# Switched differential algebraic equations: Jumps and impulses

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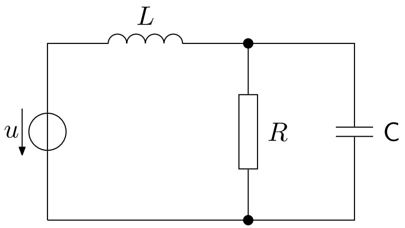
# Why DAEs?

# Electric circuit modelling




## Physical variables

voltage and current for each circuit element

## Defining equations

- element behaviors (voltage-current relation)
- Kirchhoff laws (voltage-loops, current-nodes)

Basic circuit elements:

- Resistors: $v_R(t) = Ri_R(t)$
- Capacitor: $C\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) = i_C(t)$
- Inductor: $L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) = v_L(t)$
- Voltage source: $v_S(t) = u(t)$ (current $i_S$ free)

Kirchhoff laws:

- $i_s = i_L$
- $i_L = i_R + i_C$
- $v_s = v_L + v_R$
- $v_R = v_C$

**We already have arrived at a DAE model!**

With $x = (v_R, i_R, v_C, i_C, v_L, i_L, v_S, i_S)$ we have $\boxed{E\dot{x} = Ax + Bu}$

# Different circuit modeling frameworks

DAE-model:

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&C&0&0&0&0&0\\ 0&0&0&0&0&L&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1&R&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&1\\ 0&-1&0&-1&0&1&0&0\\ -1&0&0&0&-1&0&1&0\\ 1&0&-1&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0&0&0&0&0&0&0&1 \end{bmatrix} x$$

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ODE-model:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} i_L \\ v_c \end{pmatrix} = \begin{bmatrix} 0 & \frac{-1}{L} \\ \frac{1}{C} & \frac{-R}{C} \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{pmatrix} i_L \\ v_c \end{pmatrix}$$

---

Transfer function: $g(s) = \dfrac{R + Cs}{CLs^2 + LRs + 1}$

**Which is the best?**

None! All have advantages and disadvantages.

# Pros and Cons of DAE formulation

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&C&0&0&0&0&0\\ 0&0&0&0&0&L&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1&R&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&1\\ 0&-1&0&-1&0&1&0&0\\ -1&0&0&0&-1&0&1&0\\ 1&0&-1&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix} u$$

## DAE-models: Advantages

- Most natural and intuitive way to model (just write down all first-principal equations)
- Inputs do not need to be specified a priori ($\rightsquigarrow E\dot{x} = Ax$ with rectangular $E$,$A$)
- Connecting two DAE models is trivial (just add new algebraic constraints)
- Sudden structural changes (switches or faults) can be modeled easily

## DAE-models: Disadvantages

- Solution theory more complicated
- Not so many standard tools available for numerical solutions, control design, ...
- Harder to work with manually

# DAEs are not ODEs

$$\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}\dot x = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix} x + \begin{pmatrix}f_1\\f_2\\f_3\end{pmatrix}$$

$$\dot x_2 = x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot f_2$$

$$0 = x_2 + f_2 \longrightarrow x_2 = -f_2 \nearrow$$

$$0 = f_3 \qquad \text{no restriction on } x_3$$

## Key differences to ODEs

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -f_1(0) - \dot f_2(0)$, $x_2(0) = f_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value ($x_3$ free)
- Inhomogeneity not arbitrary
  - structural restrictions ($f_3 = 0$)
  - differentiability restrictions ($\dot f_2$ must be well defined)

# Content

# Equivalence of matrix pairs and DAEs

## Definition (Equivalence of matrix pairs)

$(E_1, A_1)$, $(E_2, A_2)$ are called equivalent $:\Longleftrightarrow$ $(E_2, A_2) = (SE_1T, SA_1T)$

short: $(E_1, A_1) \cong (E_2, A_2)$ or $(E_1, A_1) \overset{S,T}{\cong} (E_2, A_2)$

