IMPULSE-FREE LINEAR QUADRATIC OPTIMAL CONTROL OF SWITCHED DIFFERENTIAL ALGEBRAIC EQUATIONS

PAUL WIJNBERGEN[1,*], STEPHAN TRENN[2]

[1]Department of Mechanical Engineering, Eindhoven University, 5600 MB Eindhoven, The Netherlands
[2]Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence, University of Groningen, 9700 AK Groningen, The Netherlands

Abstract. In this paper the finite horizon linear quadratic regulator (LQR) problem for switched linear differential algebraic equations is studied. It is shown that for switched DAEs with a switching signal that induces locally finitely many switches, the problem can be solved by recursively solving several LQR problems for non-switched DAE. First, it is shown how to solve the non-switched problems for index-1 DAEs followed by an extension of the results to higher index DAEs. The resulting optimal control can be computed based on the solution of a Riccati differential equation expressed in terms of the differential system matrices. The paper concludes with the extension of the results to the LQR problem for general switched DAEs.

Keywords. switched systems, differential algebraic equations, impulse-controllability, geometric control

2020 Mathematics Subject Classification. 49J21, 49J30, 93C30, 34A09.

This contribution is dedicated to Professor Ezra Zeheb on the occasion of his 85th birthday.

1. Introduction

In this paper, we aim to find necessary and sufficient conditions for the existence of an input that solves the finite horizon linear quadratic regulator problem for switched differential-algebraic equations.

Problem 1. [LQR for switched DAEs] Find an input u (from a suitable signal space specified later) that minimizes

$$J(x_0,u,t_0) = \int_{t_0}^{t_f} \|y(t)\|^2 \,\mathrm{d}t + x(t_f^-)Px(t_f^-), \tag{1.1}$$

$$\text{s.t.} \quad E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \tag{1.2a}$$

$$y = C_\sigma x + D_\sigma u, \tag{1.2b}$$

$$x(t_0^-) = x_0, \tag{1.2c}$$

$$x(t_f^-) \in \mathscr{V}^{\text{end}}, \tag{1.2d}$$

---

*Corresponding author.

E-mail address: p.wijnbergen@tue.nl (P. Wijnbergen), s.trenn@rug.nl (S. Trenn).

Received October, 30, 2023; Accepted May, 6, 2024.

This work was partially supported by the NWO Vidi-grant 639.032.733.

where $\sigma : [t_0,t_f) \to \mathbb{N}$ is a given piecewise constant switching signal, x is the state, the matrices $E_p, A_p \in \mathbb{R}^{n\times n}$ form a regular matrix pair, (i.e., $\det(sE-A)$ is not indentically zero), $B_p \in \mathbb{R}^{n\times m}$, $C_p \in \mathbb{R}^{q\times n}$ and $D_p \in \mathbb{R}^{q\times m}$, $p \in \mathbb{N}$, $P = P^\top \in \mathbb{R}^{n\times n}$ is some symmetric positive semi-definite matrix and $\mathscr{V}^{\text{end}} \subseteq \mathbb{R}^n$ is some subspace.

Switched differential algebraic equations (swDAEs) of the form (1.2a) arise naturally when modeling physical systems with certain algebraic constraints on the state variables. Examples of applications of non-switched DAEs in electrical circuits (with distributional solutions) can be found, e.g., in [37]. For non-switched DAEs, these constraints are often eliminated such that the system is described by ordinary differential equations. However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode. Therefore, there typically does not exist a description as a switched ODE with a common state variable. This problem can be overcome by studying switched DAEs directly.

In the context of linear systems the linear quadratic regulator (LQR) problem on both the finite and infinite horizon has been studied extensively, see [15, 17, 16, 50, 48] for results on ODEs and [7, 2, 25, 26, 27, 18, 11, 10, 12] for DAEs. Recent studies regarding the optimal control problem for DAEs focus on finding solutions based on the Lur'e equation, which can be interpreted as an extension of the Kalman-Yakubovich-Popov lemma [32, 42, 31]; further results have been obtained in the context of model predictive control [9, 30, 13, 14]. For switched differential algebraic equations it seems that so far only qualitative properties such as controllability, stabilizability [23, 24, 29, 28, 20, 43, 44, 46], and observability have been studied [33, 34, 35, 36, 22, 21, 19]. To the best of the authors knowledge quantitative properties such as optimal control have not been studied for switched DAEs. This paper aims to close this gap in the literature.

As trajectories of switched DAEs generally exhibit jumps (or even impulses), which may exclude classical solutions from existence, the *piece-wise smooth distributional solution framework* introduced in [38] is adopted. In particular, $(x,u) \in \mathbb{D}_{\text{pw}\mathscr{C}^\infty}^{n+m}$, where $\mathbb{D}_{\text{pw}\mathscr{C}^\infty}$ denotes the space of piece-wise smooth distributions. Due to this assumption, Problem 1 is considered in a piece-wise smooth distributional setup. Since within this setup, the integral over the norm squared of a Dirac impulse is not well defined, it follows directly that the cost is finite, if and only if, the output (1.2b) is impulse-free. Focusing on solutions that result in an impulse-free output, we denote the output as a piece-wise continuous function, whereas it is actually a distribution.

We consider Problem 1 under the assumption that the switching signal does not induce chattering behavior, i.e., we assume that it induces locally finitely many switches. In principle, the switching signal could still induce infinitely many switches. As this is troublesome for solving the problem in finitely many steps, we consider the bounded interval $[t_0,t_f)$. In this interval thus only finitely many switches are present.

For many applications, it is of interest to extend an optimal solution in an impulse-free way on the interval $[t_f,\infty)$. This is the case for example in choosing a suitable terminal cost if the LQR problem is to be solved on a receding horizon. To allow for such extensions, we impose the subspace endpoint constraint (1.2d) to the state at t_f^-. As we will show, this subspace endpoint constraint fits naturally in the LQR problem for switched DAEs as there exists a solution to Problem 1 if and only if the initial value x_0 is contained in a certain subspace.

The remainder of the paper is structured as follows. First mathematical notation and preliminaries are introduced in Section 2. Then the approach to solving Problem 1 is formulated in Section 3 and the main result is presented. In Section 4 necessary and sufficient conditions for solvability of Problem 1 for non-switched DAEs of index-1 presented and it is shown how to generalize these results to arbitrary index-DAEs in Section 5. In Section 6 the results are utilized to solve the optimal control problem for switched DAEs where each mode is given by an arbitrary index DAE. The paper is concluded in Section 7 and an appendix containing some technical proofs of the results.

2. Mathematical preliminaries

In this section we recall some notation and properties related to the non-switched DAE

$$E\dot{x} = Ax + Bu. \tag{2.1}$$

In the following, we call a matrix pair (E,A) and the associated DAE (2.1) *regular* iff the polynomial $\det(sE-A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form*[4].

Proposition 2.1. *A matrix pair $(E,A) \in \mathbb{R}^{n\times n}\times\mathbb{R}^{n\times n}$ is regular if, and only if, there exist invertible matrices $S,T\in\mathbb{R}^{n\times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right), \tag{2.2}$$

where $J\in\mathbb{R}^{n_1\times n_1}$, $0\leqslant n_1\leqslant n$, is some matrix and $N\in\mathbb{R}^{n_2\times n_2}$, $n_2 := n-n_1$, is a nilpotent matrix of order $\nu\in\mathbb{N}$. In particular, ν is referred to as the index of (2.1).

The matrices S and T can be calculated by using the so-called *Wong sequences* [4, 49]:

$$\mathcal{V}_0 := \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \quad i = 0,1,\ldots$$
$$\mathcal{W}_0 := \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \quad i = 0,1,\ldots$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup \mathcal{W}_i.$$

For any full rank matrices V,W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V,W]$ and $S := [EV, AW]^{-1}$ are invertible and (2.2) holds. Based on the Wong sequences we define the following projector and selectors.

Definition 2.2. Consider the regular matrix pair (E,A) with corresponding quasi-Weierstrass form (2.2). The *consistency projector* of (E,A) is given by

$$\Pi_{(E,A)} := T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}, \tag{2.3}$$

the *differential* and *impulse selector* are given by

$$\Pi^{\text{diff}}_{(E,A)} := T\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \quad \Pi^{\text{imp}}_{(E,A)} := T\begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S. \tag{2.4}$$

In all three cases, the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore, we define

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)}A, \qquad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)}E, \qquad B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)}B, \qquad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)}B.$$

A classical (i.e. a differentiable or locally integrable) solution to (2.1) in terms of these matrices yields $x = x^{\text{diff}} + x^{\text{imp}}$, where x^{diff} and x^{imp} satisfy

$$\dot{x}^{\text{diff}} = A^{\text{diff}}x^{\text{diff}} + B^{\text{diff}}u, \quad x^{\text{diff}}(t_0^-) = \Pi x_0, \tag{2.5a}$$

$$x^{\text{imp}} = -\sum_{i=0}^{\nu-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}. \tag{2.5b}$$

Observe that for index-1 systems we find $x^{\text{imp}} = -B^{\text{imp}}u$. Note that all the above-defined matrices do not depend on the choice of transformation matrices S and T; they are uniquely determined by the original matrix pair (E,A).

The switched DAE (1.2a) will not have classical solutions in general and $x(t_i^-) \neq x(t_i^+)$ due to the switching between modes. Consequently, the state is allowed to contain jumps or even Dirac impulses. We therefore utilize the piecewise-smooth distributional framework as introduced in [38], i.e., x and u are vectors of piecewise-smooth distributions given by

$$\mathbb{D}_{\text{pw}\mathscr{C}^\infty} := \left\{ D = f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathscr{C}^\infty_{\text{pw}}, T \subseteq \mathbb{R} \text{ is discrete}, \\ \forall t \in T : D_t \in \operatorname{span}\{\delta_t, \delta_t', \delta_t'', \ldots\} \end{array} \right\},$$

where $\mathscr{C}^\infty_{\text{pw}}$ denotes the space of piecewise-smooth functions, $f_{\mathbb{D}}$ denotes the regular distribution induced by f, δ_t denotes the Dirac impulse with support $\{t\}$ and δ_t' denotes the distributional derivative of δ_t. For $D = f_{\mathbb{D}} + \sum_{t\in T} D_t \in \mathbb{D}_{\text{pw}\mathscr{C}^\infty}$ three types of "evaluation at time t" are defined: left side evaluation $D(t^-) := f(t^-)$, right side evaluation $D(t^+) := f(t^+)$ and the impulsive part $D[t] := D_t$ if $t \in T$ and $D[t] = 0$ otherwise.

The space $\mathbb{D}_{\text{pw}\mathscr{C}^\infty}$ can be equipped with a multiplication (*cf.* [39]). In particular, the multiplication of a piecewise-constant function with a piecewise-smooth distribution is well defined and the switched DAE (1.2a) can be interpreted as an equation within the space of piecewise-smooth distributions. Within the piece-wise smooth distributional framework, restrictions of x and u to intervals, are well defined. Given the notation $x_{\mathscr{I}}$ for the restriction of x to the interval $\mathscr{I} \subseteq \mathbb{R}$, it is shown in [38] that the *initial trajectory problem* (ITP)

$$x_{(-\infty,t_0)} = x^0_{(-\infty,t_0)}, \tag{2.6a}$$

$$(E\dot{x})_{[t_0,\infty)} = (Ax)_{[t_0,\infty)} + (Bu)_{[t_0,\infty)}, \tag{2.6b}$$

has a unique solution for any initial trajectory if, and only if, the matrix pair (E,A) is regular. Note that it can be shown that the solution of (2.6) on $[t_0,\infty)$ is uniquely determined by $x(t_0^-)$, hence it is justified to replace (2.6a) by $x(t_0^-) = x_0$ for some $x_0 \in \mathbb{R}^n$.

The impulsive part of a solution (induced by an inconsistent initial value) of (2.6) is given by

$$x[t_0] = -\sum_{i=0}^{\nu-1} \left(E^{\text{imp}}\right)^{i+1} \left(x(t_0^-) - x(t_0^+)\right) \delta_{t_0}^{(i)}. \tag{2.7}$$

Additional impulses in x can occur in response to discontinuities and Dirac impulses in the input; these additional impulses are determined by (2.5) because the decomposition $x = x^{\text{diff}} + x^{\text{imp}}$ with corresponding solution formulas (2.5) remain valid also in a distributional setup. We will later discuss the situation that (distributional) solutions evolve within certain subspaces and with a slight abuse of notation we write $x[t] \in \mathscr{M}$ for some subspace $\mathscr{M} \subseteq \mathbb{R}^n$ if $x[t] = \sum_{i=0}^{k} \alpha_i \delta_t^{(i)}$ satisfies $\alpha_i \in \mathscr{M}$ for all $i = 0, 1, \ldots, k$.

For a single mode, the concept of the impulse-controllable space is defined as follows.

