Model reduction for switched linear systems

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Problem formulation

System class: Switched linear ODEs

$$\dot{x} = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x$$

$\sigma : [t_0, t_f) \to \{0, 1, 2, \dots, \mathtt{m}\}$
$A_0, A_1, \dots, A_{\mathtt{m}} \in \mathbb{R}^{n\times n}$
$B_0, B_1, \dots, B_{\mathtt{m}} \in \mathbb{R}^{n\times m}$
$C_0, C_1, \dots, C_{\mathtt{m}} \in \mathbb{R}^{p\times n}$

Reduced model

$$\dot{z} = \widehat{A}_\sigma z + \widehat{B}_\sigma u$$
$$y = \widehat{C}_\sigma z$$

$\widehat{A}_0, \widehat{A}_1, \dots, \widehat{A}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n}\times \widehat{n}}$
$\widehat{B}_0, \widehat{B}_1, \dots, \widehat{B}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n}\times m}$
$\widehat{C}_0, \widehat{C}_1, \dots, \widehat{C}_{\mathtt{m}} \in \mathbb{R}^{p\times \widehat{n}}$

Related research

- Simultaneous balancing (Monshizadeh et al. 2012)
- Output-depending switching (Papadopoulus & Prandini 2016)
- Enveloping (non-switched) system (Schulze & Unger 2018)
- Gramian-based approaches (Petrezcky, Gosea, ...)

Novel viewpoint

Consider switched linear ODE as special case of time-varying linear system

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u \\ y &= C(t)x \end{aligned}$$

In particular, consider switching signal as given time-varying system parameter

Existing approaches unsuitable

Existing approaches (Imae, Shokoohi, Silverman, Verriest):

- Smoothness of coefficients assumed
- Reduced model is fully time-varying (not piecewise-constant)

Challenge: Mode-wise reduction

Naive mode-wise reduction is not working

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Each mode is input-output equivalent to same scalar system

$$\dot{z} = u, \quad y = z$$

But outputs do not match anymore after switch!

Reducability of modes is effected by other modes

In example:
Second state is unobservable in first mode, but becomes observable in second mode

Challenge: Different reduced state-dimensions

Reduced switched system with non-equal state-dimensions

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Reduced system (with identical input-output behavior):

on $[t_0, t_1)$:

$$\dot{z}^0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} z^0$$

on $[t_1, t_f)$:

$$\dot{z}^1 = 0 \cdot z^1 + u$$

$$y = z^1$$

with concatination condition: $z^1(t_1) = \begin{bmatrix} 1 & 0 \end{bmatrix} z^0(t_1)$

New system class: Switched ODEs with jumps

Model reduction leaves original system class in general.

Challenge: Duration depend reduction

Reducability may depend on mode durations

Example:

on $[t_0, t_1)$:
$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_2)$:
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$
$$y = 0$$

on $[t_2, t_f)$
$$\dot{x} = 0$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

For $t_2 - t_1 = 2k\pi$ reduction possible to

on $[t_0, t_1)$:
$$\dot{z}^0 = 0 \cdot z^0 + u$$
$$y = z^0$$

on $[t_1, t_2)$: no state
$$y = 0$$

on $[t_2, t_f)$
$$\dot{z}^2 = 0$$
$$y = z^2$$

For almost all other switching durations: First two modes not reducible!

Duration-dependent reduction methods?

Duration dependent methods challenging (numerically expensive, non-robust)

More general system class

Switched linear ODEs with jumps

$$\Sigma_\sigma : \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0, 1, 2, \ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Key features:

- Fixed switching signal with $\sigma(t) = k$ on $[t_k, t_{k+1})$
- states $x^k : (t_k, t_{k+1}) \to \mathbb{R}^{n_k}$ may have mode-dependent dimension
- $J_k : \mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$ defines jumps at switch
- Reduced model in same system class
- Certain switched DAEs $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$, $y = C_\sigma x$ fall into this class

Content

Reduced realization

Question

Given: Switched ODE with jumps and known mode sequence
Sought: Reduced switched ODE with jumps with same input-output behavior

Key reduction intuition

States which are unreachable and unobservable can be removed
BUT: Reachability and observability are not local properties anymore
AND: Exact reachability and observability spaces are time-varying during modes

Definition (Reachability + Unobservability spaces)

$$\mathcal{R}_{[t_0,t)} := \{ x(t^-) \mid \exists \text{ sol. } (x,u) \text{ of } \Sigma_\sigma \text{ with } x(t_0^-) = 0 \}$$
$$\mathcal{U}_{[t,t_f)} := \{ x(t^+) \mid \exists \text{ sol. } (x,u=0) \text{ of } \Sigma_\sigma \text{ with } y = 0 \text{ on } [t,t_f) \}$$

Recursive expressions for reach./unobs. spaces

Theorem

For $t \in (t_k, t_{k+1}]$*:*

$$\mathcal{R}_{[t_0,t)} = e^{A_k(t-t_k)} J_k \mathcal{M}_{k-1} + \mathcal{R}_k$$

where $\mathcal{M}_{k-1} := \mathcal{R}_{[t_0,t_k)}$ *and* $\mathcal{R}_k = \operatorname{im}[B_k, A_k B_k, \ldots, A_k^{n_k-1} B_k] = \langle A_k | \operatorname{im} B_k \rangle$.

