Solution theory for switched singular systems in discrete-time

Stephan Trenn

Jan C. Willems Center for Systems and Control

University of Groningen, Netherlands

Joint work with

Do Duc Thuan, Hanoi University for Science and Technology, Vietnam

Sutrisno, Diponegoro University, Indonesia

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System class and motivation

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

- $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{n}\}$ switching signal
- $E_1, E_2, \ldots, E_{\mathtt{n}}, A_1, A_2, \ldots, A_{\mathtt{n}} \in \mathbb{R}^{n \times n}$ with E-matrix singular
- $x : \mathbb{N} \to \mathbb{R}^n$ state

Motivation

- Leontief economic model (Luenberger 1977)
- discretization of continuous-time time-varying DAEs
- sampled feedback loop for descriptor systems

Simple question

What can we say about existence and uniqueness of solutions?

Small excursion to continuous time

$$E_\sigma \dot{x} = A_\sigma x$$

Theorem (Existence and uniqueness in continuous time, TRENN 2012)

Assume (E_i, A_i) are regular, i.e. $\det(sE_i - A_i)$ is not the zero polynomial.
Then for any past trajectory $x^0(\cdot)$ and any $t_0 \in \mathbb{R}$ there exists unique $x(\cdot)$ such that

$$\begin{aligned} x_{(-\infty,t_0)} &= x^0_{(-\infty,t_0)} \\ (E_\sigma \dot{x})_{[t_0,\infty)} &= (A_\sigma x)_{[t_0,\infty)} \end{aligned}$$

In particular, solution behavior is causal w.r.t. to the switching signal.

Distributional solution framework necessary

Above solution result only holds when solution space is enlarged to allow for jumps and Dirac impulses.

A simple example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{SSS}$$

Example

Consider (SSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = I \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = I$$

Nonswitched solution behavior

$$\sigma \equiv 1: \quad \left.\begin{aligned} x_1(k+1) &= x_1(k) \\ 0 &= x_2(k) \end{aligned}\right\} \quad \rightsquigarrow \quad x(k) = \begin{pmatrix} c_1 \\ 0 \end{pmatrix} \ \forall k \in \mathbb{N}$$

$$\sigma \equiv 2: \quad \left.\begin{aligned} 0 &= x_1(k) \\ x_2(k+1) &= x_2(k) \end{aligned}\right\} \quad \rightsquigarrow \quad x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \ \forall k \in \mathbb{N}$$

A simple example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \tag{SSS}$$

Example

Consider (SSS) with

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_1 = I \quad \text{and} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},\ A_2 = I$$

Switched solution behavior $\sigma(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s \end{cases}$

For $k < k_s$ we have $x(k) = \left(\begin{smallmatrix} c_1 \\ 0 \end{smallmatrix}\right)$ and for $k = k_s - 1$ also $x_1(k_s) = x_1(k_s - 1) = c_1$

BUT: For $k = k_s$ also $0 = x_1(k_s)$, hence $c_1 = 0$ necessary!

Furthermore $x_2(k_s)$ not constraint by mode 1 $\rightsquigarrow$ $x_2(k) = c_2$ for all $k \geq k_s$

$\rightsquigarrow \quad x(k) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $k < k_s$ and $x(k) = \begin{pmatrix} 0 \\ c_2 \end{pmatrix}$ for $k \geq k_s$

Observations from example

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

No existence and uniqueness of solutions!

- Not all solutions from the past can be extended to a global solution
- Single initial value leads to multiple solutions in the future
- Loss of causality w.r.t. to switching signal

Definition

(SSS) is called causal w.r.t. the switching signal $:\Longleftrightarrow \forall \sigma, \widetilde{\sigma}\ \forall x(\cdot)$ sol. for σ $\forall \widetilde{k} \in \mathbb{N}$:

$$\sigma(k) = \widetilde{\sigma}(k)\ \forall k \leq \widetilde{k} \quad \Longrightarrow \quad \exists\, \widetilde{x}(\cdot) \text{ sol. for } \widetilde{\sigma}: \ \widetilde{x}(k) = x(k)\ \forall k \leq \widetilde{k}$$

Example not causal w.r.t. the switching signal: Let $\sigma \equiv 1$, $\widetilde{\sigma}(k) = \begin{cases} 1, & k < k_s \\ 2, & k \geq k_s \end{cases}$

$\rightsquigarrow$ no solution $\widetilde{x}$ with $\widetilde{x}(k) = c_1 = x(k) \neq 0$ for $k < k_s$.

Causality and One-Step-Map

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

Question

When is (SSS) causal w.r.t. the switching signal?

More specifically: When is $x(k+1)$ uniquely defined for all $x(k)$, $\sigma(k)$ and $\sigma(k+1)$?