Definition 2.3. The impulse-controllable space for (2.1) is given by

$$\mathscr{C}^{\mathrm{imp}} := \left\{ x_0 \;\middle|\; \begin{array}{l} \exists (x,u) \in (\mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty})^{n+m} \text{ solving (2.6)} \\ \text{s.t. } x(t_0^-) = x_0 \text{ and } (x,u)[t_0] = 0. \end{array} \right\}.$$

Furthermore, the DAE is called impulse-controllable if all initial values are impulse-controllable, i.e., $\mathscr{C}^{\mathrm{imp}} = \mathbb{R}^n$.

It can be shown (see e.g. [46, Lem. 13]), that

$$\mathscr{C}^{\mathrm{imp}} = \operatorname{im} \Pi_{(E,A)} + \langle E^{\mathrm{imp}}, B^{\mathrm{imp}} \rangle + \ker E, \tag{2.8}$$

where

$$\langle E^{\mathrm{imp}}, B^{\mathrm{imp}} \rangle := \operatorname{im}[B^{\mathrm{imp}}, E^{\mathrm{imp}} B^{\mathrm{imp}}, \ldots, (E^{\mathrm{imp}})^{n-1} B^{\mathrm{imp}}].$$

Lemma 2.4 ([8, Prop. 3])**.** *The regular DAE (2.1) is impulse controllable if and only if*

i) $\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n$,

ii) There exists a matrix L such that the closed loop with feedback $u = Lx$ results in an index-1 matrix pair $(E, A+BL)$; the latter can be characterized by $\operatorname{im} E + (A+BL)\ker E = \mathbb{R}^n$.

We conclude this section with an explicit definition of a solution to the switched DAE (1.2a).

Definition 2.5. A distribution $(x,u) \in \mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}^{(n+m)}$ is called a solution to the switched DAE (1.2a) for a given right continuous switching signal σ with switching times $t_0, t_1, \ldots,$, if (x,u) considered on each interval $[t_k, t_{k+1})$ is a local (distributional) solution to ITP (2.6) on $[t_k, t_{k+1})$ with $E = E_{\sigma(t_k)}$, $A = A_{\sigma(t_k)}$ and $B = B_{\sigma(t_k)}$, where the initial condition $x(t_k^-)$ is either given by (1.2c) or by the final value of the solution from the previous interval.

Since by assumption each matrix pair (E_p, A_p) is regular, it follows that each local ITP is uniquely solvable and hence the overall switched DAE is uniquely solvable for any given input and any given initial value x_0.

3. Problem Formulation and Approach

As mentioned in the introduction, we consider Problem 1 in a distributional setup. As such, the aim is to find a distribution $u \in \mathbb{D}_{\mathrm{pw}\mathscr{C}^\infty}^m$ that minimizes a quadratic cost functional subject to a switched differential-algebraic equation. In order to utilize the distributional solution framework and to avoid technical difficulties in general, we only consider systems with a switching signal from the following class

$$\mathscr{S} := \left\{ \sigma : \mathbb{R} \to \mathbb{N} \;\middle|\; \begin{array}{l} \sigma \text{ is right continuous with a} \\ \text{locally finite number of jumps} \end{array} \right\}.$$

By doing so, we exclude an accumulation of switching times (see [38]). Since a bounded interval $[t_0, t_f)$ is considered in Problem 1, it follows that the switching signal induces $\mathtt{n} \in \mathbb{N}$ switches on this interval. Each switch occurs at t_k, where $k \in \{1, 2, \ldots, \mathtt{n}\}$. Furthermore, the switching signal is assumed to be known a priory; in particular, solvability and the solution of

Problem 1 depends on the specific switching signal. By appropriately relabeling the matrices we can therefore assume without loss of generality that

$$\sigma(t) = k, \qquad \text{for } t_k \leqslant t < t_{k+1}, \tag{3.1}$$

where $t_{\mathrm{n}+1} := t_f$.

In order to prove the main result regarding the existence of an input that solves Problem 1, some technical auxiliary results are utilized. To prevent the reader from having to go through many technical details in order to arrive at the main result, we will state the main result at the end of this section. Before presenting the main result, we will first introduce some general result on the form of the optimal input, assuming it exists, and the cost associated with this input. In particular we will show that this optimal input is a feedback and leads to a quadratic cost. Then we will show how these results allow for a dynamic-programming approach. This approach leads to the reduction of the optimal control for switched DAEs to a recursive optimal control problem for non-switched DAEs. To limit the notation that needs to be introduced, we present the main result for Problem 1 under the assumption that each mode is given by an index-1 DAE. Later in the paper we will present more general results where each mode is of arbitrary index.

3.1. **Optimal feedback and quadratic cost.** To show that the optimal input is a feedback, we start by pointing out that the switching signal is *not* regarded as a control input. Consequently, a switched differential algebraic equation of the form (1.2a) with a switching signal $\sigma \in \mathscr{S}$ can be regarded as a (piecewise-constant) time-varying linear system. Such systems have a linear solutions space where the sum of solutions is also a solution. Furthermore, the subspace endpoint constraint (1.2d) is also a linear constraint and hence the sum of solutions satisfying (1.2d) will also satisfy (1.2d). Together with the fact that the cost functional (1.1) is quadratic in the state and input all ingredients are present to prove several important properties of Problem 1. Namely, if there exists an input that solves Problem 1 the optimal cost is quadratic in the initial value and the optimal input is linear in the state, i.e., it is a feedback.

Lemma 3.1. *If there exists an input $u \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ that solves Problem 1 then $u(t^+) = F(t)x(t^-)$ for some $F : \mathbb{R} \to \mathbb{R}^{m\times n}$.*

The (technical) proof can be found in the Appendix. Note that we only consider piecewise-smooth solutions, hence $x(t^-) \neq x(t^+)$ for only finitely many $t \in [t_0, t_f)$ and hence we can assume that the input is right-continuous and we can simply write $u(t) = F(t)x(t)$ in the following.

Corollary 3.2. *If there exists an input u that solves Problem 1 then the optimal cost $J(x_0, u, t_0)$ is quadratic in x_0, i.e.,*

$$J(x_0, u, t_0) = x_0^\top K(t_0)x_0,$$

for some $K : \mathbb{R} \to \mathbb{R}^{n\times n}$.

Proof. In the proof of Lemma 3.1 it was shown that the map $x_0 \mapsto V(x_0, t_0)$ for the optimal cost satisfies the parallelogram equality (A.2). Hence it is a (semi-) norm induced by an inner product. Therefore, there exists a (positive semi-definite) matrix $K(t_0) \in \mathbb{R}^{n\times n}$ such that $V(x_0, t_0) = x_0^\top K(t_0)x_0$. $\square$

The result of Lemma 3.1 also leads to the observation that the space of initial values for which Problem 1 is solvable must be a subspace.

Definition 3.3. The set of initial values for which Problem 1 is solvable on $[t_0,t_f)$ is given by

$$\mathscr{V}_{t_0}^{\text{init}} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\, u \text{ that solves Problem 1 on} \\ [t_0,t_f) \text{ satisfying } x(t_0^-) = x_0 \end{array} \right\}.$$

Corollary 3.4. *The set $\mathscr{V}_{t_0}^{\text{init}}$ is a subspace.*

Proof. Suppose that $x_0, y_0 \in \mathscr{V}_{t_0}^{\text{init}}$. Since the inputs u_{x_0} and u_{y_0}that solve Problem 1 for $x(t_0^-) = x_0$ and $x(t_0^-) = y_0$ are feedbacks, it follows that $\alpha u_{x_0} + \beta u_{y_0}$ is the optimal input that solves Problem 1 for $x(t_0^-) = z_0 = \alpha x_0 + \beta y_0$. Consequently, $z_0 \in \mathscr{V}_{t_0}^{\text{init}}$ and thus $\mathscr{V}_{t_0}^{\text{init}}$ is a subspace. □

3.2. **A repeated optimal control problem.** Next we will show how the results in the previous subsection allow for a dynamic-programming approach for Problem 1. Let $\mathscr{V}_{t_i}^{\text{init}}$ be the subspace of initial values for which there exists a solution to Problem 1 on the interval $[t_i,t_f)$ with terminal subspace $\mathscr{V}^{\text{end}}$ and terminal cost matrix P. Furthermore, let the optimal cost matrix be given by $K(t_i)$, that is, the solution to Problem 1 yields an optimal cost $J(x_i,u,t_i) = x_i^\top K(t_i) x_i$. Then the following lemma is a reformulation of the Bellman principle of optimality.

Lemma 3.5. *Problem 1 with initial value x_0, terminal cost matrix P and terminal subspace $\mathscr{V}^{\text{end}}$ has a solution on $[t_0,t_f)$ if and only if Problem 1 on the interval $[t_0,t_i)$ with initial value x_0, terminal cost matrix $K(t_i)$ and terminal subspace $\mathscr{V}_{t_i}^{\text{init}}$ has a solution.*

Proof. The statement follows directly from the Bellman principle of optimality [1]. □

As a consequence of Lemma 3.5, it follows that if we can characterize $\mathscr{V}_{t_i}^{\text{init}}$ and we are able to compute the corresponding cost matrix $K(t)$ and corresponding optimal control, we can reduce the problem of solving Problem 1 on the interval $[t_0,t_f)$ to solving Problem 1 on the interval $[t_0,t_i)$. Moreover, by choosing $t_i = t_{\mathtt{n}}$, Problem 1 on the interval $[t_{\mathtt{n}},t_f)$ reduces to an optimal control problem subject to a non-switched DAE. By applying Lemma 3.5 recursively and choosing each t_i to be a switching time, it follows that we can solve Problem 1 by solving $\mathtt{n}$ optimal control problems for non-switched DAEs, each defined on the interval $[t_{i-1},t_i)$, $i \in \{\mathtt{n},\mathtt{n}-1,\ldots,1\}$. This leads to the following control problem for non-switched DAEs.

Problem 2. Find an input $u \in \mathbb{D}^m_{\text{pw}\mathscr{C}^\infty}$ that minimizes

$$J(x_0,u,t_0) = \int_{t_0}^{t_f} \|y(t)\|^2 \,\mathrm{d}t + x(t_f^-) P x(t_f^-), \tag{3.2}$$

$$\text{s.t.} \quad E\dot{x} = Ax + Bu, \tag{3.3a}$$

$$y = Cx + Du, \tag{3.3b}$$

$$x(t_0^-) = x_0 \in \mathbb{R}^n, \tag{3.3c}$$

$$x(t_f^-) \in \mathscr{V}^{\text{end}}, \tag{3.3d}$$

on the interval $[t_0,t_f)$, $x \in \mathbb{D}^n_{\text{pw}\mathscr{C}^\infty}$ is the state, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{q\times n}$ and $D \in \mathbb{R}^{q\times m}$, $P = P^\top \in \mathbb{R}^{n\times n}$ is some symmetric positive semi-definite matrix and $\mathscr{V}^{\text{end}} \subseteq \mathbb{R}^n$ is some subspace.

Remark 3.6. So far, none of the results have implied that the optimal input is an impulse-free distribution. As such, the optimal solution is potentially impulsive and Dirac impulses resulting from the input might cancel out Dirac impulses in the state.

A phenomenon already well-known for ODE optimal control problems is that the cost for the input (given by D) needs to be non-singular to avoid impulsive optimal controls. We will make a similar assumption here as well, which reads as follows:

$$\operatorname{rank}\,(D - CB^{\text{imp}}) = m. \tag{3.4}$$

The main result for the index-1 case of Problem 2 is then given by the forthcoming Theorem 4.10, which shows that Problem 2 is solvable if, and only if, the initial value x_0 is an element of a subspace $\mathscr{V}^{\text{init}}$ which is defined in terms of the given final subspace $\mathscr{V}^{\text{end}}$. Furthermore, an explicit solution for the optimal control is provided.

3.3. **Main result Problem 1.** Returning to Problem 1 we can, utilizing Lemma 3.5, define a sequence of subspaces $\mathscr{V}_{t_i}^{\text{init}}$ as the subspace of feasible initial values for Problem 2 for mode i considered on the time interval $[t_i, t_{i+1})$ with final subspace $\mathscr{V}_{t_{i+1}}^{\text{init}}$ (where $\mathscr{V}_{t_{\mathtt{n}+1}}^{\text{init}} := \mathscr{V}^{\text{end}}$). Furthermore, extending the non-singular input-cost assumption to the switched case yields assuming

$$\operatorname{rank}\,(D_p - C_p B_p^{\text{imp}}) = m. \tag{3.5}$$

for $p \in \{0, 1, ..., \mathtt{n}\}$. Given this assumption, we are ready to present the main result for solvability of Problem 1 in the case each mode of the switched DAE is given by an index-1 DAE.

