mathcal{V} \oplus A\mathcal{W} = \mathbb{R}^n$
*and any bases* $V, W$ *of* $\mathcal{V}, \mathcal{W}$ *lead to QWF with* $T = [V, W]$ *and* $S = [EV, AW]^{-1}$

$\iff$ *the Wong limits* $\mathcal{V}_*$ *and* $\mathcal{W}_*$ *satisfy* $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$ *and* $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$, *where*

$$\mathcal{V}_0 = \mathbb{R}^n, \quad \mathcal{V}_{i+1} = A^{-1}(E\mathcal{V}_i), \quad \mathcal{V}_* := \bigcap_i \mathcal{V}_i$$
$$\mathcal{W}_0 = \{0\}, \quad \mathcal{W}_{j+1} = E^{-1}(A\mathcal{W}_j), \quad \mathcal{W}_* := \bigcup_j \mathcal{W}_j$$

$\iff$ $E\dot{x} = Ax + Bu$ *is solvable for all* $B$ *and all smooth* $u$
*and each solution is uniquely determined by* $x(0)$

$\iff$ $E\dot{x} = Ax, x(0) = 0$ *has only the trivial solution (and* $E, A$ *square)*

# Summary

- Different forms of DAEs (nonlinear, semi-linear, semi-explicit, linear (homogeneous, inhom., with inputs/outputs))
- DAEs are not ODEs
  - non-existence of solutions
  - non-uniqueness of solutions
  - differentiability requirements on inhomogeneities
- Equivalence
- Four basic types of linear DAEs $E\dot{x} = Ax + v$
  - type ODE (square and $E$ invertible)
  - type nilpotent DAE (square, $A$ invertible and $A^{-1}E$ nilpotent)
  - type underdetermined DAE (full row rank of $\lambda E - A$)
  - type overdetermined DAE (full column rank of $\lambda E - A$)
- Quasi-Kronecker form for general linear DAEs $E\dot{x} = Ax + v$
  - Any DAE can be decoupled into above four types $\rightsquigarrow$ QKF
  - Quasi-Weierstrass form: no uDAE and oDAE parts $\rightsquigarrow$ regularity
  - Regularity characterizations