## Equivalence and solution behavior

For $(E_1, A_1) \cong (E_2, A_2)$ and $B_2 = SB_1$, $C_2 = C_1T$ we have:

$$(x, u, y) \text{ solves } \begin{cases} E_1\dot{x} = A_1x + B_1u \\ y = C_1x \end{cases} \overset{x=Tz}{\Longleftrightarrow} (z, u, y) \text{ solves } \begin{cases} E_2\dot{z} = A_2z + B_2u \\ y = C_2z \end{cases}$$

## Goal: Reveal inner structure of DAEs

Find $S$ and $T$ such that $(SET, SAT)$ has simple structure

# Four types of DAEs

## Definition

- $(E,A)$ is of type ODE $:\Longleftrightarrow (E,A)\cong(I,J)$
- $(E,A)$ is of type nDAE $:\Longleftrightarrow (E,A)\cong(N,I)$, $N$ nilpotent (i.e. $N^\nu=0$)
- $(E,A)$ is of type uDAE $:\Longleftrightarrow (E,A)\cong(\operatorname{diag}(E_1,\ldots,E_k),\operatorname{diag}(A_1,\ldots,A_k))$,

  where $\quad (E_i,A_i)=\left(\begin{bmatrix}1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0\end{bmatrix},\begin{bmatrix}0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1\end{bmatrix}\right)$ underdetermined prototypes
- $(E,A)$ is of type oDAE $:\Longleftrightarrow (E,A)\cong(\operatorname{diag}(E_1,\ldots,E_k),\operatorname{diag}(A_1,\ldots,A_k))$,

  where $\quad (E_i,A_i)=\left(\begin{bmatrix}0 & & \\ 1 & \ddots & \\ & \ddots & 0\\ & & 1\end{bmatrix},\begin{bmatrix}1 & & \\ 0 & \ddots & \\ & \ddots & 1\\ & & 0\end{bmatrix}\right)$ overdetermined prototypes

Every DAE can be decoupled in these four types! $\rightsquigarrow$ Quasi-Kronecker form

# Quasi-Kronecker form

**Theorem (Quasi-Kronecker Form, Berger & T. '12,'13)**

*For any* $E, A \in \mathbb{R}^{\ell \times n}$, $\exists$ *invertible* $S \in \mathbb{R}^{\ell \times \ell}$ *and invertible* $T \in \mathbb{R}^{n \times n}$:

$$(E, A) \overset{S,T}{\cong} \left( \begin{bmatrix} E_U & & & \\ & E_J & & \\ & & E_N & \\ & & & E_O \end{bmatrix}, \begin{bmatrix} A_U & & & \\ & A_J & & \\ & & A_N & \\ & & & A_O \end{bmatrix} \right)$$

*where*

- $(E_U, A_U)$ *is of type uDAE (underdetermined part)*
- $(E_J, A_J)$ *is of type ODE (ODE part)*
- $(E_N, A_N)$ *is of type nDAE (nilpotent part)*
- $(E_O, A_O)$ *is of type oDAE (overdetermined part)*

# Regularity

## Definition

$(E, A)$ is regular $:\Longleftrightarrow$ $\ell = n$ and $\det(sE - A) \not\equiv 0$

## Theorem (Regularity characterizations)

*The following statements are equivalent:*

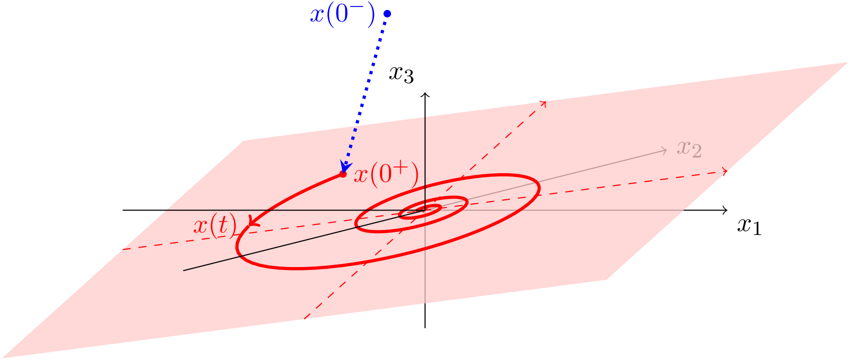
- $(E, A)$ *is regular*
- $(E, A) \cong \left( \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$ *(quasi-Weierstrass form)*
- $E\dot{x} = Ax + Bu$ *has solution for all* $u$ *and is uniquely determined by* $x(0)$