Theorem 3.7. *Consider the regular, index-1, switched DAE (1.2) satisfying (3.5) for which Problem 1 is solvable, i.e.* $x_0 \in \mathscr{V}_{t_0}^{\text{init}}$. *Then the optimal input is given by*

$$u(t) = -R_{\sigma(t)}^{-1}((B_{\sigma(t)}^{\text{diff}})^\top K(t) + S_{\sigma(t)}^\top)\Pi_{\sigma(t)} x(t),$$

where $R_\sigma = (D_\sigma - C_\sigma B_\sigma^{\text{imp}})^\top (D_\sigma - C_\sigma B_\sigma^{\text{imp}})$, $S_\sigma = (D_\sigma - C_\sigma B_\sigma^{\text{imp}})^\top C_\sigma$ *and* Π_i *is a projector resulting from the Wong sequence based on* (E_i, A_i); $K(t)$ *is given by the (symmetric) solution of*

$$\dot{K} = -(A_i^{\text{diff}})^\top K - K A_i^{\text{diff}} + (S_i + K B_i^{\text{diff}}) R_i^{-1}((B_i^{\text{diff}})^\top K + S_i^\top) - C_i^\top C_i,$$

on $[t_i, t_{i+1})$ *with boundary conditions* $K(t_{i+1}^-) = \Psi_i^\top P_i \Psi_i$, *where*

$$P_{\mathtt{n}} := P, \qquad P_i := K(t_{i+1}^+),\ i = \mathtt{n}-1, \mathtt{n}-2, \ldots, 0,$$

and $\Psi_i := (I - B_i^{\text{imp}} N_i)\Pi_i$, *for some* N_i *that satisfies* $\begin{bmatrix} I & 0 & N_i\Pi_i \end{bmatrix} \ker \mathscr{H}_i = 0$, *with*

$$\mathscr{H}_i := \begin{bmatrix} B_i^{\text{imp}\top} P_i B_i^{\text{imp}} & B_i^{\text{imp}\top}(I - \Pi_{\mathscr{V}_i^{\text{end}}})^\top \\ (I - \Pi_{\mathscr{V}_i^{\text{end}}}) B_i^{\text{imp}} & 0 \\ -\Pi_i^\top P_i B_i^{\text{imp}} & -\Pi_i^\top (I - \Pi_{\mathscr{V}_i^{\text{end}}})^\top \end{bmatrix}^\top. \tag{3.6}$$

and $\Pi_{\mathscr{V}_i^{\text{end}}}$ *is a projector onto the subspace* $\mathscr{V}_i^{\text{end}} := \mathscr{V}_{t_{i+1}}^{\text{init}}$. *Finally, the optimal cost is given by*

$$\min_u J(x_0, u, t_0) = x_0^\top K(t_0) x_0.$$

We conclude this section by stressing that initially, we did not impose any assumptions on the index of each mode of the switched DAE in Problem 1. However, allowing for modes with arbitrary index only leads to some technicalities. As such, in the presentation of the technical results in the next sections, we will first focus on solving Problem 2 under the assumption that the DAE at hand is of index-1. Then we will show how this result can be generalized to solvability of Problem 2 without this assumption. Finally, we will show how this general result recursively leads to the solution to Problem 1 where each mode is of arbitrary index. As can be expected, the general main result reduces Theorem 3.7 in the case each mode is of index-1 and to a result for Problem 2 in the case no switches occur.

4. Optimal control for non-switched index-1 DAEs

As mentioned previously, we will consider first the optimal control problem for non-switched DAEs, i.e., Problem 2. Furthermore, we will first consider Problem 2 subject to an index-1 DAE. As such, the state can be decomposed as

$$x = x^{\text{diff}} + x^{\text{imp}} = x^{\text{diff}} - B^{\text{imp}} u, \tag{4.1}$$

where the differential state component satisfies

$$\dot{x}^{\text{diff}} = A^{\text{diff}} x^{\text{diff}} + B^{\text{diff}} u, \qquad x^{\text{diff}}(t_0^-) = \Pi_{(E,A)} x_0. \tag{4.2}$$

respectively. As a consequence, we can state the following result which follows from Lemma 3.1.

Corollary 4.1. *If there exists an input $u \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ that solves Problem 2 where the DAE (3.3) is of index-1, then $u(t) = F(t)x^{\text{diff}}(t)$ for some $F : \mathbb{R} \to \mathbb{R}^{m\times n}$.*

Proof. After decomposing the state as (4.1) and considering the ODE dynamics (4.2) the proof is analogous to the proof of Lemma 3.1. ☐

4.1. **Terminal cost.** Decomposing the state as in (4.1) allows us to express the terminal cost as a quadratic function of the differential state $x^{\text{diff}}(t_f^-)$ and the input. Consequently, an input u with a value $u(t_f^-)$ that minimizes the terminal cost with respect to the resulting $x^{\text{diff}}(t_f^-)$ can be chosen. However, as the terminal cost penalizes the value of u at t_f^- from the left and this value needs to be well-defined, the input u needs to be continuous on at least $[t_f - \varepsilon, t_f)$ for some $\varepsilon > 0$. Therefore altering a solution (x^{diff}, u) such that the output has a desired value at t_f^- will in general influence the running cost. As a result, we can not optimize the running cost and the terminal cost independently of each other. However, the following result shows that the value of the optimal input $u(t_f^-)$ minimizes the terminal cost with respect to the value $x^{\text{diff}}(t_f^-) \in \operatorname{im}\Pi_{(E,A)}$.

Lemma 4.2. *Let u be an input that solves Problem 2 and let x^{diff} be the corresponding optimal trajectory. Denote $u(t_f^-) = \psi^* \in \mathbb{R}^m$ and $x^{\text{diff}}(t_f^-) = \zeta^* \in \operatorname{im}\Pi_{(E,A)}$. Then ψ^* is a minimizer of the following problem.*

$$\begin{aligned} \min_{\psi\in\mathbb{R}^m} \quad & (\zeta^* - B^{\text{imp}}\psi)^\top P(\zeta^* - B^{\text{imp}}\psi), \\ s.t. \quad & \zeta^* - B^{\text{imp}}\psi \in \mathscr{V}^{\text{end}}. \end{aligned} \tag{4.3}$$

The problem with $\mathscr{V}^{\text{end}} = \mathbb{R}^n$ was already studied in [45]; but the consideration of a general subspace $\mathscr{V}^{\text{end}}$ increases the difficulty significantly and is crucial for utilizing the result in the context of switched DAEs.

The proof can be found in the Appendix.

Lemma 4.3. *For a given* $\zeta \in \left(\mathscr{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}\right) \cap \operatorname{im} \Pi_{(E,A)}$ *the vector* $\psi^* \in \mathbb{R}^m$ *solves*

$$\begin{aligned} \min_{\psi \in \mathbb{R}^m} \quad & (\zeta - B^{\text{imp}}\psi)^\top P(\zeta - B^{\text{imp}}\psi), \\ \text{s.t.} \quad & \zeta - B^{\text{imp}}\psi \in \mathscr{V}^{\text{end}}, \end{aligned} \tag{4.4}$$

if and only if $\zeta = \left[\begin{smallmatrix} 0 & 0 & \Pi_{(E,A)} \end{smallmatrix}\right] h$ *and* $\psi^* = \left[\begin{smallmatrix} I & 0 & 0 \end{smallmatrix}\right] h$ *for some* $h \in \ker \mathscr{H}$*, where*

$$\mathscr{H} := \begin{bmatrix} B^{\text{imp}\top} P B^{\text{imp}} & B^{\text{imp}\top}(I - \Pi_{\mathscr{V}^{\text{end}}})^\top \\ (I - \Pi_{\mathscr{V}^{\text{end}}})B^{\text{imp}} & 0 \\ -\Pi_{(E,A)}^\top P B^{\text{imp}} & -\Pi_{(E,A)}^\top (I - \Pi_{\mathscr{V}^{\text{end}}})^\top \end{bmatrix}^\top \tag{4.5}$$

and $\Pi_{\mathscr{V}^{\text{end}}}$ *is any projector onto* $\mathscr{V}^{\text{end}}$.

The proof can be found in the Appendix.

Given the result of Lemma 4.3, we can compute which states $\zeta \in \operatorname{im} \Pi_{(E,A)}$ are possibly an endpoint of an optimal trajectory. Moreover, for each potential endpoint $\zeta \in \operatorname{im} \Pi_{(E,A)}$ we can compute a value of ψ that solves (4.4). Consequently, for a given optimal solution (x^{diff}, u) where $x^{\text{diff}}(t_f^-) = \zeta$, we are able to express the terminal cost of this solution in terms of $x^{\text{diff}}(t_f^-)$ only.

Corollary 4.4. *If there exists an input u that solves Problem 2 then the optimal terminal cost satisfies*

$$x(t_f^-)^\top P x(t_f^-) = x^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi x^{\text{diff}}(t_f^-),$$

where $\Psi = (I - B^{\text{imp}} N)\Pi_{(E,A)}$*, for any N satisfying*

$$\begin{bmatrix} I & 0 & -N\Pi_{(E,A)} \end{bmatrix} \ker \mathscr{H} = 0, \tag{4.6}$$

where $\mathscr{H}$ *is given by* (4.5).

Proof. Since (x, u) is solving Problem 2 it follows from Lemma 4.2 that $\psi = u(t_f^-)$ minimizes (4.3) for $\zeta = x^{\text{diff}}(t_f^-)$. By Corollary 4.1 the optimal input is linear in x^{diff}, i.e., $u = Nx^{\text{diff}}$ for some linear map N. Hence by Lemma 4.3, N satisfies $\left[\begin{smallmatrix} I & 0 & -N\Pi_{(E,A)} \end{smallmatrix}\right] h = 0$ for any $h \in \ker \mathscr{H}$, i.e. (4.6) actually has a solution. Furthermore, for any other $\bar{N}$ which satisfies (4.6) it follows that $\bar{N}\zeta = \left[\begin{smallmatrix} 0 & 0 & \bar{N}\Pi_{(E,A)} \end{smallmatrix}\right] h = \left[\begin{smallmatrix} I & 0 & 0 \end{smallmatrix}\right] h = \left[\begin{smallmatrix} 0 & 0 & N\Pi_{(E,A)} \end{smallmatrix}\right] h = N\zeta$, hence the effective optimal feedback does not depend on the specific choice of N satisfying (4.6). ☐

Although the minimum of the objective function in (4.4) is uniquely given for a particular $x^{\text{diff}} \in \mathbb{R}^n$, a minimizer $u \in \mathbb{R}^m$ is not necessarily unique. However, the following result can still be concluded regarding an optimal input.

Corollary 4.5. *If an input u solves Problem 2 then the final optimal feedback satisfies* $u(t_f^-) = Nx^{\text{diff}}(t_f^-)$ *for some N satisfying* (4.6)*. Furthermore, the optimal solution satisfies* $x(t_f^-) = \Psi x^{\text{diff}}(t_f^-) \in \mathscr{V}^{\text{end}}$.

4.2. **Running cost.** We will now turn our attention to the running cost and the optimal control given on the interval $[t_0, t_f)$. To that extent, we will write

$$\|y(t)\|^2 = \begin{bmatrix} x \\ u \end{bmatrix}^\top \begin{bmatrix} C \\ D \end{bmatrix} \begin{bmatrix} C & D \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} C^\top \\ \hat{D}^\top \end{bmatrix} \begin{bmatrix} C & \hat{D} \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix},$$

where $\hat{D} = D - CB^{\mathrm{imp}}$. Then, after defining $Q = C^\top C$, $S = \hat{D}^\top C$ and $R = \hat{D}^\top \hat{D}$, we can rewrite the cost functional as

$$J(x_0, u, t_0) = \int_{t_0}^{t_f} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix} \,\mathrm{d}t + x(t_f^-)^\top P x(t_f^-).$$

Lemma 4.6. *Assume that the matrices C and D satisfy (3.4). Then $\hat{D} := D - CB^{\mathrm{imp}}$ has full column rank and $\hat{D}^\top \hat{D}$ is positive definite.*

Proof. Since $D - CB^{\mathrm{imp}}$ has full column rank it follows directly that $\hat{D}^\top \hat{D}$ is invertible. □

Remark 4.7. As already mentioned in the introduction, the assumption (3.4) can be regarded as the differential-algebraic version of the assumption that $D^\top D$ is positive definite, which is commonly made in the LQR problem for ordinary differential equations. The assumption that $D^\top D$ is positive definite is usually made to penalize every input action in the cost. As the solution x of a DAE has a component that is directly determined by the input, the cost functional can penalize the input also indirectly via penalizing the corresponding state component. Hence penalizing all input actions is equivalent to the condition (3.4).

Lemma 4.8. *If an input $u \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ solves Problem 2 then*

$$u(t) = -R^{-1}\left((B^{\mathrm{diff}})^\top K(t) + S^\top\right) x^{\mathrm{diff}}(t), \tag{4.7}$$

where K is the (symmetric) solution of

$$\dot{K} = -(A^{\mathrm{diff}})^\top K - KA^{\mathrm{diff}} - Q + (S + KB^{\mathrm{diff}})R^{-1}((B^{\mathrm{diff}})^\top K + S^\top), \tag{4.8}$$

with terminal condition $K(t_f^-) = \Psi^\top P \Psi$ with Ψ as in Corollary 4.4.