For $t \in [t_k, t_{k+1})$*:*

$$\mathcal{U}_{[t,t_f)} = e^{-A_k(t_{k+1}-t)} J_{k+1}^{-1} \mathcal{N}_{k+1} \cap \mathcal{U}_k$$

where $\mathcal{N}_{k+1} := \mathcal{U}_{[t_{k+1},t_f)}$ *and* $\mathcal{U}_k = \ker[C_k / C_k A_k / \ldots / C_k A_k^{n_k-1}] = \langle \ker C_k | A_k \rangle$

Eliminate time-dependence

For any subspace $\mathcal{S} \subseteq \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n\times n}$:

$$\forall t \in \mathbb{R} : \langle \mathcal{S} \mid A \rangle \subseteq e^{At}\mathcal{S} \subseteq \langle A \mid \mathcal{S} \rangle$$

Extended reach. / restricted unobs. space

$\mathcal{M}_k = e^{A_k \tau_k} J_k \mathcal{M}_{k-1} + \mathcal{R}_k$

$\mathcal{N}_k = e^{-A_k \tau_k} J_{k+1}^{-1} \mathcal{N}_{k+1} \cap \mathcal{U}_k$

Extended reachable space

$\overline{\mathcal{R}}_0 := \mathcal{R}_0$
$\overline{\mathcal{R}}_k := \langle A_k \mid J_k \overline{\mathcal{R}}_{k-1} \rangle + \mathcal{R}_k$

Restricted unobservable space

$\overline{\mathcal{U}}_{\mathtt{m}} := \mathcal{U}_{\mathtt{m}}$
$\overline{\mathcal{U}}_k := \langle J_{k+1}^{-1} \overline{\mathcal{U}}_{k+1} \mid A_k \rangle + \mathcal{U}_k$

Theorem (Properties of extended reach. / restricted unobs. space)

- $\overline{\mathcal{R}}_k \supseteq \mathcal{M}_k = \mathcal{R}_{[t_0, t_{k+1})}$, *in fact*, $\forall t \in (t_k, t_{k+1}) : \overline{\mathcal{R}}_k \supseteq \mathcal{R}_{[t_0,t)}$
- $\overline{\mathcal{U}}_k \subseteq \mathcal{N}_k = \mathcal{U}_{[t_k, t_f)}$, *in fact*, $\forall t \in (t_k, t_{k+1}) : \overline{\mathcal{U}}_k \subseteq \mathcal{U}_{[t,t_f)}$
- $\overline{\mathcal{R}}_k \neq \mathbb{R}^n \rightsquigarrow$ *uniformly unreachable states in mode* k
- $\overline{\mathcal{U}}_k \neq \{0\} \rightsquigarrow$ *uniformly unobservable states in mode* k
- $\overline{\mathcal{R}}_k$ *and* $\overline{\mathcal{U}}_k$ *are both* A_k*-invariant*

Weak Kalman decomposition

Theorem (Weak Kalman decomposition)

For (A, B, C) *let* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ *and* $\overline{\mathcal{U}} \subseteq \ker C$ *be two* A*-invariant subspaces.*
Let $T = [T_1, T_2, T_3, T_4]$ *invertible with*

$$\operatorname{im} T_1 = \overline{\mathcal{R}} \cap \overline{\mathcal{U}}, \quad \operatorname{im}[T_1, T_2] = \overline{\mathcal{R}}, \quad \operatorname{im}[T_1, T_3] = \overline{\mathcal{U}}$$

then

$$(T^{-1}AT, T^{-1}B, CT) = \left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right).$$

In particular, $Ce^{At}B = C_2 e^{A_{22}t} B_2$

With above notation let $V := T_2$ be the weak-KD-right-projector and W the corresponding rows in T^{-1} be the weak-KD-left projector.

Proposed reduction method based on weak KD

$$\Sigma_\sigma : \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0, 1, 2, \ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Reduction algorithm

Step 1a: Calculate extended reachable spaces $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \ldots, \overline{\mathcal{R}}_{\mathtt{m}}$
Step 1b: Calculate restricted unobservable spaces $\overline{\mathcal{U}}_{\mathtt{m}}, \overline{\mathcal{U}}_{\mathtt{m}-1}, \ldots, \overline{\mathcal{U}}_0$

Step 2: Calculate weak-KD-left/right-projectors W_k, V_k

Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (W_k A_k V_k, W_k B_k, C_k V_k)$

Step 4: Calculate reduced jump map $\widehat{J}_k := W_k J_k V_{k-1}$

Properties of this reduction method

Reduction method is implementable

Method only depends on mode sequence of switching signal, not mode duration.