## Regularity means existence and uniqueness of solutions

BUT not for all initial conditions $x(0) = x_0$!

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\rightsquigarrow$ regular, but $x_2(t) = 0$ for all $t$

# Jump and flow




## Questions

- How to find consistency space?
- What determines the jump $x(0^-) \mapsto x(0^+)$?

# Wong-sequences and Wong limits

## Definition (Wong sequences)

For $E, A \in \mathbb{R}^{\ell \times n}$ let

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0,1,2,\dots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{j+1} := E^{-1}(A\mathcal{W}_j) \quad , \qquad j = 0,1,2,\dots$$

Here $M\mathcal{S} := \{Mx \mid x \in \mathcal{S}\}$ and $M^{-1}\mathcal{S} := \{x \mid Mx \in \mathcal{S}\}$

## Wong limits

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \dots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \dots$$
$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \dots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \dots$$

Then we can define: $\mathcal{V}^* := \bigcap_{i \in \mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}$ and $\mathcal{W}^* := \bigcup_{j \in \mathbb{N}} \mathcal{W}_j = \mathcal{W}_{j^*}$

# Motivation of first Wong sequence

## Definition (Consistency space)

The consistency space of $E\dot{x} = Ax$ is

$$\mathfrak{C}_{(E,A)} := \{x_0 \in \mathbb{R}^n \mid \exists \text{ sol. } x \text{ of } E\dot{x} = Ax \text{ with } x(0) = x_0\}$$

## Inductive refinement of consistency space

- Initially no knowledge $\rightsquigarrow$ $\mathcal{V}_0 = \mathbb{R}^n \supseteq \mathfrak{C}_{(E,A)}$ $\rightsquigarrow$ trivial constraint $\dot{x} \in \mathcal{V}_0$
- $E\dot{x} = Ax$ constraints $x$ to $x \in A^{-1}\{E\dot{x}\} \subseteq A^{-1}(E\mathcal{V}_0) =: \mathcal{V}_1 \supseteq \mathfrak{C}_{(E,A)}$
- $\dot{x}(t) := \lim_{h\to 0} \frac{x(t+h)-x(t)}{h} \in \mathcal{V}_1$
- $E\dot{x} = Ax$ constraints $x$ to $x \in A^{-1}\{E\dot{x}\} \subseteq A^{-1}(E\mathcal{V}_1) =: \mathcal{V}_2 \supseteq \mathfrak{C}_{(E,A)}$
- $\dot{x} \in \mathcal{V}_2 \rightsquigarrow x \in A^{-1}(E\mathcal{V}_2) =: \mathcal{V}_3 \subseteq \mathfrak{C}_{(E,A)}$ ...
- $\mathcal{V}^* \supseteq \mathfrak{C}_{(E,A)}$, in fact, it turns out that $\mathcal{V}^* = \mathfrak{C}_{(E,A)}$

# Regularity and Wong limits

## Theorem (Ilchmann et al. 2012)

- $(E, A)$ *is regular* $\iff$ $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$ *and* $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^\ell$
- $T := [V, W]$, $S = [EV, AW]^{-1}$ *where* $\operatorname{im} V = \mathcal{V}^*$ *and* $\operatorname{im} W = \mathcal{W}^*$ *gives QWF*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)$$