Proof. For any symmetric-matrix-valued continuously differentiable function $K(t)$ defined on $[t_0, t_f)$ we can write the cost-functional as

$$J(x_0, u, t_0) - x_0^\top K(t_0) x_0 = \int_{t_0}^{t_f} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix}^\top \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix} \begin{bmatrix} x^{\mathrm{diff}} \\ u \end{bmatrix} + \frac{d}{dt}(x^{\mathrm{diff}}(t)^\top K(t) x^{\mathrm{diff}}(t)) \,\mathrm{d}t$$
$$+ x^{\mathrm{diff}}(t_f^-)^\top \left(\Psi^\top P \Psi - K(t_f^-)\right) x^{\mathrm{diff}}(t_f^-).$$

Taking the two integrands together and computing the second integrand using the differential equation and the completion of the squares formula, we obtain after defining

$$W := \dot{K} + (A^{\mathrm{diff}})^\top K + KA^{\mathrm{diff}} - (S + KB^{\mathrm{diff}})R^{-1}((B^{\mathrm{diff}})^\top K + S^\top) + Q$$

and omitting the dependence on t:

$$\begin{aligned}(x^{\text{diff}})^\top Q x^{\text{diff}} &+ 2(x^{\text{diff}})^\top S^\top u + u^\top R u + \tfrac{\mathrm{d}}{\mathrm{d}t}((x^{\text{diff}})^\top K x^{\text{diff}})\\
&= (x^{\text{diff}})^\top (Q + (A^{\text{diff}})^\top K + K A^{\text{diff}} + \dot{K}) x^{\text{diff}} + 2u^\top (B^{\text{diff}} K + S) x^{\text{diff}} + u^\top R u,\\
&= (x^{\text{diff}})^\top (S + K B^{\text{diff}}) R^{-1} ((B^{\text{diff}})^\top K + S^\top) x^{\text{diff}} + 2u^\top (B^{\text{diff}} K + S) x^{\text{diff}}\\
&\quad + u^\top R u + (x^{\text{diff}})^\top W x^{\text{diff}},\\
&= \| R^{1/2} u + R^{-1/2}((B^{\text{diff}})^\top K + S^\top) x^{\text{diff}} \|^2 + (x^{\text{diff}})^\top W x^{\text{diff}},\end{aligned}$$

Consequently, we can rewrite the cost in Problem 2 as

$$\begin{aligned}J(x_0,u,t_0) = x_0^\top K(t_0^-) x_0 &+ \int_{t_0}^{t_f} \| R^{1/2} u(t) + R^{-1/2}((B^{\text{diff}})^\top K(t) + S^\top) x^{\text{diff}}(t) \|^2\\
&+ x^{\text{diff}}(t)^\top W(t) x^{\text{diff}}(t)\, \mathrm{d}t + x^{\text{diff}}(t_f^-)^\top \left(\Psi^\top P \Psi - K(t_f^-) \right) x^{\text{diff}}(t_f^-).\end{aligned}$$

Under the assumption (3.4), it follows from the literature on solutions on the Riccati differential equation (*cf.* Theorem 10.7 in [40]) that a function K satisfying $K(t_f^-) = \Psi^\top P \Psi$ such that $W = 0$ can always be chosen. Hence by choosing $K(t)$ such that $W = 0$ and $K(t_f^-) = \Psi^\top P \Psi$ we obtain that the cost $J(x^{\text{diff}}, u)$ can be expressed as

$$J(x_0,u,t_0) - x_0^\top K(t_0^-) x_0 = \int_{t_0}^{t_f} \| R^{1/2} u(t) + R^{-1/2}((B^{\text{diff}})^\top K(t) + S^\top) x^{\text{diff}}(t) \|^2 \, \mathrm{d}t. \tag{4.9}$$

Clearly without the constraint $x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-) \in \mathscr{V}^{\text{end}}$ it follows that $J(x_0,u,t_0)$ is minimized if the input is given by

$$u = -R^{-1}((B^{\text{diff}})^\top K + S^\top) x^{\text{diff}}.$$

In any case, (4.9) shows that $x_0^\top K(t_0^-) x_0$ is a lower bound for the optimal cost.

Next, we will show that for the problem with the constraint $x^{\text{diff}}(t_f^-) - B^{\text{imp}} u(t_f^-) \in \mathscr{V}^{\text{end}}$ we have $\inf_u J(x_0,u,t_0) = x_0^\top K(t_0^-) x_0$. For that let x^{diff} be the solution of (4.2) with input u given by (4.7).

Case 1: $x^{\text{diff}}(t_f^-) \in \mathscr{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$
Consider the input $u_\delta = u + v_\delta$ where v_δ is defined as

$$v_\delta(t) = \begin{cases} 0, & \text{if } t \in [t_0, t_f - \delta),\\ \phi(t), & \text{if } t \in [t_f - \delta, t_f - \frac{\delta}{2}),\\ N x^{\text{diff}}(t_f^-) - u(t_f^-), & \text{if } t \in [t_f - \frac{\delta}{2}, t_f),\end{cases}$$

for some N satisfying $\left[I \;\; 0 \;\; -N\Pi_{(E,A)} \right] \ker \mathscr{H} = 0$ and $\phi(t)$ is chosen in such a way that the corresponding solution x_δ^{diff} satisfies $x_\delta^{\text{diff}}(t_f^-) = x^{\text{diff}}(t_f^-)$ (which is always possible, *cf.* Lemma A.1 in the Appendix).

Note that, with Ψ as in Corollary 4.4,

$$x_\delta(t_f^-) = x_\delta^{\text{diff}}(t_f^-) - B^{\text{imp}}(u(t_f^-) + v_\delta(t_f^-)) = x^{\text{diff}}(t_f^-) - B^{\text{imp}} N x^{\text{diff}}(t_f^-) = \Psi x^{\text{diff}}(t_f^-) \in \mathscr{V}^{\text{end}}$$

and thus u_δ is a feasible input. Furthermore, using Corollary 4.4, we have

$$x_\delta(t_f^-)^\top P x_\delta(t_f^-) = x^{\text{diff}}(t_f^-)^\top \Psi^\top P \Psi x^{\text{diff}}(t_f^-) \leqslant x(t_f^-)^\top P x(t_f^-).$$

Let $M := x(t_f^-)^\top P x(t_f^-) - x_\delta(t_f^-)^\top P x_\delta(t_f^-) \geqslant 0$, then

$$J(x_0,u_\delta,t_0) - J(x_0,u,t_0) \leqslant \int_{t_f-\delta}^{t_f} \|y_\delta(t)\|^2 - \|y(t)\|^2 \,\mathrm{d}t - M$$

and for every $\varepsilon > 0$ we can choose $\delta > 0$ such that

$$J(x_0,u_\delta,t_0) - J(x_0,u,t_0) \leqslant \varepsilon - M.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small and because $J(x_0,u_\delta,t_0) \geqslant x_0^\top K(t_0^-)x_0 = J(x_0,u,t_0)$ we can thus conclude that $M = 0$ and

$$\inf_{\bar{u}} J(x_0,\bar{u},t_0) = x_0^\top K(t_0^-)x_0.$$

Case 2: $x^{\text{diff}}(t_f^-) \notin \mathscr{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$

Suppose Problem 2 is solvable. Let (x^*,u^*) be the optimal solution satisfying $x^*(t_0^-) = x_0$ and $x^*(t_f^-) = x^{\text{diff}*}(t_f^-) - B^{\text{imp}}u^*(t_f^-)$, where $x^{\text{diff}*}(t_f^-) = x_1$ for some $x_1 \in \operatorname{im}\Pi$.

We will first prove that the input u^* solving Problem 2 for the initial value x_0 also minimizes Problem 2 with $\mathscr{V}^{\text{end}}$ replaced by $\mathscr{V}^{\text{end}*} := \mathscr{V}^{\text{end}} + \operatorname{im} B^{\text{imp}}$.

Suppose there exists an input $\bar{u}$ such that $\bar{x}(t_f^-) \in \mathscr{V}^{\text{end}*}$ and $J(x_0,\bar{u},t_0) < J(x_0,u^*,t_0)$. Using similar arguments as in Case 1, for any $\varepsilon > 0$ we can find an input $\bar{u}^*$ such that the corresponding solution $\bar{x}^*$ satisfies $\bar{x}^* \in \mathscr{V}^{\text{end}}$ and $J(x_0,\bar{u}^*,t_0) < J(x_0,\bar{u},t_0) + \varepsilon$. Hence for sufficiently small $\varepsilon > 0$ we arrive at the contradiction $J(x_0,\bar{u}^*,t_0) < J(x_0,u^*,t_0)$. Hence we can conclude that u^* also solves Problem 2 with $\mathscr{V}^{\text{end}}$ replaced by $\mathscr{V}^{\text{end}*}$.

Next, let $\tilde{x}_0 \neq x_0$ be the initial value for which the solution $(\tilde{x},\tilde{u})$, with $\tilde{u}$ given by (4.7) satisfies $\tilde{x}(t_f^-) \in \mathscr{V}^{\text{end}*}$. Note that this implies $\tilde{x}^{\text{diff}}(t_f^-) = x_1$. Then clearly it follows from the expression (4.9) that $\tilde{u}$ solves Problem 2 with $\mathscr{V}^{\text{end}*}$ for the initial value $\tilde{x}_0$. It follows from the linearity of the optimal solution that $v := u^* - \tilde{u}$ must be the optimal solution for the initial value $z_0 := x_0 - \tilde{x}_0$ for Problem 2 with $\mathscr{V}^{\text{end}*}$. Then the solution (z^{diff},v) satisfies

$$\begin{aligned}
z^{\text{diff}}(t_f^-) &= e^{A^{\text{diff}}(t_f-t_0)}z_0 + \int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)}B^{\text{diff}}v(\tau)\,\mathrm{d}\tau \\
&= e^{A^{\text{diff}}(t_f-t_0)}x_0 + \int_{t_0}^{t_f} e^{A^{\text{diff}}(t-\tau)}B^{\text{diff}}u^*(\tau)\,\mathrm{d}\tau - e^{A^{\text{diff}}(t_f-t_0)}\tilde{x}_0 - \int_{t_0}^{t} e^{A^{\text{diff}}(t-\tau)}B^{\text{diff}}\tilde{u}(\tau)\,\mathrm{d}\tau \\
&= x^{\text{diff}*}(t_f^-) - \tilde{x}^{\text{diff}}(t_f^-)
\end{aligned}$$

and consequently $z^{\text{diff}}(t_f^-) = 0$. However, this implies that $z_0 = 0$, because a linear state-feedback, cannot control an initial condition to zero unless it is zero. Hence we arrive at the contradiction $\tilde{x}_0 = x_0$. □

4.3. **Combining the results.** Until now we have only been concerned with necessary conditions for solvability of Problem 2. The reason that the conditions in Corollary 4.8 are not sufficient in general is that a feedback of the form (4.7) does not necessarily ensure that all the constraints are satisfied. A solution (x^{diff},u) with u given by (4.7) and $x^{\text{diff}}(t_0^-) = x_0 \in \operatorname{im}\Pi_{(E,A)}$ does not necessarily satisfy

$$x(t_f^-) = x^{\text{diff}}(t_f^-) - B^{\text{imp}}u(t_f^-) \in \mathscr{V}^{\text{end}},$$

nor

$$x(t_f^-)^\top P x(t_f^-) = x^{\mathrm{diff}}(t_f^-)\Psi^\top P \Psi x^{\mathrm{diff}}(t_f^-),$$

for any N for which $\left[\begin{smallmatrix} I & 0-N\Pi_{(E,A)} \end{smallmatrix}\right] \ker \mathscr{H} = 0$. Both these conditions can be rewritten equivalently as

$$(I - \Pi_{\mathscr{V}^{\mathrm{end}}})\Theta_{t_f} x^{\mathrm{diff}}(t_f^-) = 0 \tag{4.10}$$

and

$$\left(\Theta_{t_f}^\top P \Theta_{t_f} - \Psi^\top P \Psi\right) x^{\mathrm{diff}}(t_f^-) = 0, \tag{4.11}$$

where

$$\Theta_{t_f} := (I + B^{\mathrm{imp}} R^{-1}\left((B^{\mathrm{diff}})^\top \Psi^\top P \Psi + S^\top\right)); \tag{4.12}$$

here, we utilized the fact that under the feedback (4.7) we have $x(t_f^-) = \Theta_{t_f} x^{\mathrm{diff}}(t_f^-)$

However, it follows straightforwardly that if a solution (x^{diff}, u) with $x^{\mathrm{diff}}(t_0^-) = x_0$ and u satisfying (4.7) is such that (4.10) and (4.11) are satisfied the input is optimal. To prove this, we will first introduce the backward state-transition matrix, defined similarly to [15] or [41] and which also appears in [3].