Theorem

Original and reduced systems have identical input-output behavior.

Theorem

Applying procedure on reduced system doesn't lead to further reduction.

Open question

Does this procedure lead to minimal realization for almost all switching durations?

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Further reduction

Next step: Remove difficult to observe and difficult to reach states

Define suitable reachability and observability Gramians

$$\mathcal{P}_0^\sigma(t) := \int_{t_0}^{t} e^{A_0(\tau-t_0)} B_0 B_0^\top e^{A_0^\top(\tau-t_0)} d\tau, \qquad t \in [t_0, t_1],$$

$$\mathcal{P}_k^\sigma(t) := e^{A_k(t-t_k)} J_k \mathcal{P}_{k-1}^\sigma(t_k) J_k^\top e^{A_k^\top(t-t_k)} + \int_{s_k}^{t} e^{A_k(\tau-t_k)} B_k B_k^\top e^{A_k^\top(\tau-t_k)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

$$\mathcal{Q}_{\mathtt{m}}^\sigma(t) := \int_{t}^{t_f} e^{A_{\mathtt{m}}^\top(t_f-\tau)} C_{\mathtt{m}}^\top C_{\mathtt{m}} e^{A_{\mathtt{m}}(t_f-\tau)} d\tau, \qquad t \in [t_{\mathtt{m}}, t_f],$$

$$\mathcal{Q}_k^\sigma(t) := e^{A_k^\top(s_{k+1}-t)} J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(t_{k+1}) J_{k+1} e^{A_k(t_{k+1}-t)} + \int_{t}^{t_{k+1}} e^{A_k^\top(t_{k+1}-\tau)} C_k^\top C_k e^{A_k(t_{k+1}-\tau)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

Reachability/unobservability spaces and Gramians

Theorem

$\forall t \in [t_k, t_{k+1}): \quad \operatorname{im} \mathcal{P}_k^\sigma(t) = \mathcal{R}_{[t_0,t)}$ *and* $\ker \mathcal{Q}_k^\sigma(t) = \mathcal{U}_{[t,t_f)}$

Directly using these for balanced truncation?

- Fully time-varying: Balance-based projectors time-varying
  $\rightsquigarrow$ leaves system class of switched ODEs (with jumps)
- Quantitative reachability / observability properties depend on mode durations
  $\rightsquigarrow$ Duration dependence cannot be eliminated
- $\mathcal{P}_k^\sigma(t_k + \varepsilon)$ is dominated by reachability properties $\mathcal{P}_k^\sigma(t_k)$ of past
  $\rightsquigarrow$ reachability properties of current mode not significantly visible in $\mathcal{P}_k^\sigma(t_k + \varepsilon)$
- Analogously for $\mathcal{Q}_k^\sigma(t_{k+1} - \varepsilon)$ which is dominated by the future $\rightsquigarrow$ observability properties of current mode not significantly visible in $\mathcal{Q}_k^\sigma(t_{k+1} - \varepsilon)$

Midpoint based balanced truncation

$m_k := (t_k + t_{k+1})/2$

Use midpoint-Gramians

Use $\mathcal{P}_k^\sigma(m_k)$ and $\mathcal{Q}_k^\sigma(m_k)$ for balance truncation of mode k!

Reachability and observability assumption required

Invertibility (positive definiteness) of Gramians
$\iff \quad \mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ and $\mathcal{U}_{[t,t_f)} = \{0\}$ for all $t \in (t_0, t_f)$

Minimal realization

Similar as in time-invariant setup: Remove unobservable and unreachable states first
$\rightsquigarrow$ reduced realization discussed earlier

Overall reduction algorithm

$$\Sigma_\sigma : \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0,1,2,\ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Algorithm

Step 0: If necessary reduce system via weak Kalman decomposition

Step 1: Calculate midpoint Gramians $\mathcal{P}_k^\sigma(m_k)$ and $\mathcal{Q}_k^\sigma(m_k)$

Step 2a: Based on singular values of $\mathcal{P}_k^\sigma(m_k)\mathcal{Q}_k^\sigma(m_k)$ decide on reduction order $\widehat{n}_k$
Step 2b: Calculate left/right projectors W_k, V_k via standard balanced truncation

Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (W_k A_k V_k, W_k B_k, C_k V_k)$

Step 4: Calculate reduced jump map $\widehat{J}_k := W_k J_k V_{k-1}$