## Definition (Index, consistency projector and diff./imp. selectors)

- Index of regular $(E, A)$ := nilpotency index of $N$ (hence: index one $\iff$ $N = 0$)
- Consistency projector $\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$
- Differential selector $\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$
- Impulse selector $\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$

# Explicit solution formula for regular DAEs

$$E\dot{x} = Ax + Bu \qquad (E,A) \overset{S,T}{\cong} \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix}\right)$$

$$A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A, \quad B^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} B, \quad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)}, \quad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B$$

## Theorem (Solution formula, T. 2012)

$(x,u)$ *is a smooth solution of* $E\dot{x} = Ax + Bu$ $\iff$

$$x(t) = e^{A^{\text{diff}} t} \Pi_{(E,A)} x(0) + \int_0^t e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s) \mathrm{d}s - \sum_{i=0}^{\nu-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t)$$

$\iff$ $x = x^{\text{diff}} \oplus x^{\text{imp}}$ *where*

$$\dot{x}^{\text{diff}} = A^{\text{diff}} x + B^{\text{diff}} u, \quad x^{\text{diff}}(0) \in \operatorname{im} \Pi_{(E,A)}, \quad x^{\text{diff}}(t) \in \mathcal{V}^*$$

$$E^{\text{imp}} \dot{x}^{\text{imp}} = x^{\text{imp}} + B^{\text{imp}} u, \qquad x^{\text{imp}}(t) \in \mathcal{W}^*$$

# Consistency projector

## "Corollary" (Response to inconsistent initial value)

For $u = 0$ we have

$$x(0^+) = \Pi_{(E,A)} x(0^-), \qquad \Pi_{(E,A)} = T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1} = \Pi_{\mathcal{V}^*}^{\mathcal{W}^*}$$

Index 1: Jump uniquely determined by $x(0^+) \in \mathcal{V}^*$ and $x(0^+) - x(0^-) \in \ker E = \mathcal{W}^*$

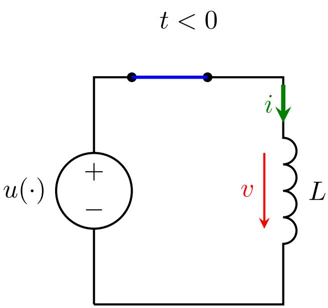
## Other jump rules

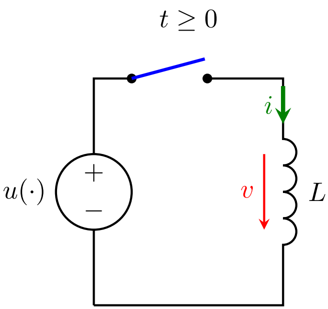
Wong-sequence based jump rule coincides with (Costantini et al. 2013):

- passivity based energy minimization jump rule (Frasca et al. 2010)
- Conservation of charge/flux (Liou 1972)
- Laplace transform approach (Opal & Vlach 1990)

# Content

# Motivating example



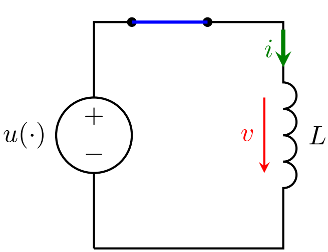




inductivity law: $L\frac{\mathrm{d}}{\mathrm{d}t}i = v$

switch dependent: $0 = v - u$ $\quad$ $0 = i$

# Motivating example

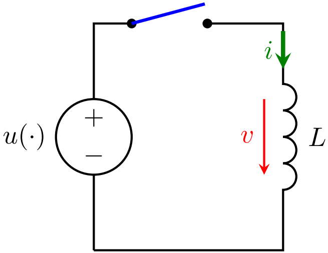
$t < 0$




$x = [i, v]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$t \geq 0$




$x = [i, v]^\top$

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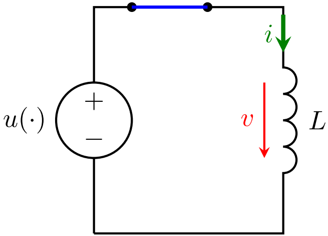
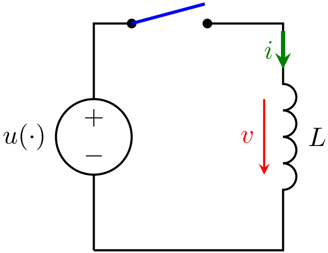