In other words: Is there a one-step-map $\Phi_{i,j} \in \mathbb{R}^{n\times n}$, $i,j \in \{1,2,\ldots,\mathtt{n}\}$ such that

$$\forall \text{ sol. } x(\cdot) \text{ of (SSS)}: \quad x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$$

Regularity and index

Theorem (Quasi-Weierstrass Form)

(E, A) *is regular* $\iff$ $\exists S, T$ *invertible with*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \tag{QWF}$$

where N *is nilpotent*

Definition

(E, A) has index-1 $:\iff$ $N = 0$ in (QWF)

Index-1 (together with regularity) is also called:

- causal
- admissable
- impulse-free

Index-1 characterization

Theorem (see e.g. Griepentrog & März 1986)

(E, A) *is regular and index-1*
$\iff$ $\mathcal{S} \oplus \ker E = \mathbb{R}^n$, *where* $\mathcal{S} := A^{-1}(\operatorname{im} E) := \{\xi \in \mathbb{R}^n \mid A\xi \in \operatorname{im} E\}$
$\iff$ $\mathcal{S} \cap \ker E = \{0\}$
Furthermore, $T = [T_1, T_2]$ *and* $S = [ET_1, AT_2]^{-1}$ *with* $\operatorname{im} T_1 = \mathcal{S}$ *and* $\operatorname{im} T_2 = \ker E$*:*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right) \qquad \text{(QWF)}$$

Corollary

$Ex(k+1) = Ax(k)$ *being regular + index-1 has unique solution with* $x(0) = x_0 \in \mathbb{R}$
$\iff$ $x_0 \in \mathcal{S}$

In fact, $x(k+1) = \Phi_{(E,A)} x(k)$ *with* $\Phi_{(E,A)} := T \begin{bmatrix} J & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$

Is this the sought one-step map already?

Attention

$\Phi_{(E,A)}$ is one-step-map for $Ex(k+1) = Ax(k)$
BUT: Only true when system is active for at least two time-steps:

$$Ex(1) = Ax(0) \implies x(1) \in E^{-1}(Ax(0)) = \{\Phi_{(E,A)}x(0)\} + \ker E$$
$$Ex(2) = Ax(1) \implies x(1) \in A^{-1}(Ex(2)) \subseteq \mathcal{S}$$

Hence, invoking $\mathcal{S} \cap \ker E = \{0\}$,

$$Ex(1) = Ax(0) \quad \wedge \quad Ex(2) = Ax(1) \quad \implies \quad x(1) = \Phi_{(E,A)}x(0)$$

⇝ Not suitable for switched systems!

Both modes in Example were regular+index-1, but no one-step-map exists!

Problem seems to be overlooked in the some past literature!

A key definition

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) \qquad \text{(SSS)}$$

Definition

(SSS) or $\{(E_1, A_1), (E_2, A_2), \ldots, (E_{\mathtt{n}}, A_{\mathtt{n}})\}$ is called jointly index-1 $:\Longleftrightarrow$

$$\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n \qquad \forall i, j \in \{1, 2, \ldots, \mathtt{n}\},\ \mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$$

- Clearly ($i = j$) each pair (E_i, A_i) must be index-1
- In general, (E_j, A_i) is not index-1 (not even regular)

Example

$$E_1 = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right],\ A_1 = I \quad \rightsquigarrow \quad \ker E_1 = \operatorname{span}\left\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right\},\ \mathcal{S}_1 = A_1^{-1}(\operatorname{im} E_1) = \operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right\}$$

$$E_2 = \left[\begin{smallmatrix}0 & 0\\ 0 & 1\end{smallmatrix}\right],\ A_2 = I \quad \rightsquigarrow \quad \ker E_2 = \operatorname{span}\left\{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)\right\},\ \mathcal{S}_2 = A_2^{-1}(\operatorname{im} E_2) = \operatorname{span}\left\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\right\}$$

Clearly, $\mathcal{S}_i \oplus \ker E_j \neq \mathbb{R}^n$ for $i \neq j$.

The one-step-map

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \tag{SSS}$$

Theorem (Anh, Linh, Thuan, Trenn; CDC 2019, Automatica 2020)

Assume (SSS) is jointly index-1. Then $\forall \sigma \ \forall x_0 \in \mathbb{R}^n$*:*

$$x(\cdot) \text{ solves (SSS)} \quad \Longleftrightarrow \quad x_0 \in \mathcal{S}_{\sigma(0)} \ \wedge \ x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$$

where

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \cdot \Phi_{(E_j,A_j)}$$

and $\Pi_{\mathcal{S}_i}^{\ker E_j}$ *is the projector onto* $\mathcal{S}_i$ *along* $\ker E_j$*.*

Skip proof idea

Proof idea

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k) \quad \text{with} \quad \Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \cdot \Phi_{(E_j,A_j)}$$