Definition 4.9. The backward state transition matrix for the closed loop time-varying differential equation

$$\dot{x}^{\mathrm{diff}} = \left(A^{\mathrm{diff}} - B^{\mathrm{diff}} R^{-1}((B^{\mathrm{diff})^\top} K + S^\top)\right) x^{\mathrm{diff}},$$

where K is a solution to (4.8) with terminal condition $K(t_f^-) = \Psi^\top P \Psi$, is defined as the map $\Omega(\cdot, t_f) : \mathbb{R} \to \mathbb{R}^{n\times n}$, i.e. all solutions x^{diff} of the above closed-loop satisfy $x^{\mathrm{diff}}(t) = \Omega(t, t_f) x^{\mathrm{diff}}(t_f^-)$.

Theorem 4.10. *Problem 2 is solvable if and only if*

$$x_0 \in \mathscr{V}^{\mathrm{init}} := \Omega(t_0, t_f) \ker \Xi \Pi_{(E,A)}, \tag{4.13}$$

with

$$\Xi = \begin{bmatrix} (I - \Pi_{\mathscr{V}^{\mathrm{end}}})\Theta_{t_f} \\ \Theta_{t_f}^\top P \Theta_{t_f} - \Psi^\top P \Psi \end{bmatrix},$$

where $\Omega(t_0, t_f)$ is the backward state transition matrix as defined in Definition 4.9, Θ_{t_f} is given by (4.12) and Ψ is given in Corollary 4.4. The optimal control is then given by

$$u(t) = -R^{-1}((B^{\mathrm{diff}})^\top K(t) + S^\top) x^{\mathrm{diff}}(t), \tag{4.14}$$

where K is a solution to (4.8) with terminal condition $K(t_f^-) = \Psi^\top P \Psi$. Finally, the optimal cost is given by

$$J^*(x_0, u, t_f) = x^{\mathrm{diff}}(t_0^-) K(t_0) x^{\mathrm{diff}}(t_0^-)$$

and is quadratic in $x^{\mathrm{diff}}(t_0^-)$.

5. LQR FOR HIGHER-INDEX DAES

In the previous section, Problem 2 has been considered where the DAE was assumed to be of index-1. This assumption allowed us to decompose the state into a component that solves an ODE and a feed-trough term depending directly on the input. Furthermore, the solution (x,u) was impulse-free regardless of the initial value as long as the input was impulse-free. This decomposition can not be made anymore if a higher index DAE is considered. As a result of the higher index of the DAE, the state will also depend on the derivatives of the input u and the state will not necessarily be impulse-free if the input is impulse-free.

In fact, there exists an input that results in an impulse-free solution (x,u) satisfying $x(t_0^-) = x_0$ if and only if the initial value is contained in the impulse-controllable space $\mathscr{C}^{\text{imp}}$. For such initial values, we will show in the following a particular impulse-controllable DAE can be considered equivalently instead of (3.3). Specifically, after applying a preliminary feedback, an index-1 DAE can be considered.

For initial values $x_0 \notin \mathscr{C}^{\text{imp}}$, i.e., initial values that are not contained in the impulse-controllable space, the corresponding solution will inevitably contain a Dirac impulse, i.e., regardless of the choice of input. However, an optimal control might still exist for these initial values, as long as the corresponding Dirac impulses are not visible in the output. Combining these observations leads to the following result.

Lemma 5.1. *Consider the DAE (3.3) and assume it is of arbitrary index. There exists an impulse-free input $u \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ such that for the solution (x,u) satisfying $x(t_0^-) = x_0$ of (3.3) the output is impulse-free at t_0, i.e., $y[t_0] = Cx[t_0] + Du[t_0] = 0$, if and only if $x_0 \in \mathscr{C}^{\text{imp}} + \mathscr{O}^{\text{imp}}$ where $\mathscr{O}^{\text{imp}}$ is the impulse-unobservable space defined as*

$$\mathscr{O}^{\text{imp}} := \ker \begin{bmatrix} CE^{\text{imp}} \\ C(E^{\text{imp}})^2 \\ \vdots \\ C(E^{\text{imp}})^{\nu-1} \end{bmatrix} \tag{5.1}$$

and ν is the index of nilpotency of E^{imp}.

Proof. The proof can be found in the Appendix. □

As the condition $x_0 \in \mathscr{C}^{\text{imp}} + \mathscr{O}^{\text{imp}}$ is necessary and sufficient for the existence of an impulse-free output, it is a necessary condition for the existence of an impulse-free input that minimizes (3.2), subject to (3.3). However, it suffices to only consider initial values contained in $\mathscr{C}^{\text{imp}}$. To see this, we first observe that $\mathscr{C}^{\text{imp}} \subseteq \mathscr{V}^*$ and $\mathscr{O}^{\text{imp}} \subseteq \mathscr{W}^*$ where the Wong limits for the regular matrix pair (E,A) satisfy $\mathscr{V}^* \oplus \mathscr{W}^* = \mathbb{R}^n$. Therefore, we can choose a projector W such that $\operatorname{im} W = \mathscr{C}^{\text{imp}}$ and $\operatorname{im}(I-W) = \ker W \subseteq \mathscr{W}^*$, note that then $W\Pi = \Pi$ and $\Pi(I-W) = 0$. Now we can decompose the initial value as $x_0 = Wx_0 + (I-W)x_0 =: \bar{x}_0 + \tilde{x}_0$ with $\bar{x}_0 \in \mathscr{C}^{\text{imp}}$ and $\tilde{x}_0 \in \mathscr{O}^{\text{imp}}$. Applying the same decomposition on the solution x for some input u we can conclude that $\tilde{x} = 0$ on (t_0, t_f) (independently from the input u); this follows from the solution formula given in (2.5) together with $\operatorname{im}(E^{\text{imp}})^i B^{\text{imp}} \subseteq \mathscr{C}^{\text{imp}} = \operatorname{im} W = \ker(I-W)$ and $\Pi\tilde{x}_0 = \Pi(I-W)x_0 = 0$. The impulsive response $\tilde{x}[t_0]$ due to the initial value $\tilde{x}_0$ is in general non-zero, but these Dirac impulses are not visible in the output because by construction $\tilde{x}_0 \in \mathscr{O}^{\text{imp}}$. Hence for the optimal control problem, we only need to consider $x_0 \in \mathscr{C}^{\text{imp}}$.

Corollary 5.2. *Consider the DAE* (3.3)*. For any* $x_0 \in \mathscr{C}^{\text{imp}} + \mathscr{O}^{\text{imp}}$ *a solution* (x,u) *with* $x(t_0^-) = x_0$ *satisfies* $y(t) = \bar{y}(t)$ *where* $\bar{y}(t)$ *is the output corresponding to the solution* $(\bar{x},u)$ *with* $\bar{x}(t_0^-) = Wx_0$ *where* W *is a projector onto* $\mathscr{C}^{\text{imp}}$ *with* $\ker W \subseteq \mathscr{W}^*$*.*

Hence in the remainder of the paper, we will consider initial values contained in the impulse-controllable space of (3.3). However, instead of considering (3.3), which is not impulse-controllable and of higher index, we can consider an auxiliary impulse-controllable DAE. The latter has the same input-output behavior as (3.3) for initial values $x_0 \in \mathscr{C}^{\text{imp}}$.

Lemma 5.3. *Let* $\mathscr{C}^{\text{imp}}$ *be the impulse-controllable space of* (3.3)*. A distribution* (x,u) *satisfying* $x(t_0^-) \in \mathscr{C}^{\text{imp}}$*, solves* (3.3) *if and only if it solves*

$$EW\dot{x} = Ax + Bu, \tag{5.2}$$

$$y = Cx + Du, \tag{5.3}$$

where W *is a projector onto* $\mathscr{C}^{\text{imp}}$ *with* $\ker W \subseteq \mathscr{W}^*$*. Moreover the pair* (EW,A) *is regular and* (5.2) *is an impulse-controllable DAE.*

Proof. The proof can be found in the Appendix. □

The auxiliary DAE (5.2) is much easier to analyze with respect to the optimal control problem as for impulse-controllable DAEs there exists a feedback that reduces the index to 1, *cf.* Lemma 2.4. Let $u = Lx + v$ be such a feedback. After applying this feedback we obtain

$$\Sigma^{\text{aux}} : \begin{cases} EW\dot{x} &= (A+BL)x + Bv, \\ \quad y &= (C+DL)x + Dv, \end{cases} \tag{5.4}$$

which is of index-1. For index-1 DAEs the results have already been established and the following result shows that these results can be carried over to (5.4). As such, to solve Problem 2 subject to a higher index DAE, it suffices to find an optimal input v that solves the following auxiliary Problem.

Problem 3. Consider the DAE (3.3) and let W be a projector onto $\mathscr{C}^{\text{imp}}$ with $\ker W \subseteq \mathscr{W}^*$. Find an input $v \in \mathbb{D}^m_{\text{pw}\mathscr{C}^\infty}$ that minimizes

$$J(x_0, v, t_0) = \int_{t_0}^{t_f} \|\bar{y}(t)\|^2 \, \mathrm{d}t + x(t_f^-)Px(t_f^-), \tag{5.5}$$

$$\text{s.t.} \quad EW\dot{x} = (A+BL)x + Bv, \tag{5.6a}$$

$$\bar{y} = (C+DL)x + Dv, \tag{5.6b}$$

$$x(t_0^-) = x_0 \in \mathbb{R}^n, \tag{5.6c}$$

$$x(t_f^-) \in \mathscr{V}^{\text{end}}, \tag{5.6d}$$

on the interval $[t_0, t_f)$, where L is a matrix, such that $(EW, A+BL)$ is of index-1.

Lemma 5.4. *Let* $\mathscr{C}^{\text{imp}}$ *be the impulse-controllable space corresponding to* (3.3)*. There exists an input* $u \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ *that solves Problem* 2 *subject to* $x_0 \in \mathscr{C}^{\text{imp}}$ *if and only if there exists an input* $v \in \mathbb{D}^m_{pw\mathscr{C}^\infty}$ *that solves Problem* 3 *subject to* $x_0 \in \mathscr{C}^{\text{imp}}$*. Furthermore, the optimal input that solves Problem* 2 *subject to* (3.3) *satisfies* $u = Lx + v$*, where* v *is the optimal input that solves Problem* 3*.*

Proof. As $x_0 \in \mathscr{C}^{\text{imp}}$ it follows form Lemma 5.3 that the solution (x,u) solves (3.3) if and only if it solves (5.2). Hence we will consider solutions of (5.2). Applying a feedback to (5.2) can be regarded as a change of coordinates

$$\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}. \tag{5.7}$$

Writing (5.2) as

$$\begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix},$$

enables us to write

$$\begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{\bar{x}} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} EW & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{u} \end{bmatrix} = \begin{bmatrix} A & B \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix} = \begin{bmatrix} (A+BL) & B \end{bmatrix} \begin{bmatrix} \bar{x} \\ v \end{bmatrix}.$$

Hence (x,u) solves (5.2) if and only if, $(\bar{x},v)$ satisfying (5.7) solves (5.4).

Furthermore, it follows that in Problem 3 the cost resulting from applying an input v to (5.6a) equals the cost in Problem 2 resulting from applying an input $u = Lx - v$ to (3.3). Therefore we can conclude that v solves Problem 3 then $u = v - L\bar{x} = v - Lx$ solves Problem 2 and vice-versa. □

Given a method to compute the optimal input to Problem 2, it remains to characterize the space for which the problem can be solved. This space can easily be computed based on the computation of the optimal input for Problem 3.

Lemma 5.5. *Let $\overline{\mathscr{V}}^{\text{init}}$ be the space of initial values for which Problem 3 can be solved. Then the space of initial values for which Problem 2 can be solved is given by*

$$\mathscr{V}^{\text{init}} = \overline{\mathscr{V}}^{\text{init}} \cap (\mathscr{C}^{\text{imp}} + \mathscr{O}^{\text{imp}}), \tag{5.8}$$

where $\mathscr{C}^{\text{imp}}$ and $\mathscr{O}^{\text{imp}}$ are the impulse-controllable space and impulse-observable space corresponding to (3.3).

6. LQR FOR GENERAL SWITCHED DAES

Given the results regarding Problem 2 where the DAE is assumed to be of arbitrary index, the results for Problem 1 where each mode of (1.2a) is of arbitrary index follow straightforwardly. A summarizing algorithm is presented in Algorithm 1.

We illustrate the overall procedure with the following illustrative (academic) example.