$E_1\dot{x} = A_1x + B_1u$
on $(-\infty, 0)$

$t \geq 0$



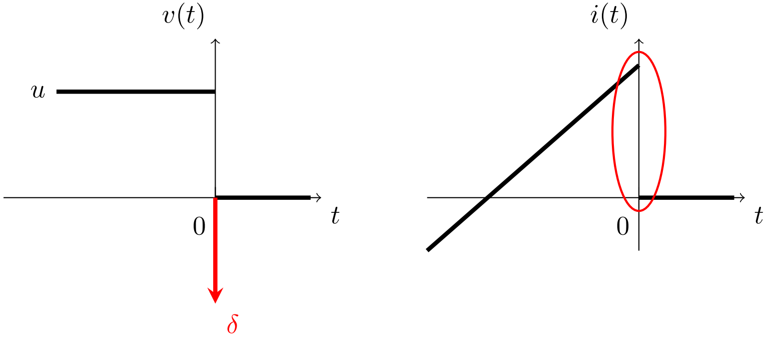

$E_2\dot{x} = A_2x + B_2u$
on $[0, \infty)$

$\to$ switched differential-algebraic equation

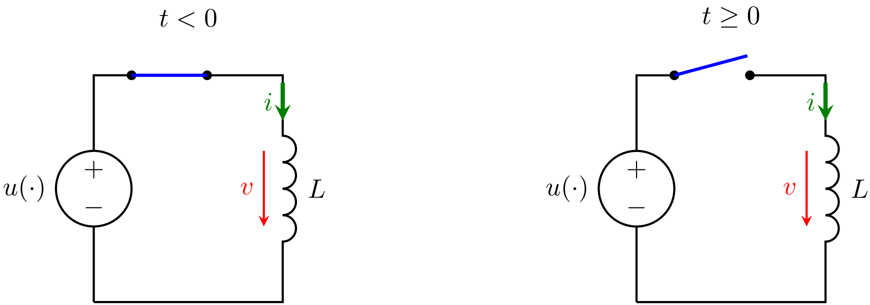
# Solution of circuit example

| $t < 0$ | $t \geq 0$ |
|---|---|
| $v = u$ | $i = 0$ |
| $L\frac{\mathrm{d}}{\mathrm{d}t}i = v$ | $v = L\frac{\mathrm{d}}{\mathrm{d}t}i$ |

Solution (assume constant input $u$):



# Observations



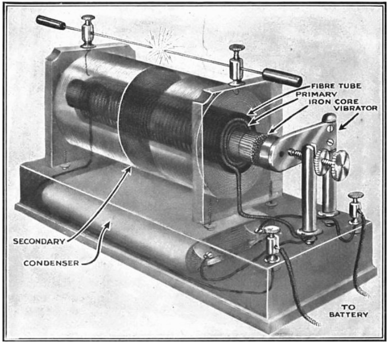

## Observations

- $x(0^-) \neq 0$ inconsistent for $E_2\dot{x} = A_2x + B_2u$
- unique jump from $x(0^-)$ to $x(0^+)$
- derivative of jump = Dirac impulse appears in solution

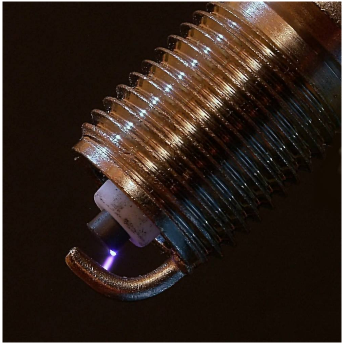
# Dirac impulse is "real"

## Dirac impulse

Not just a mathematical artifact!




