Inhomogeneous Singular Linear Switched Systems in Discrete Time: Solvability, Reachability, and Controllability Characterizations

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System class

Discrete-time Inhomogeneous Linear Switched Singular Systems

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$$

Why?

- Dynamic and time-variant Leontief model, cf. LUENBERGER 1977,1978
- Discretization of switched DAEs e.g. from electrical circuits with switches
- Mathematical curiosity

Challenges

- Solution theory (existence, uniqueness, causality)
- Controllability / reachability notions and characterizations

Solvability issues

No canonical solvability definition

Classical "$\forall x_0 \in \mathbb{R}^n$ and $\forall u(\cdot)$ there exists a unique $x(\cdot)$" is too restrictive and causality is not addressed!

Example (non-switched): Non-causality and non-existence

$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = x(k) - \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right] u(k)$ has (unique!) solution: $x(k) = \begin{pmatrix} u(k+1) \\ u(k) \end{pmatrix}$

- Solution not causal w.r.t. input!
- Initial value $x(0)$ cannot be chosen independently from input

Solvability issues

No canonical solvability definition

Classical "$\forall x_0 \in \mathbb{R}^n$ and $\forall u(\cdot)$ there exists a unique $x(\cdot)$" is too restrictive and causality is not addressed!

Example (homogeneous): Non-existence due to switching

Mode 1: $x(k+1) = x(k)$ active on $[0, k_s)$ with arbitrary initial value x_0
Mode 2: $0 \cdot x(k+1) = x(k)$ active on $[k_s, \infty)$ for some $k_s > 0$

- Each mode has a regular matrix pair (E_i, A_i), i.e. existence and uniqueness of solutions of non-switched system is guaranteed (for consistent initial values)
- When switching from mode 1 to 2 at $k = k_s$ there is no solution for any (consistent) initial value $x_0 \neq 0$, because
  - Mode 1 at $k = k_s - 1$ yields $x(k_s) = x(k_s - 1) = x_0 \neq 0$
  - Mode 2 at $k = k_s$ yields $0 = x(k_s)$

Solvability definition

$$\boxed{E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)}$$

Define: $\boxed{\widehat{\mathcal{S}}_i := A_i^{-1}(\operatorname{im}[E_i, B_i])} = \{\xi \in \mathbb{R}^n : A_i\xi \in \operatorname{im}[E_i, B_i]\}$

Definition (Solvability for arbitrary switching signals)

An InhSLSS is solvable $\iff$ $\forall k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$, $\forall x_{k_0} \in \widehat{\mathcal{S}}_{\sigma(k_0)}$, $\forall (u(k_0), u(k_0+1), \ldots, u(k_1-1))$, and $\forall \sigma$, $\exists!$ a solution on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$.

Remarks

- Local solutions: Solvability on any interval $[k_0, k_1]$ is required
- Consistent initial value: All values in $\widehat{\mathcal{S}}_{\sigma(k_0)}$ are considered as initial values
- Strict causality: $x(k_1)$ is not allowed to depend on $u(k_1)$

Solvability characterization

Theorem (Necessary and Sufficient Condition for Solvability)

An InhSLSS is solvable $\iff$

$$\boxed{E_j^+ A_j \widehat{\mathcal{S}}_j + \operatorname{im} E_j^+ B_j \subseteq \ker E_j \oplus \widehat{\mathcal{S}}_i} \qquad \forall i,j \in \{0,1,\dots,\mathtt{p}\}$$

If solvable, all solutions are also solutions of the surrogate system

$$\boxed{x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)} x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)} u(k)}$$

where

- $\widehat{\Phi}_{i,j} = \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+ A_j$ *and* $\widehat{\Theta}_{i,j} = \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+ B_j$
- $\Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j}$ *is the canonical projector from* $\ker E_j \oplus \widehat{\mathcal{S}}_i$ *to* $\widehat{\mathcal{S}}_i$.

In particular, $x(k) \in \widehat{\mathcal{S}}_{\sigma(k)}$ *for all* $k \in \mathbb{N}$.

Non-solvable example

Example A

Consider an InhSLSS composed of:

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$
$$(E_1, A_1, B_1) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Geometric computations provide

$$\ker E_0 = \operatorname{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \ker E_1 = \operatorname{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$\widehat{\mathcal{S}}_0 = \operatorname{span}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \widehat{\mathcal{S}}_1 = \operatorname{span}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- $E_i^+ A_i \widehat{\mathcal{S}}_i + \operatorname{im}[E_i^+ B_i] \subseteq \ker E_i \oplus \widehat{\mathcal{S}}_i, \ \forall i = 0, 1$
  $\rightsquigarrow$ individual modes (non-switched) are solvable
- $\widehat{\mathcal{S}}_1 \cap \ker E_0 \neq \{0\}$ and also $\widehat{\mathcal{S}}_0 \cap \ker E_1 \neq \{0\}$
  $\rightsquigarrow$ switched system is not solvable

Solvable example

Example B

$$(E_0, A_0, B_0) = \left(\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right)$$

$$(E_1, A_1, B_1) = \left(\begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right)$$

$E_j^+ A_j \widehat{\mathcal{S}}_j + \operatorname{im}[E_j^+ B_j] \subseteq \ker E_j \oplus \widehat{\mathcal{S}}_i, \ \forall i,j = 0,1 \rightsquigarrow$ switched system is solvable