Example 6.1. Consider the switched DAE given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, && 0 \leqslant t < 1, \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dot{x} &= -x + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u, && 1 \leqslant t < 2, \\ \dot{x} &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u, && 2 \leqslant t < 3, \end{aligned}$$

together with the output

$$y = x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u.$$

Algorithm 1 LQR with subspace constraint

Input : E_i, A_i, B_i, C_i, D_i, $i = 0, 1, \ldots, \mathtt{n}$, P, $\mathscr{V}^{\text{end}}$,
switching signal $\sigma : [t_0, t_f) \to \{1, \ldots, \mathtt{n}\}$ in standard form (3.1)

Output: Subspace $\mathscr{V}_{t_0}^{\text{init}}$ of feasible initial states, optimal state feedback $u(t) = F_\sigma(t)x(t)$

Set $\mathscr{V}_{t_{\mathtt{n}+1}}^{\text{end}} := \mathscr{V}^{\text{end}}$
for $i = \mathtt{n}, \mathtt{n}-1, \ldots, 0$ **do**

Step 1: Preconditioning

Compute $\mathscr{C}_i^{\text{imp}}$ via (2.8) and $\mathscr{O}_i^{\text{imp}}$ via (5.1)
Choose any projector W_i onto $\mathscr{C}_i^{\text{imp}}$
Utilizing Lemmas 2.4 and 5.3, choose L_i such that $(EW_i, A_i + B_iL_i)$ is of index 1
Define

$$(\bar{E}_i, \bar{A}_i, \bar{B}_i, \bar{C}_i, \bar{D}_i) := (E_iW_i, A_i + B_iL_i, B_i, C_i + D_iL_i, D_i)$$

Compute $\bar{\Pi}_i$, $\bar{A}_i^{\text{diff}}$, $\bar{B}_i^{\text{diff}}$,$\bar{B}_i^{\text{imp}}$ via the Wong sequences of $(\bar{E}_i, \bar{A}_i)$

Step 2: Solve Riccati equations

If $i < \mathtt{n}$ then $\mathscr{V}_{t_{i+1}}^{\text{end}} = \mathscr{V}_{t_{i+1}}^{\text{init}}$
Solve (backwards in time)

$$\dot{K} = -(\bar{A}_i^{\text{diff}})^\top K - K\bar{A}_i^{\text{diff}} + (\bar{S}_i + K_i\bar{B}_i^{\text{diff}})\bar{R}_i^{-1}((\bar{B}_i^{\text{diff}})^\top K + \bar{S}_i^\top) - \bar{Q}_i,$$

on $[t_i, t_{i+1})$, with $\bar{R}_i := (\bar{D}_i - \bar{C}_i\bar{B}_i^{\text{imp}})^\top(\bar{D}_i - \bar{C}_i\bar{B}_i^{\text{imp}})$, $\bar{S}_i := (\bar{D}_i - \bar{C}_i\bar{B}_i^{\text{imp}})^\top \bar{C}^\top$, $\bar{Q}_i := \bar{C}_i^\top \bar{C}_i$ and final condition $P_i = P$ or

$$P_i = \Psi_i^\top K(t_{i+1}^+)\Psi_i, \qquad \text{for } i < \mathtt{n},$$

where $\Psi_i = (I - \bar{B}_i^{\text{imp}}N_i)\bar{\Pi}_i$, for some N_i that satisfies $\begin{bmatrix} I & 0 & N_i\bar{\Pi}_i \end{bmatrix} \ker \mathscr{H}_i = 0$, with

$$\mathscr{H}_i = \begin{bmatrix} \bar{B}_i^{\text{imp}\top} P_i \bar{B}_i^{\text{imp}} & \bar{B}_i^{\text{imp}\top}(I - \Pi_{\mathscr{V}_{t_{i+1}}^{\text{end}}})^\top \\ (I - \Pi_{\mathscr{V}_{t_{i+1}}^{\text{end}}})\bar{B}_i^{\text{imp}} & 0 \\ -\bar{\Pi}_{(\bar{E}_i,\bar{A}_i)}^\top P_i \bar{B}_i^{\text{imp}} & -\bar{\Pi}_i^\top (I - \Pi_{\mathscr{V}_{t_{i+1}}^{\text{end}}})^\top \end{bmatrix}^\top$$

and $\Pi_{\mathscr{V}_{t_{i+1}}^{\text{end}}}$ is a projector onto the subspace $\mathscr{V}_{t_{i+1}}^{\text{end}}$

Step 3: Compute subspace $\mathscr{V}_{t_i}^{\text{init}}$

Compute $\Omega_i(t_i, t_{i+1})$ (see Def. 4.9) for the system

$$\dot{x}^{\text{diff}} = \left(\bar{A}_i^{\text{diff}} - \bar{B}_i^{\text{diff}}\bar{R}_i^{-1}((\bar{B}_i^{\text{diff}})^\top K + \bar{S}_i^\top)\right)x^{\text{diff}}.$$

Compute $\bar{\mathscr{V}}_{t_i}^{\text{init}} = \Omega_i(t_i, t_{i+1}) \ker \Xi_i \bar{\Pi}_i$, with .

$$\Xi_i = \begin{bmatrix} (I - \Pi_{\mathscr{V}_{t_{i+1}}^{\text{end}}})\Theta_i \\ \Theta_i^\top K \Theta_i - P_i \end{bmatrix}$$

where $\Theta_i := I + \bar{B}^{\text{imp}}(\bar{R}_i^{-1}\left((\bar{B}_i^{\text{diff}})^\top P_i + \bar{S}_i^\top\right)$
Compute $\mathscr{V}_{t_i}^{\text{init}} = \overline{\mathscr{V}}_{t_i}^{\text{init}} \cap (\mathscr{C}_i^{\text{imp}} + \mathscr{O}_i^{\text{imp}})$.
end for

Step 4: Compute optimal control

Compute

$$u(t) = -\bar{R}_{\sigma(t)}^{-1}\left((\bar{B}_{\sigma(t)}^{\text{diff}})^\top K(t) + \bar{S}_{\sigma(t)}^\top\right)\bar{\Pi}_{\sigma(t)}x(t),$$

The cost functional to be minimized is thus given by

$$J(x_0,u,0) = \int_0^3 \|y(t)\| \, \mathrm{d}t,$$

subject to $x(3^-) \in \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \right\} := \mathscr{V}^{\text{end}}$. In this particular problem, the terminal cost matrix is given by $P = 0$.

Note that the mode active on $1 \leqslant t < 2$ is impulse-controllable, but not index-1. To that extent a preliminary index-reducing feedback given by

$$u(t) = \begin{cases} \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x(t) + v(t), & 1 \leqslant t < 2, \\ v(t), & \text{otherwise}, \end{cases}$$

is applied, resulting in

$$\begin{aligned}
(\bar{E}_0,\bar{A}_0,\bar{B}_0,\bar{C}_0,\bar{D}_0) &= \left(I, \left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], I, \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \right), \\
(\bar{E}_1,\bar{A}_1,\bar{B}_1,\bar{C}_1,\bar{D}_1) &= \Bigg(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right], \\
&\qquad \left[\begin{smallmatrix} 1 \\ 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \Bigg), \\
(\bar{E}_2,\bar{A}_2,\bar{B}_2,C_2,D_2) &= \left(I, \left[\begin{smallmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], I, \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \right),
\end{aligned}$$

The optimal feedback matrix on each interval $[t_i, t_{i+1})$, $i \in \{0,1,2\}$ is computed after solving

$$\begin{aligned}
\dot{K}_i &= -(\bar{A}_i^{\text{diff}})^\top K_i - K_i \bar{A}_i^{\text{diff}} + (\bar{S}_i + K_i \bar{B}_i^{\text{diff}}) \bar{R}_i^{-1} ((\bar{B}_i^{\text{diff}})^\top K + \bar{S}_i^\top) - \bar{Q}_i, \\
K_i(t_{i+1}^-) &= \Psi_i^\top K_{i+1}(t_{i+1}^+) \Psi_i,
\end{aligned}$$

where $\Psi_i = (I - B_i^{\text{imp}} N_i) \bar{\Pi}_i$ for some N_i which satisfies $\left[\begin{smallmatrix} I & 0 & -N_i \bar{\Pi} \end{smallmatrix}\right] \ker \mathscr{H}_i = 0$ and $K_2(3^-) = 0$. The computation yields

$$\begin{aligned}
K_1(2^-) &= \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right], \; K_0(1^-) = \left[\begin{smallmatrix} 0.39 & 0 & 0.38 \\ 0 & 0 & 0 \\ 0.38 & 0 & 2.40 \end{smallmatrix}\right], \\
K_0(0^+) &= \left[\begin{smallmatrix} 0.21 & -0.03 & 0.07 \\ -0.03 & 0.03 & -0.19 \\ 0.07 & -0.19 & 1.59 \end{smallmatrix}\right].
\end{aligned}$$

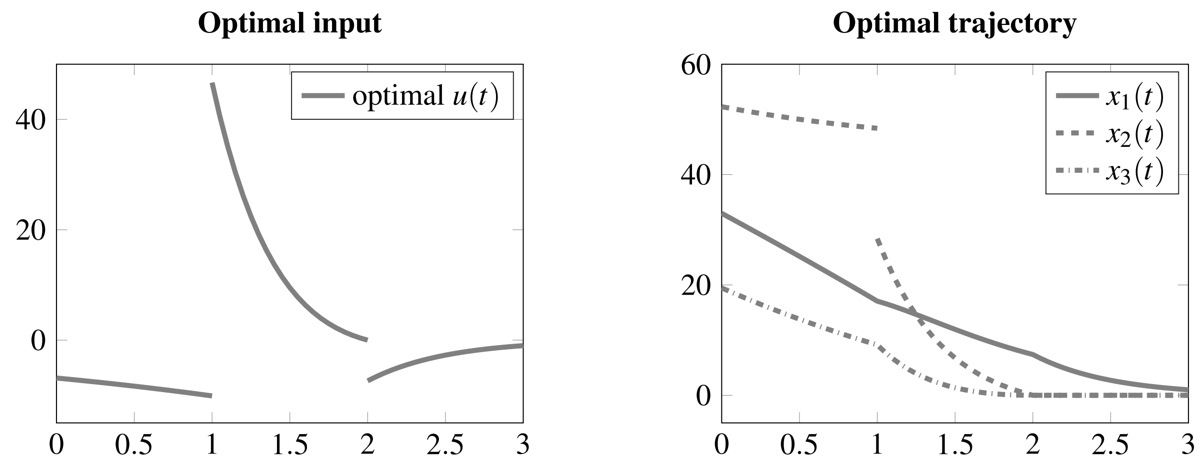
After computing the backward state transition matrices Ξ_i it follows that

$$\begin{aligned}
\mathscr{V}_2^{\text{init}} &= \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \right\}, \quad \mathscr{V}_1^{\text{init}} = \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 0 \\ 0.54 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right] \right\}, \\
\mathscr{V}_0^{\text{init}} &= \operatorname{span}\left\{ \left[\begin{smallmatrix} 1 \\ 0 \\ 0.49 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \\ 0.06 \end{smallmatrix}\right] \right\}.
\end{aligned}$$

Given the solution K_i we can compute the optimal input and optimal state trajectory, which are shown in Figure 1a and 1b, respectively. As can be seen, both the optimal input and the optimal trajectory are piecewise continuous and contain jumps. ◇

7. Conclusion

In this paper, the finite horizon LQR problem for switched linear differential-algebraic equations has been studied. It was shown that for switched DAEs with a switching signal that induces locally finitely many switches the problem can be solved by solving LQR problems for non-switched DAE repeatedly. First, it was shown how to solve the non-switched problems for



(a) The optimal input $u(t)$ that solves Problem 1

(b) The corresponding optimal trajectory $x(t)$ that solves Problem 1 with initial value $x_0 = \begin{bmatrix} 32.98 & 52.30 & 19.46 \end{bmatrix}^\top$.

FIGURE 1. Simulation results for Example 6.1

index-1 DAEs followed by an extension of the results to higher index DAEs. The resulting optimal control can be computed based on the solution of a Riccati differential equation expressed in terms of the differential system matrices. Although these differential systems matrices do not depend on a particular coordinate transformation, it remains a future direction of research to express the results in terms of the original system matrices,

Another natural direction of future research is to explore the admission of impulsive inputs. The authors suspect however that the results in this direction would not be much different than the ones already obtained in this paper and the results on singular optimal control obtained by Willems et all [47].

APPENDIX A. PROOFS

Proof of Lemma 3.1. First we will show that the map $x_0 \mapsto u$ is linear, where $x(t_0^-) = x_0$ and u solves Problem 1; in particular, we will show that λu is the optimal control for any initial value λx_0 and that for any initial values $x_0, z_0 \in \mathbb{R}^n$ for which optimal inputs u_x, u_z exists, the input $u_x + u_z$ is optimal for the initial value $x_0 + z_0$.