Drawing: Harry Winfield Secor, public domain



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# Definition

Switch $\rightarrow$ Different DAE models (=modes)
depending on time-varying position of switch

## Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \rightarrow \{1, \dots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \qquad \text{(swDAE)}$$

## Attention

Each mode might have different consistency spaces
$\Rightarrow$ inconsistent initial values at each switch
$\Rightarrow$ Dirac impulses, in particular distributional solutions

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# Distribution theory - basic ideas

## Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse $\delta$ is "derivative" of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1) Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2) Axiomatic: Space of all "derivatives" of continuous functions (J. Sebastião e Silva 1954)

# Distributions - formal

## Definition (Test functions)

$\mathcal{C}_0^\infty := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is smooth with compact support}\}$

## Definition (Distributions)

$\mathbb{D} := \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous}\}$

## Definition (Regular distributions)

$f \in \mathcal{L}_{1,\text{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_\mathbb{R} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

## Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

## Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

$$(\mathbb{1}_{[0,\infty)_\mathbb{D}})'(\varphi) = -\int_\mathbb{R} \mathbb{1}_{[0,\infty)}\varphi' = -\int_0^\infty \varphi' = -(\varphi(\infty) - \varphi(0)) = \varphi(0)$$

# Multiplication with functions

**Definition (Multiplication with smooth functions)**

$\alpha \in \mathcal{C}^\infty: \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

**Coefficients not smooth**

Problem: $E_\sigma, A_\sigma, C_\sigma \notin \mathcal{C}^\infty$

Observation, for $\sigma_{[t_i,t_{i+1})} \equiv p_i$, $i \in \mathbb{Z}$:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z}: \begin{aligned} (E_{p_i}\dot{x})_{[t_i,t_{i+1})} &= (A_{p_i}x + B_{p_i}u)_{[t_i,t_{i+1})} \\ y_{[t_i,t_{i+1})} &= (C_{p_i}x + D_{p_i}u)_{[t_i,t_{i+1})} \end{aligned}$$

BUT: Distributional restriction impossible to define (T. 2022)

# Dilemma

**Switched DAEs**

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

**Distributions**

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

**Underlying problem**

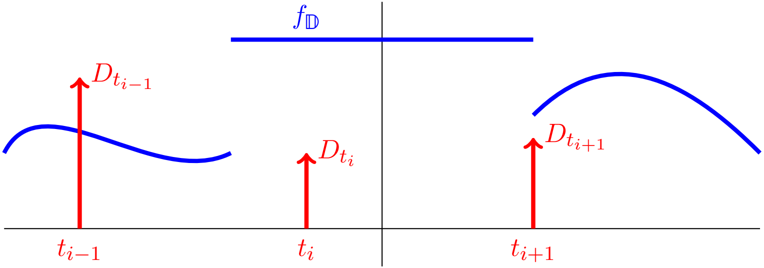
Space of distributions too big.

# Piecewise smooth distributions

Define a suitable smaller space:

**Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, T. 2009)**

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



# Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\text{pw}}$ "$\subseteq$" $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ and $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Well definded restriction $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

$$D = f_{\mathbb{D}} + \sum_{t \in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t \in T \cap M} D_t$$

- Multiplication with $\alpha = \sum_{i\in\mathbb{Z}} \alpha_{i[t_i,t_{i+1})} \in \mathcal{C}^\infty_{\text{pw}}$ well defined:

$$\alpha D := \sum_{i \in \mathbb{Z}} \alpha_i D_{[t_i,t_{i+1})}$$

- Evaluation at $t \in \mathbb{R}$: $D(t^-) := f(t^-)$, $D(t^+) := f(t^+)$
- Impulses at $t \in \mathbb{R}$: $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$

## Application to (swDAE)

$(x,u)$ solves (swDAE) $\quad :\Leftrightarrow \quad$ (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