Lemma

For any subspace $\mathcal{V},\mathcal{W} \subseteq \mathbb{R}^n$ it holds that

$$\mathcal{V} \oplus \mathcal{W} = \mathbb{R}^n \quad \Longrightarrow \quad \mathcal{V} \cap (\{z\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} z\}$$

- jointly index-1 $\Longrightarrow$ $\mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n \rightsquigarrow \Pi_{\mathcal{S}_i}^{\ker E_j}$ well defined
- $E_{\sigma(0)} x(1) = A_{\sigma(0)} x(0) \Longrightarrow x(0) \in \mathcal{S}_{\sigma(0)}$
- Show by induction that $x(k) \in \mathcal{S}_{\sigma(k)} \Longrightarrow \exists!\, x(k+1) \in \mathcal{S}_{\sigma(k+1)}$
  - $E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k) \Longrightarrow x(k+1) \in \{\Phi_{(E_{\sigma(k)},A_{\sigma(k)})} x(k)\} + \ker E_{\sigma(k)}$
  - $E_{\sigma(k+1)} x(k+2) = A_{\sigma(k+1)} x(k+1) \Longrightarrow x(k+1) \in A_{\sigma(k+1)}^{-1}(\operatorname{im} E_{\sigma(k+1)}) = \mathcal{S}_{\sigma(k+1)}$
  - $\overset{\text{Lemma}}{\Longrightarrow} x(k+1) = \Pi_{\mathcal{S}_{\sigma(k+1)}}^{\ker E_{\sigma(k)}} \Phi_{(E_{\sigma(k)},A_{\sigma(k)})} x(k)$

Necessity of index-1 asssumption

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = 0 \qquad \text{(SSS)}$$

Theorem

$\forall\sigma\ x(1) = 0$ *is only solution of (SSS) for* $k = 0, 1$
$\implies \mathcal{S}_i \cap \ker E_j = \{0\}$ *for* $i, j \in \{1, 2, \ldots, \mathtt{n}\}$
$\implies \mathcal{S}_i \oplus \ker E_j = \mathbb{R}^n$ *for* $i, j \in \{1, 2, \ldots, \mathtt{n}\}$

Skip proof sketch

Proof sketch:

- $k = 0$: $E_j x(1) = A_j x(0) = 0 \iff x(1) \in \ker E_j$
- $k = 1$: $E_i x(2) = A_i x(1) \iff x(1) \in \mathcal{S}_i$
- $x(1) = 0$ is only solution $\implies \mathcal{S}_i \cap \ker E_j = \{0\}$
- $\mathcal{S}_i \cap \ker E_j = \{0\}\ \forall i, j \implies \dim \mathcal{S}_i = \dim \mathcal{S}_j$ and $\operatorname{rank} E_i = \operatorname{rank} E_j\ \forall i, j$
  $\implies \dim \mathcal{S}_i + \dim \ker E_j = n\ \forall i, j$

Solvability characterization

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \tag{SSS}$$

Corollary

(SSS) is uniquely solvable $\forall\sigma$ *and* $\forall x_0 \in \mathcal{S}_{\sigma(0)}$ $\iff$ $\{(E_i, A_i)\}$ *is jointly index-1*

Question

For given σ what is actually necessary for solvability?

Example

$k = 0 : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x(1) = x(0) = \begin{pmatrix} 0 \\ a \end{pmatrix}$, $\quad k \geq 1 : \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x(k+1) = x(k)$

Uniquely solvable with $x(k) = \begin{pmatrix} a \\ 0 \end{pmatrix}$ for $k \geq 1$.

BUT: Initial mode is not index-1

Weaker solvability notion

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \qquad \text{(SSS)}$$

Definition (Sutrisno & Trenn 2024)

(SSS) is called (locally uniquely) solvable w.r.t. σ $:\Longleftrightarrow$
$\forall k_0 < k_1$ $\forall x_{k_0} \in \mathcal{S}_{\sigma(k_0)}$ $\exists$ unique $x_{[k_0,k_1]}$ satisfying (SSS) on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$

Motivation

For $k_1 = k_0 + 1$ above definition implies existence of a one-step-map

$$\Phi_{\sigma(k_0+1),\sigma(k_0)} : \mathcal{S}_{\sigma(k_0)} \to \mathcal{S}_{\sigma(k_0+1)}, x(k_0) \mapsto x(k_0+1)$$

only depending on local information (i.e. $(E_{\sigma(k_0)}, A_{\sigma(k_0)})$ and $(E_{\sigma(k_0+1)}, A_{\sigma(k_0+1)})$)

Remark: Global vs. local solvability

Uniquely solvable on $[0, \infty)$ $\overset{\Longleftarrow}{\not\Longrightarrow}$ locally uniquely solvable