To that extent, let $V(x_0,t_0)$ be the value function as

$$V(x_0,t_0) = \inf_u J(x_0,u,t_0) \tag{A.1}$$

Applying the input λu to an initial condition λx_0 results in a trajectory λx, due to the linearity of solutions of the switched DAE. This means that $J(\lambda x_0, \lambda u) = \lambda^2 J(x_0,u)$ for any $\lambda \in \mathbb{R}$ and we can conclude that

$$\lambda^2 V(x_0,t_0) = \lambda^2 J(x_0,u) = J(\lambda x_0, \lambda u) = V(\lambda x_0, t_0).$$

Hence we can conclude that if u is the optimal input for x_0 that λu is the optimal input for λx_0. In the following, we will prove if u_x and u_z are the optimal inputs for x_0 and z_0 respectively, that

$u_x + u_z$ is the optimal input for $x_0 + z_0$. To do so, it will be first shown that

$$V(x_0+z_0,t_0)+V(x_0-z_0,t_0) = 2V(x_0,t_0)+2V(z_0,t_0). \tag{A.2}$$

Observe that

$$\begin{aligned}\|C_\sigma(x+z)+D_\sigma(u_x+u_z)\|^2 + \|C_\sigma(x-z)+D_\sigma(u_x-u_z)\|^2 \\ = 2\|C_\sigma x+D_\sigma u_x\|^2+2\|C_\sigma z+D_\sigma u_z\|^2,\end{aligned}$$

and

$$\begin{aligned}(x(t_f^-)+z(t_f^-))^\top P(x(t_f^-)+z(t_f^-))+(x(t_f^-)-z(t_f^-))^\top P(x(t_f^-)-z(t_f^-)) \\ = 2x(t_f^-)^\top Px(t_f^-)+2z(t_f^-)^\top Pz(t_f^-),\end{aligned}$$

from which we can conclude that

$$J(x_0+z_0,u_x+u_z,t_0)+J(x_0-z_0,u_x-u_z,t_0) = 2J(x_0,u_x,t_0)+2J(z_0,u_z,t_0).$$

Hence for all input u_x and u_z (and thus not necessarily the optimal ones) we obtain

$$\begin{aligned}&V(x_0+z_0,t_0)+V(x_0-z_0,t_0)\\ &\qquad\leqslant J(x_0+z_0,u_x+u_z)+J(x_0-z_0,u_x-u_z)\\ &\qquad= 2J(x_0,u_x)+2J(z_0,u_z),\end{aligned}$$

which means that $V(x_0+z_0,t_0)+V(x_0-z_0,t_0) \leqslant 2V(x_0,t_0)+2V(z_0,t_0)$. Conversely,

$$\begin{aligned}&2V(x_0,t_0)+2V(z_0,t_0)\\ &\qquad\leqslant 2J(x_0,u_x,t_0)+2J(z_0,u_z,t_0)\\ &\qquad= J(x_0+z_0,u_x+u_z,t_0)+J(x_0-z_0,u_x-u_z,t_0),\end{aligned}$$

from which we can conclude that $2V(x_0,t_0)+2V(z_0,t_0) \leqslant V(x_0+z_0,t_0)+V(x_0-z_0,t_0)$ and therefore the equality $V(x_0+z_0,t_0)+V(x_0-z_0,t_0) = 2V(x_0,t_0)+2V(z_0,t_0)$ follows. Furthermore, if u_x is the optimal input for x_0 and u_z is the optimal input for z_0 then

$$\begin{aligned}&V(x_0-z_0,t_0)+V(x_0+z_0,t_0)\\ &\qquad= 2V(x_0,t_0)+2V(z_0,t_0)\\ &\qquad= 2J(x_0,u_x,t_0)+2J(z_0,u_z,t_0)\\ &\qquad= J(x_0+z_0,u_x+u_z,t_0)+J(x_0-z_0,u_x-u_z,t_0).\end{aligned}$$

Since $V(x_0+z_0,t_0) \leqslant J(x_0+z_0,u_x+u_z)$ and similarly $V(x_0-z_0,t_0) \leqslant J(x_0-z_0,u_x-u_z)$, it follows that

$$0 \leqslant J(x_0+z_0,u_x+u_z)-V(x_0+z_0,t_0) = V(x_0-z_0,t_0)-J(x_0-z_0,u_x-u_z) \leqslant 0,$$

and thus

$$V(x_0+z_0,t_0) = J(x_0+z_0,u_x+u_z),$$

which shows that u_x+u_z is optimal for x_0+z_0.

Hence there exists a linear map between the optimal trajectory and the optimal input. In particular, the map $x(t_0^-) \mapsto u(t_0^+)$ is linear, i.e., there exists a matrix $F(t_0) \in \mathbb{R}^{m\times n}$ such that $u(t_0^+) = F(t_0)x(t_0^-)$.

From the dynamic programming principle [1, 6] it follows that $u_{[\tau,t_f)}$ is the optimal control for the cost function in Problem 1 considered on the interval $[\tau,t_f)$ for any $\tau\in[t_0,t_f)$, hence by replacing the initial time t_0 in the above argumentation by $\tau\in[t_0,t_f)$ we can conclude that for every $\tau\in[t_0,t_f)$ a matrix $F(\tau)\in\mathbb{R}^{m\times n}$ exists such that the optimal control satisfies $u(\tau^+)=F(\tau)x(\tau^-)$.

Proof of Lemma 4.2. Before proving Lemma 4.2 we need the following technical lemma:

Lemma A.1. *Consider the ODE (2.5a) on the interval $[0,\delta]$ and with zero initial condition. Then for any $\alpha\in\mathbb{R}^m$, there exists $\phi:[0,\delta/2]\to\mathbb{R}^m$ such that the input*

$$u(t)=\begin{cases}\phi(t), & t\in[0,\delta/2)\\ \alpha, & t\in[\delta/2,\delta)\end{cases}$$

has a corresponding solution x^{diff} with $x^{\text{diff}}(\delta^-)=0$.

Proof. Let $x_1^{\text{diff}}:=-e^{-A^{\text{diff}}\delta/2}\int_0^{\delta/2}e^{A^{\text{diff}}(\delta/2-\tau)}B^{\text{diff}}\alpha\,\mathrm{d}\tau$, then applying $u(t)=\alpha$ on $[\delta/2,\delta)$ with initial value x_1^{diff} will result in a solution which reaches zero at $t=\delta$. Furthermore, by definition $e^{A^{\text{diff}}\delta/2}x_1$ is reachable, and since the reachable space is A^{diff}-invariant, it follows that also x_1^{diff} is reachable from zero, which guarantees the existence of ϕ as claimed. □

In order to prove Lemma 4.2, assume now that u solves Problem 2 for some fixed x_0. Let x^{diff} be the corresponding optimal trajectory on $[t_0,t_f)$. Denote $u(t_f^-)=\psi\in\mathbb{R}^m$ and $x^{\text{diff}}(t_f^-)=\zeta\in\operatorname{im}\Pi$. Seeking a contradiction, assume there exists a value w for which $\zeta-B^{\text{imp}}w\in\mathscr{V}^{\text{end}}$ and

$$(\zeta-B^{\text{imp}}w)^\top P(\zeta-B^{\text{imp}}w)=(\zeta-B^{\text{imp}}\psi)^\top P(\zeta-B^{\text{imp}}\psi)-M,$$

for some $M>0$. Consider the solution (x_δ,u_δ) of (3.3) where $u_\delta=u+v_\delta$ and v_δ is defined as

$$v_\delta(t)=\begin{cases}0, & \text{if } t\in[t_0,t_f-\delta),\\ \phi(t), & \text{if } t\in[t_f-\delta,t_f-\frac{\delta}{2}),\\ w-\psi & \text{if } t\in[t_f-\frac{\delta}{2},t_f),\end{cases}$$

where $\phi(t)$ is chosen in such a way that $x_\delta^{\text{diff}}(t_f^-)=x^{\text{diff}}(t_f^-)$, which is always possible according to Lemma A.1.

Furthermore, for any $\varepsilon>0$ there exists a sufficiently small $\delta>0$ such that the output y_δ resulting from the solution $(x_\delta^{\text{diff}},u_\delta)$ satisfies

$$\begin{aligned}\int_{t_0}^{t_f}\|y_\delta(t)\|^2\,\mathrm{d}t&=\int_{t_0}^{t_f-\delta}\|y_\delta(t)\|^2\,\mathrm{d}t+\int_{t_f-s}^{t_f}\|y_\delta(t)\|^2\,\mathrm{d}t=\int_{t_0}^{t_f-\delta}\|y(t)\|^2\,\mathrm{d}t+\int_{t_f-\delta}^{t_f}\|y_\delta(t)\|^2\,\mathrm{d}t\\&\leqslant\int_{t_0}^{t_f}\|y(t)\|^2\,\mathrm{d}t+\varepsilon,\end{aligned}$$

As $u_\delta(t_f^-)=u(t_f^-)+v_\delta(t_f^-)=w$ we find that $x_\delta^{\text{diff}}(t_f^-)-B^{\text{imp}}u_\delta(t_f^-)\in\mathscr{V}^{\text{end}}$ and

$$J(x_0,u_\delta)=J(x_0,u)+\varepsilon-M.$$

Hence for $\varepsilon<M$ there exists an δ such that the solution $(x_\delta^{\text{diff}},u_\delta)$ satisfies $J(x_0,u_\delta,t_0)<J(x_0,u,t_0)$, which contradicts the optimality of (x^{diff},u). Therefore the result follows.

Proof of Lemma 4.3. Note that the terminal cost function

$$(\zeta - B^{\text{imp}}\psi)^\top P(\zeta - B^{\text{imp}}\psi), \tag{A.3}$$

for a given $\zeta \in \operatorname{im}\Pi$ (A.3) is a convex function of $\psi \in \mathbb{R}^m$. Furthermore $\psi \in \mathbb{R}^m$ minimizes (A.3) if and only if ψ minimizes

$$\tfrac{1}{2}\psi^\top B^{\text{imp}\top} P B^{\text{imp}}\psi - \zeta^\top \Pi^\top P B^{\text{imp}}\psi,$$

where here and in the following we replace ζ by $\Pi\zeta$ to enforce that $\zeta = \Pi\zeta \in \operatorname{im}\Pi$. The constraint $\Pi\zeta - B^{\text{imp}}\psi \in \mathscr{V}^{\text{end}}$ is satisfied if and only if $(I - \Pi_{\mathscr{V}^{\text{end}}})(\Pi\zeta - B^{\text{imp}}\psi) = 0$, where $\Pi_{\mathscr{V}^{\text{end}}}$ is a projector onto $\mathscr{V}^{\text{end}}$. This condition can be written equivalently as

$$(I - \Pi_{\mathscr{V}^{\text{end}}})B^{\text{imp}}\psi = (I - \Pi_{\mathscr{V}^{\text{end}}})\Pi\zeta.$$

As this constraint is linear and P is positive semi-definite, it follows that this optimization problem is a convex problem. Hence any local minimizer is a global minimizer. The first-order necessary conditions are thus also sufficient. Hence ψ is a minimizer that satisfies the constraints if and only if there exists a Lagrange multiplier λ such that

$$\begin{bmatrix} B^{\text{imp}\top} P B^{\text{imp}} & B^{\text{imp}\top}(I - \Pi_{\mathscr{V}^{\text{end}}})^\top \\ (I - \Pi_{\mathscr{V}^{\text{end}}})B^{\text{imp}} & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \lambda \end{bmatrix} = \begin{bmatrix} B^{\text{imp}\top} P \\ (I - \Pi_{\mathscr{V}^{\text{end}}}) \end{bmatrix} \Pi\zeta.$$

This can equivalently be written as $\mathscr{H}\xi = 0$ where

$$\mathscr{H} := \begin{bmatrix} B^{\text{imp}\top} P B^{\text{imp}} & B^{\text{imp}\top}(I - \Pi_{\mathscr{V}^{\text{end}}})^\top \\ (I - \Pi_{\mathscr{V}^{\text{end}}})B^{\text{imp}} & 0 \\ -\Pi^\top P B^{\text{imp}} & -\Pi^\top (I - \Pi_{\mathscr{V}^{\text{end}}})^\top \end{bmatrix}^\top \tag{A.4}$$

and $\xi^\top = \begin{bmatrix} \psi^\top & \lambda^\top & \zeta^\top \end{bmatrix}^\top$. Since $\zeta \in \operatorname{im}\Pi$ and hence $\zeta = \Pi\zeta$ the result follows.