# Relevant questions

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

**Piecewise-smooth distributional solution framework**

$x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

- Existence and uniqueness of solutions?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?
- Control theoretical questions
  - Stability and stabilization
  - Observability and observer design
  - Controllability and controller design

# Existence and uniqueness of solutions for (swDAE)

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \text{(swDAE)}$$

## Basic assumptions

- $\sigma \in \Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1,\ldots,N\} \;\middle|\; \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty,0)} \text{ is constant} \end{array} \right\}$.
- $(E_p, A_p)$ is regular $\forall p \in \{1,\ldots,N\}$, i.e. $\det(sE_p - A_p) \not\equiv 0$

## Theorem (T. 2009)

*Consider* (swDAE) *satisfying the basic assumptions. Then*

$$\forall\, u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}\ \forall\, \sigma \in \Sigma_0\ \exists \text{ solution } x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$$

*and* $x(0^-)$ *uniquely determines* $x$.

# Inconsistent initial values

$$E\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n$$

## Initial trajectory problem = special switched DAE

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax+Bu)_{[0,\infty)} \end{aligned} \qquad \text{(ITP)}$$

## Corollary (Consistency projector and Dirac impulses)

*Unique jumps and impulses for ITP, in particular, for $u = 0$,*

$$x(0^+) = \Pi_{(E,A)} x^0(0^-)$$

$$x[0] = -\sum_{i=0}^{\nu-2} (E^{\text{imp}})^{i+1} x^0(0^-) \delta^{(i)}$$

# Sufficient conditions for impulse-freeness

## Question

When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ impulse free?

Note: Jumps are OK.

## Lemma (Sufficient conditions)

- *$(E_p, A_p)$ all have index one (i.e. $(sE_p - A_p)^{-1}$ is proper)*
  *$\Rightarrow$ (swDAE) impulse free*
- *all consistency spaces of $(E_p, A_p)$ coincide*
  *$\Rightarrow$ (swDAE) impulse free*

# Characterization of impulse-freeness

**Theorem (Impulse-freeness, T. 2009)**

*The switched DAE* $E_\sigma \dot{x} = A_\sigma x$ *is impulse free* $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \ldots, N\}$$

*where* $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \ldots, N\}$ *is the* $p$*-th consistency projector.*

**Remark**

- Index-1-case $\Rightarrow$ $E_q(I - \Pi_q) = 0 \; \forall q$
- Consistency spaces equal $\Rightarrow$ $(I - \Pi_q)\Pi_p = 0 \; \forall p, q$

# Content

# Nonlinear Wong-sequence and geometric index

Nonlinear DAE: $E(x)\dot{x} = F(x), \quad x \in X$

## Definition (Nonlinear Wong-sequence)

- $M_0^c := X$ or $M_0^c := U_0$ open neighborhood of some $x_p \in X$ (initial submanifold)
- $M_k := \left\{x \in M_{k-1}^c \,\middle|\, F(x) \in E(x) T_x M_{k-1}^c\right\}$, where $T_x M_{k-1}^c$ denotes the tangent-space of $M_{k-1}^c$ at $x$
- Choose $M_k^c \subseteq M_k$ to be smooth connected submanifold (of same dimension)

## Theorem (Chen & T. 2021)

*Under some local constant rank assumptions:*

- $\exists$ *minimal* $k^* \in \mathbb{N}$ : $\quad M_{k^*}^c = M_{k^*+1}^c$ *(geometric index)*
- $k^*$ *equals the well-known differential index (Gear 1988)*
- $M_{k^*}^c$ *equals locally the set of consistent initial values*

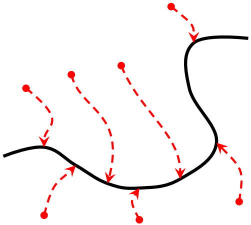
# Impulse free jumps

$E(x)\dot{x} = F(x), \quad x \in X$ with consistency space $S_c \subset X$,

## Definition (Impulse free jump)

Let $x_0 \in X \setminus S_c$ (inconsistent initial value).
A $\mathcal{C}^1$ curve $J : [0,a] \to X$ is called impulse-free jump path $\iff$

$$J(0) = x_0, \quad J(a) \in S_c, \quad \forall \tau \in [0,a] : \tfrac{\mathrm{d}}{\mathrm{d}\tau} J(\tau) \in \ker E(J(\tau))$$



## Atttention

$\tau$ is not a time-paramter, but a path-parameter, in particular, $a > 0$ doesn't have to be small!