Surrogate system matrices:

$$\widehat{\Phi}_{0,0} = \begin{bmatrix} -1 & 1 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -2 \end{bmatrix}, \quad \widehat{\Phi}_{1,0} = \begin{bmatrix} -1 & \frac{1}{2} & -\frac{3}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -2 \end{bmatrix}, \quad \widehat{\Phi}_{0,1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & -1 \end{bmatrix}, \quad \widehat{\Phi}_{1,1} = \begin{bmatrix} 0 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & -1 & -2 \end{bmatrix},$$

$$\widehat{\Theta}_{0,0} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \widehat{\Theta}_{1,0} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \widehat{\Theta}_{0,1} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \quad \widehat{\Theta}_{1,1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Solvability for general time-varying system

Consider time-varying singular system:

$$E_k x(k+1) = A_k x(k) + B_k u(k) \qquad \text{(tvLSS)}$$

Corollary

(tvLSS) *is solvable* $\iff$ $\boxed{E_k^+ A_k \widehat{\mathcal{S}}_k + \operatorname{im} E_k^+ B_k \subseteq \ker E_k \oplus \widehat{\mathcal{S}}_{k+1}}$ $\forall k$

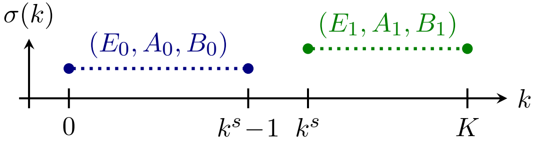
Solvability and regularity of (E_k, A_k)

Regularity of (E_k, A_k) is neither necessary nor sufficient for solvability!
$\rightsquigarrow$ crucial difference to the continuous time case

Reachability

Single Switch Switching Signal:

$$\sigma(k) = \begin{cases} 0, & 0 \leq k < k^s, \\ 1, & k^s \leq k \leq K. \end{cases}$$



$$\mathcal{P}_0 := \widehat{\mathcal{S}}_0 \cap \mathcal{R}_0(k^s - 1)$$

$$\mathcal{P}_1 := \widehat{\mathcal{S}}_1 \cap \left(\widehat{\Phi}_1^{K-k^s}\widehat{\Phi}_{1,0}\mathcal{P}_0 + \operatorname{im}\widehat{\Phi}_{1,0}^{K-k^s}\widehat{\Theta}_{1,0} + \mathcal{R}_1(K-k^s)\right)$$

where $\mathcal{R}_i(k) = \operatorname{im} R_i(k) = \operatorname{im}\left[\widehat{\Theta}_i, \widehat{\Phi}_i\widehat{\Theta}_i, \cdots, \widehat{\Phi}_i^{k-1}\widehat{\Theta}_i\right]$, $i = 0, 1$.

Theorem (Necessary and Sufficient Condition for Reachability)

Let $\mathcal{R}^\sigma_{[0,K]}$ be the reachable subspace on $[0, K]$ w.r.t. σ of a solvable InhSLSS. Then

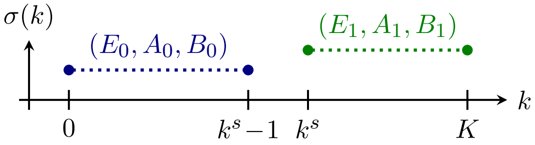
$$\boxed{\mathcal{P}_1 = \mathcal{R}^\sigma_{[0,K]}}$$

In particular, the system is reachable $\iff$ $\mathcal{P}_1 = \widehat{\mathcal{S}}_1$.

Controllability

Single Switch Switching Signal:

$$\sigma(k) = \begin{cases} 0, & 0 \leq k < k^s, \\ 1, & k^s \leq k \leq K. \end{cases}$$



$$\mathcal{Q}_1 := \widehat{\mathcal{S}}_1 \cap \left[\widehat{\Phi}_1^{K-k^s}\right]^{-1} \mathcal{R}_1(K - k^s)$$

$$\mathcal{Q}_0 := \widehat{\mathcal{S}}_0 \cap \left[\widehat{\Phi}_{1,0}\widehat{\Phi}_0^{k^s-1}\right]^{-1} \left[\mathcal{Q}_1 + \widehat{\Phi}_{1,0}\mathcal{R}_0(k^s - 1) + \operatorname{im}\widehat{\Theta}_{1,0}\right]$$

Theorem (Controllability space)

Let $\mathcal{C}^{\sigma}_{[0,K]}$ be the controllable subspace on $[0, K]$ w.r.t. σ of a solvable InhSLSS. Then

$$\boxed{\mathcal{C}^{\sigma}_{[0,K]} = \mathcal{Q}_0}$$

In particular, the system is controllable (to zero) $\iff$ $\mathcal{Q}_0 = \widehat{\mathcal{S}}_0$.

Summary

Inhomogeneous switched linear singular systems in discrete time

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k)$$

- Novel solution characterization for arbitrary switching signals
- Solvable system ⇝ surrogate system

$$x(k+1) = \widehat{\Phi}_{\sigma(k+1),\sigma(k)}x(k) + \widehat{\Theta}_{\sigma(k+1),\sigma(k)}u(k)$$

- For fixed switching signal (or general time-varying case):
  Regularity (and index 1) of (E_i, A_i) neither necessary nor sufficient!
- Surrogate system can be utilized to characterize reachability and controllability