Proof of Lemma 5.1. ($\Rightarrow$) Suppose that there exists an impulse-free input such that $y[t] = 0$. Then since the input u is impulse-free, i.e., $u[t] = 0$, it follows that $y[t] = Cx[t] + Du[t] = Cx[t]$. Consequently, the output is impulse-free for a given impulse-free input if and only if $x[t] \in \ker C$. In the case $u[t] = 0$ then it follows from the solution formula (2.7) and observing that $E^{\text{imp}} = E^{\text{imp}}(I - \Pi)$ that

$$Cx[t] = -C\sum_{i=0}^{\nu-1}(E^{\text{imp}})^{i+1}(I - \Pi)(x_0 - x(t_0^+))\delta^{(i)} = 0.$$

Consequently $(I - \Pi)(x_0 - x(t_0^+)) \in \ker C(E^{\text{imp}})^i$, for $i \in \{1, 2, \dots, \nu - 1\}$. Hence we can conclude that $(I - \Pi)(x_0 - x(t_0^+)) \in \mathscr{O}^{\text{imp}}$. Since $(I - \Pi)x(t_0^+) \in \mathscr{C}^{\text{imp}}$ it follows that $(I - \Pi)x_0 \in \mathscr{O}^{\text{imp}} + \mathscr{C}^{\text{imp}}$. Finally, from $\operatorname{im}\Pi \subseteq \mathscr{C}^{\text{imp}}$ we can conclude that

$$x_0 = \Pi x_0 + (I - \Pi)x_0 \in \mathscr{C}^{\text{imp}} + \mathscr{O}^{\text{imp}},$$

which proves the desired result.

($\Leftarrow$). Let $x_0 = p_0 + q_0$ for some $p_0 \in \mathscr{C}^{\text{imp}}$ and $q_0 \in \mathscr{O}^{\text{imp}}$. Then by definition of $\mathscr{C}^{\text{imp}}$ there exists an impulse-free input u such that (p, u) satisfying $p(t_0^-) = p_0$ is impulse-free, i.e., $p[t] = 0$ for all $t \geqslant t_0$. As $E^{\text{imp}}(I - \Pi) = E^{\text{imp}}$ the solution $(q, 0)$ with $q(t_0^-) = q_0$ will satisfy

$$Cq[t_0] = -C\sum_{i=0}^{\nu-1}\left(E^{\text{imp}}\right)^{i+1}(I - \Pi)\,q_0\delta^{(i)} = -C\sum_{i=0}^{\nu-1}\left(E^{\text{imp}}\right)^{i+1} q_0\delta^{(i)} = 0.$$

Hence the solution $(q,0)$ with $q(t_0^-) = q_0$ will only generate a Dirac impulse at t_0, which will not appear in the output y. By linearity of solutions, (x,u) with $x(t_0^-) = x_0$ will satisfy $x(t) = p(t) + q(t)$ and hence

$$y[t] = Cx[t] + Du[t] = C(p[t] + q[t]) = Cq[t] = 0.$$

Hence u is an impulse-free input such that (x,u) with $x(t_0^-) = x_0$ ensures $y[t] = 0$.

Proof of Lemma 5.3. We first show that the pair (EW,A) is regular. To do so, let S,T, be matrices such that the quasi-Weiserstrass form of (E,A) is given by

$$(SET,SAT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & N_{11} & N_{12} \\ 0 & N_{21} & N_{22} \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right).$$

and where the block columns of $T = [T_1, T_2, T_3]$ are such that $\mathscr{C}^{\text{imp}} = \operatorname{im}[T_1, T_2]$ and $\operatorname{im} T_3 = \ker W$. Such a choice is always possible because $\mathbb{R}^n = \mathscr{V}^* \oplus \mathscr{W}^* = \mathscr{C}^{\text{imp}} \oplus \ker W = \mathscr{V}^* \oplus (\mathscr{C}^{\text{imp}} \cap \mathscr{W}^*) \oplus \ker W$. Furthermore, $SB = [B_1^\top, B_2^\top, 0]$ and invariance of $\mathscr{C}^{\text{imp}}$ implies that the coordinate transformation can be chosen such that $N_{21} = 0$, *cf.* [5].

Consequently, $WT = [T_1, T_2, 0]$ and hence

$$(SEWT,SAT) = \left(\begin{bmatrix} I & 0 & 0 \\ 0 & N_{11} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} J & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \right),$$

from which regularity of (EW,A) follows.

From the block structure of the QWF forms of (E,A,B) and (EW,A,B) it is clear that solutions restricted to the subspace $\mathscr{C}^{\text{imp}} = \operatorname{im}[T_1, T_2]$ are identically determined by the same DAEs given in QWF-coordinates by

$$\left(\begin{bmatrix} I & 0 \\ 0 & N_{11} \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right).$$

Finally, the DAE (EW,A,B) is impulse controllable on $\mathscr{C}^{\text{imp}}$ and from the QWF it is clear that the remaining coordinates are governed by the trivially impulse controllable index-1 DAE $0 = x_3$. This concludes the proof.

REFERENCES

[1] BELLMAN, R. *Dynamic Programming*. Princeton University Press, Princeton, NJ, 1957.

[2] BENDER, D., AND LAUB, A. The linear quadratic optimal regulator problem for descriptor systems. *IEEE Trans. Autom. Control 32* (1987), 672–688.

[3] BENDER, D. J., AND LAUB, A. J. The linear-quadratic optimal regulator for descriptor systems. In *Proc. 24th IEEE Conf. Decis. Control, Ft. Lauderdale, FL* (1985), pp. 957–962.

[4] BERGER, T., ILCHMANN, A., AND TRENN, S. The quasi-Weierstraß form for regular matrix pencils. *Linear Algebra Appl. 436*, 10 (2012), 4052–4069.

[5] BERGER, T., AND TRENN, S. Kalman controllability decompositions for differential-algebraic systems. *Syst. Control Lett. 71* (2014), 54–61.

[6] BERTSEKAS, D. P. *Dynamic programming and optimal control*, vol. 1. Athena scientific Belmont, MA, 1995.

[7] COBB, J. D. Descriptor variable systems and optimal state regulation. *IEEE Trans. Autom. Control 28* (1983), 601–611.

[8] COBB, J. D. Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control 29* (1984), 1076–1082.

[9] GARCIA, C. E., PRETT, D. M., AND MORARI, M. Model predictive control: Theory and practice - a survey. *Automatica 25*, 3 (1989), 335–348.

[10] GERDTS, M. *Optimal control of ODEs and DAEs*. Walter de Gruyter, 2012.

[11] GERDTS, M. A survey on optimal control problems with differential-algebraic equations. In *Surveys in Differential-Algebraic Equations II*. Springer, 2015, pp. 103–161.

[12] ILCHMANN, A., LEBEN, L., WITSCHEL, J., AND WORTHMANN, K. Optimal control of differential-algebraic equations from an ordinary differential equation perspective. *Optimal Control Applications and Methods 40*, 2 (2019), 351–366.

[13] ILCHMANN, A., WITSCHEL, J., AND WORTHMANN, K. Model predictive control for linear differential-algebraic equations. *IFAC-PapersOnLine 51*, 20 (2018), 98–103.

[14] ILCHMANN, A., WITSCHEL, J., AND WORTHMANN, K. Model predictive control for singular differential-algebraic equations. *International Journal of Control* (2021), 1–10.

[15] KALMAN, R. E. Contributions to the theory of optimal control. *Bol. Soc. Matem. Mexico II. Ser. 5* (1960), 102–119.

[16] KALMAN, R. E. When is a linear control system optimal? *Trans. ASME J. Basic Eng. 86D* (1964), 51–60.

[17] KALMAN, R. E., AND BUCY, R. S. New results in linear filtering and prediction theory. *ASME Trans., Part D 80* (1961), 95–108.

[18] KUNKEL, P., AND MEHRMANN, V. The linear quadratic control problem for linear descriptor systems with variable coefficients. *Math. Control Signals Syst. 10* (1997), 247–264.

[19] KÜSTERS, F. *Switch observability for differential-algebraic systems*. PhD thesis, Department of Mathematics, University of Kaiserslautern, 2018.

[20] KÜSTERS, F., RUPPERT, M. G.-M., AND TRENN, S. Controllability of switched differential-algebraic equations. *Syst. Control Lett. 78*, 0 (2015), 32 – 39.

[21] KÜSTERS, F., AND TRENN, S. Switch observability for switched linear systems. *Automatica 87* (2018), 121–127.

[22] KÜSTERS, F., TRENN, S., AND WIRSEN, A. Switch-observer for switched linear systems. In *Proc. 56th IEEE Conf. Decis. Control, Melbourne, Australia* (2017). to appear.

[23] LIBERZON, D., AND TRENN, S. On stability of linear switched differential algebraic equations. In *Proc. IEEE 48th Conf. on Decision and Control* (December 2009), pp. 2156–2161.

[24] LIBERZON, D., AND TRENN, S. Switched nonlinear differential algebraic equations: Solution theory, Lyapunov functions, and stability. *Automatica 48*, 5 (May 2012), 954–963.

[25] MEHRMANN, V. *The Linear Quadratic Control Problem: Theory and Numerical Algorithms*. Habilitationsschrift, Universität Bielefeld, Bielefeld, FRG, 1987.

[26] MEHRMANN, V. Existence, uniqueness and stability of solutions to singular, linear-quadratic control problems. *Linear Algebra Appl.* (1989), 291–331.

[27] MEHRMANN, V. *The Autonomous Linear Quadratic Control Problem, Theory and Numerical Solution*. No. 163 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Heidelberg, 1991.

[28] MIRONCHENKO, A., WIRTH, F., AND WULFF, K. Stabilization of switched linear differential algebraic equations and periodic switching. *IEEE Transactions on Automatic Control 60*, 8 (2015), 2102–2113.

[29] MIRONCHENKO, A., WIRTH, F. R., AND WULFF, K. Stabilization of switched linear differential-algebraic equations via time-dependent switching signals. In *Proc. 52nd IEEE Conf. Decis. Control, Florence, Italy* (2013), pp. 5975–5980.

[30] MORARI, M., AND LEE, J. H. Model predictive control: past, present and future. *Computers and Chemical Engineering 23*, 4-5 (1999), 667–682.

[31] REIS, T., RENDEL, O., AND VOIGT, M. The Kalman-Yakubovich-Popov inequality for differential-algebraic systems. *Linear Algebra Appl. 86* (2015), 153–193.

[32] REIS, T., AND VOIGT, M. Linear-quadratic infinite time horizon optimal control for differential-algebraic equations - a new algebraic criterion. In *Proceedings of MTNS-2012* (2012).

[33] TANWANI, A., AND TRENN, S. On observability of switched differential-algebraic equations. In *Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA* (2010), pp. 5656–5661.

[34] TANWANI, A., AND TRENN, S. Observability of switched differential-algebraic equations for general switching signals. In *Proc. 51st IEEE Conf. Decis. Control, Maui, USA* (2012), pp. 2648–2653.

[35] TANWANI, A., AND TRENN, S. Determinability and state estimation for switched differential–algebraic equations. *Automatica 76* (2017), 17–31.

[36] TANWANI, A., AND TRENN, S. Detectability and observer design for switched differential algebraic equations. *Automatica 99* (2019), 289–300.

[37] TOLSA, J., AND SALICHS, M. Analysis of linear networks with inconsistent initial conditions. *IEEE Trans. Circuits Syst. 40*, 12 (Dec 1993), 885 – 894.

[38] TRENN, S. *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Germany, 2009.

[39] TRENN, S. Regularity of distributional differential algebraic equations. *Math. Control Signals Syst. 21*, 3 (2009), 229–264.

[40] TRENTELMAN, H. L., STOORVOGEL, A. A., AND HAUTUS, M. L. J. *Control Theory for Linear Systems*. Communications and Control Engineering. Springer-Verlag, London, 2001.

[41] VAUGHAN, D. R. A negative exponential solution for the matrix Riccati equation. *IEEE Trans. Autom. Control 14* (1969), 72–75.

[42] VOIGT, M. *On Linear-Quadratic Optimal Control and Robustness of Differential-Algebraic Systems*. PhD thesis, Otto-von-Guericke-Universität Magdeburg, publ. by Logos Verlag Berlin, Germany, 2015.

[43] WIJNBERGEN, P., AND TRENN, S. Impulse controllability of switched differential-algebraic equations. In *2020 European Control Conference (ECC)* (2020), IEEE, pp. 1561–1566.

[44] WIJNBERGEN, P., AND TRENN, S. Impulse-free interval-stabilization of switched differential algebraic equations. *Syst. Control Lett. 149* (2021), 104870.

[45] WIJNBERGEN, P., AND TRENN, S. Optimal control of daes with unconstrained terminal costs. In *2021 60th IEEE Conference on Decision and Control (CDC)* (2021), IEEE, pp. 5275–5280.

[46] WIJNBERGEN, P., AND TRENN, S. Impulse-controllability of system classes of switched differential algebraic equations. *Math. Control Signals Syst.* (2023), 1–30. to appear.

[47] WILLEMS, J., KITAPCI, A., AND SILVERMAN, L. Singular optimal control: a geometric approach. *SIAM Journal on Control and Optimization 24*, 2 (1986), 323–337.

[48] WILLEMS, J. C. Least squares optimal control and the algebraic Riccati equation. *IEEE Trans. Autom. Control 16* (1971), 621–634.

[49] WONG, K.-T. The eigenvalue problem $\lambda Tx + Sx$. *J. Diff. Eqns. 16* (1974), 270–280.

[50] WONHAM, W. M. Optimal stationary control of a linear system with state-dependent noise. *SIAM J. Cont. 5*, 3 (1967), 486–500.