# Index one and impulse-free jump path

## Theorem (Index one end unique jump map, Chen &T. 22)

*Assume index one, some local constant rank assumption and a reachability assumption, then*

$$\forall x_0^- \in U \setminus S_c \; \exists \text{ impulse-free jump-path } \; J : [0, a] \to U$$

*Furthermore the following statements are equivalent:*

1. *The map $x_0^- \mapsto x_0^+$ is unique (non-linear consistency "projector")*
2. $\ker E$ *is involutive*
3. *The system is equivalent to an index one nonlinear Weierstrass form:*

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} \dot{v} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} f(v) \\ w \end{pmatrix}$$

# Some comments on jump path

## Unique consistency projector vs. nonunique jump-path

Although the map $x_0^- \mapsto x_0^+$ is unique, the jump-path $J : [0, a] \to U$ connection both is not unique!

$\rightsquigarrow$ Normalize via e.g. shortest path and $\left|\frac{\mathrm{d}}{\mathrm{d}\tau} J(\tau)\right| = 1$ (future research)

or: limit of singular perturbation system (Chen & T. 22)

## Jump-map invariant under coordinate transfmation

Jump map "invariant" under coordinate transformation $z = \psi(x)$ and left multiplication with $Q(x)$, i.e.

$$x_0^- \mapsto x_0^+ \quad \Longleftrightarrow \quad z_0^- = \psi(x_0^-) \mapsto z_0^+ = \psi(x_0^+)$$

Major advantage compared to existing approaches, e.g. in Matlab's `decic` and in Liberzon & T. 12.

# Next steps: sliding jumps in PWA-DAEs

Jump paths for linear DAEs
$\implies$ straight lines

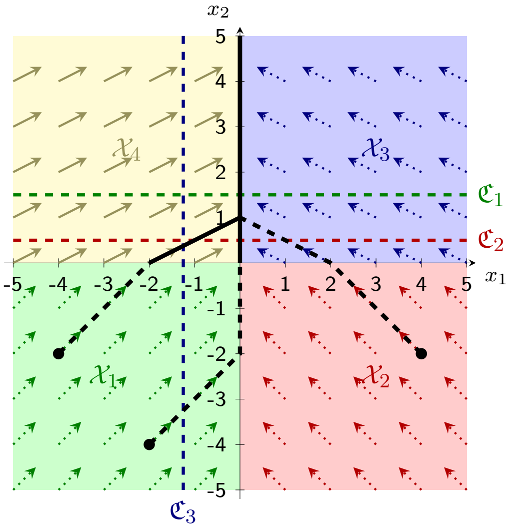
Consider piecewise-affine DAEs:

$$E_i\dot{x} = A_i x + b_i, \quad x \in X_i$$

where $\bigcup_i X_i = \mathbb{R}^n$

**Question**

What happens if jump path wants to leave active region?



# Summary

- Linear DAEs: Structural analysis
  - Wong sequences and Quasi-Kronecker form
  - Regularity $\iff$ Existence and uniqueness of solutions
- Inconsistent initial values
  - Piecwise-smooth distributions as solution space
  - Jumps and Dirac impulses
- Switched DAEs (time-dependent)
  - Existence and uniqueness of solutions
  - Impulse-freeness condition
- Nonlinear DAEs
  - Nonlinear Wong-sequence $\implies$ geometric index
  - Jump-path (coordinate free definition)
  - Index one $\implies$ unique jump-map
  - Outlook: State-dependent switched DAEs