Solvability characterization

Reminder: M^+ is generalized inverse of M $\iff$ $MM^+M = M$

Definition

$\{(E_i, A_i)\}$ is called switched index-1 w.r.t. σ $:\iff$

$$E^+_{\sigma(k)}(\operatorname{im} E_{\sigma(k)} \cap \operatorname{im} A_{\sigma(k)}) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \quad \forall k$$

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \qquad \text{(SSS)}$$

Theorem (Solvability characterization)

(SSS) is solvable w.r.t. σ $\iff$ $\{(E_i, A_i)\}$ is switched index-1 w.r.t. σ

In that case: $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}x(k)$ *with* $\Phi_{i,j} := \Pi^{\ker E_j}_{\mathcal{S}_i} E^+_j A_j$

Discussion of switched index-1 condition

$$E^+_{\sigma(k)}(\operatorname{im} E_{\sigma(k)} \cap \operatorname{im} A_{\sigma(k)}) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \quad \forall k$$

- jointly index-1 $\implies$ switched index-1 w.r.t. any σ
- switched index-1 $\not\Longrightarrow$ index-1 or regularity of each mode
- switched index-1 and $i = \sigma(k) = \sigma(k+1) \implies$ index-1 and regularity of mode i
- $\sigma(k) = k \rightsquigarrow$ solvability characterization of general time-varying descriptor system:

$$E(k)x(k+1) = A(k)x(k)$$

- What about $E(k+1)x(k+1) = A(k)x(k)$?
  $\rightsquigarrow$ loss of causality: one-step map $x(k) \mapsto x(k+1)$ then depends on $E(k+2)$!

Sequentiell index-1

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad x(0) = x_0 \tag{SSS}$$

What if we know the mode sequence $(\sigma_j)_{j\in\mathbb{N}}$ of σ but not the exact switching times?

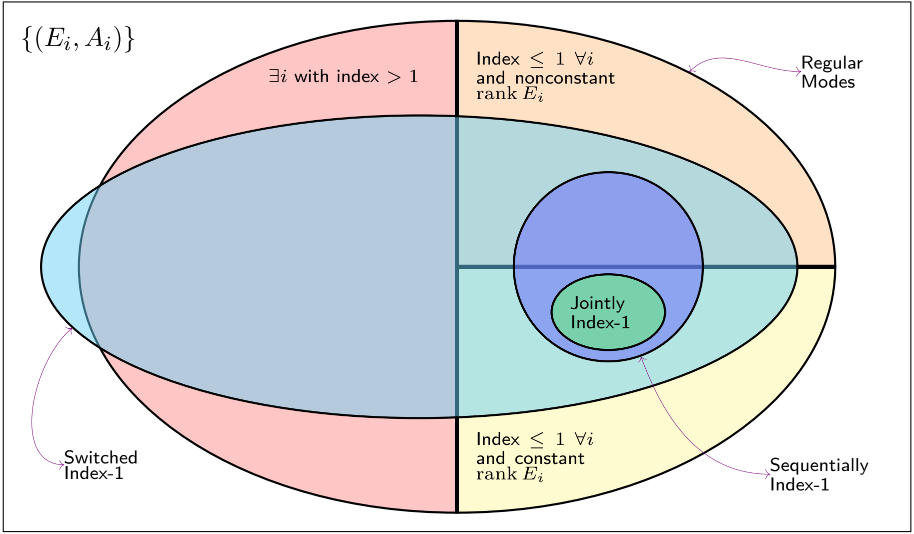
Corollary

(SSS) is solvable for all σ with given mode sequence $(\sigma_j)_{j\in\mathbb{N}}$
$\iff$ *Each individual mode (E_i, A_i) is regular and index-1 and*

$$E_{\sigma_j}^{+}(\operatorname{im} E_{\sigma_j} \cap \operatorname{im} A_{\sigma_j}) \subseteq \ker E_{\sigma_j} \oplus \mathcal{S}_{\sigma_{j+1}} \quad \forall j$$

$\Longleftrightarrow:$ $\{(E_i, A_i)\}$ *is sequentiell index-1 w.r.t. mode sequence $(\sigma_j)_{j\in\mathbb{N}}$*

Summary


$\{(E_i, A_i)\}$
$\exists i$ with index > 1
Index ≤ 1 $\forall i$
and nonconstant
rank E_i
Regular
Modes
Jointly
Index-1
Switched
Index-1
Index ≤ 1 $\forall i$
and constant
rank E_i
Sequentially
Index-1

Outlook

- Stability, observability, model reduction
- Extension to nonlinear case possible
- Extension to systems with input:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k), \quad x(0) = x_0 \tag{InhSSS}$$

  - Many different solution definitions possible!
  - Causality w.r.t. input tricky
  - Consistency dependent on input tricky