Systems & Control Letters 183 (2024) 105674

Contents lists available at ScienceDirect

Systems & Control Letters

journal homepage: www.elsevier.com/locate/sysconle

Switched linear singular systems in discrete time: Solution theory and observability notions[☆]

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ARTICLE INFO

Keywords:
Descriptor systems
Difference-algebraic systems
Implicit systems
Sequentially index-1
Switched index-1
Switched systems

ABSTRACT

We study the solution theory of linear switched singular systems. In a recent paper by Anh et al. (2019), it was highlighted that the common assumption that each mode of the switched system is index-1 is *not* sufficient to guarantee existence and uniqueness of solutions of the corresponding switched system and the notion of "jointly index-1" was introduced. However, until now it was not clear what conditions are actually required to guarantee existence and uniqueness of solutions if the switching signal is not considered arbitrary. In particular, we study the two relevant situations where the mode sequence is fixed (and the switching times are arbitrary) and where the whole switching signal is fixed. In both cases, we provide conditions in terms of the original system matrices which ensure existence and uniqueness of solutions. We also extend the idea of the one-step map introduced by Anh et al. (2019) to these two cases. It turns out that in the case of a fixed switching signal, the index-1 condition for the individual modes is also *not* necessary (in addition to not being sufficient). Furthermore, we utilize the established solution theory to provide characterizations of observability and determinability of switched singular systems.

1. Introduction

We study the solution theory as well as observability notions of a class of switched systems where each mode is a discrete-time singular linear system. To be specific, Switched Linear Singular Systems (SLSSs) of the following form are considered:

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k), \quad k \in \mathbb{N} \tag{1a}$$

$$y(k) = C_{\sigma(k)}x(k), \tag{1b}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $y(k) \in \mathbb{R}^p$, $p \in \mathbb{N}$ is the output, $\sigma : \mathbb{N} \to \{0, 1, 2, \dots, \mathtt{s}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k, and $E_i, A_i \in \mathbb{R}^{n\times n}$ and $C_i \in \mathbb{R}^{p\times n}$ are real constant matrices. The matrices E_i are singular in general, i.e. $x(k+1)$ is not explicitly expressed as a function of $x(k)$. Note that the switching is triggered only by the time and not by the state, in particular, (1) can also be interpreted as a time-varying linear system.

Under the assumption that all E_i are invertible, the system (1) can be rewritten as a (non-singular/ordinary) switched linear system, which has been broadly studied (see e.g. [1–3]). The consideration of singular systems (also called descriptor systems) goes back at least to the 70s [4] and is motivated by the observation that most first principle models of realistic systems involve dynamic as well as algebraic equations. SLSSs are still an ongoing research field with a strong focus on stability results (see e.g. [5–12]). Based on these stability results, certain control designs have been proposed, e.g. iterative learning [13] and state feedback [14]. However, as pointed out in [15] most of these results lack a thorough discussion of the existence and uniqueness of solutions, in fact, it was shown in [15] that the standard index-1 assumption for each mode is *not* sufficient to guarantee existence and uniqueness of solutions of SLSSs of the form (1), instead the stronger notion of jointly index-1 is sufficient (and necessary). Our first main contribution is the extension of this important result (which was concerned with arbitrary switching signals) to the situation for constrained switching signals.

The second main contribution is the investigation of observability notions of (1), i.e. the ability to reconstruct the state from the knowledge of the output. Apart from our own preliminary results [16,17], this topic has not been studied in the literature so far for singular systems; for non-singular systems, some characterizations were carried out using some different approaches such as Kalman rank condition-based (see e.g. [3,18,19] and geometric approach (see e.g. [1]). We

☆ This work was partially supported by the NWO Vidi-grant 639.032.733 and the UNDIP PhD Scholarship-grant 7989/UN7.P2/KS/2019.
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https://doi.org/10.1016/j.sysconle.2023.105674
Received 9 March 2023; Received in revised form 31 August 2023; Accepted 5 November 2023
Available online 16 November 2023
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are however strongly inspired by the results from the continuous time in the context of switched differential algebraic equations (see e.g. [20–22]) and by the results for nonsingular systems as the basic foundation in developing the observability characterizations for singular systems.

This paper is arranged as follows. After providing some preliminaries in Section 2, we recall the notion of "jointly index-1" in Section 3 and propose the two generalizations "sequential index-1" and "switched index-1" (Definition 3.5). Then we show that these two notions are precisely the ones that are necessary and sufficient for solvability in the sense of Definition 3.8 and we also provide explicit formulas for the corresponding one-step map. The one-step map is then utilized in Section 4 to characterize the observability and determinability of an SLSS with a fixed switching signal.

2. Preliminaries

In the following, some general mathematical preliminaries are collected, which are needed to formulate and prove our main results.

Definition 2.1 (*Generalized Inverse*). For a matrix $M \in \mathbb{R}^{m\times n}$, a generalized inverse of M is defined as a matrix $M^+ \in \mathbb{R}^{n\times m}$ that satisfies $MM^+M = M$.

A generalized matrix always exists, but is not necessarily unique, one possible choice is the well-known Moore–Penrose pseudoinverse [23]. Furthermore, for two generalized inverses M_1 and M_2 of M, it is easy to see that $(M_1 - M_2)y \in \ker M$ for all $y \in \operatorname{im} M$.

In the following, let $M^{-1}\mathcal{Y}$ denote the preimage of a (possibly singular) matrix $M \in \mathbb{R}^{n\times n}$ over a set $\mathcal{Y} \subseteq \mathbb{R}^n$, i.e. $M^{-1}\mathcal{Y} := \{x \in \mathbb{R}^n : Mx \in \mathcal{Y}\}$. The next result about the preimage of a singleton is probably not new, but a proof is difficult to find in the standard literature, which is why we include one for the convenience of the reader.

Lemma 2.2. *For any matrix $M \in \mathbb{R}^{n\times n}$ and $y \in \operatorname{im} M$, we have that*

$$M^{-1}\{y\} = \{M^+y\} + \ker M$$

where M^+ is any generalized inverse of M.

Proof. By definition $M^{-1}\{y\}$ is the solution set of $Mx = y$. Let $M^+y =: x_p$, then multiplying both sides with M gives $Mx_p = MM^+y$. Since $y \in \operatorname{im} M$, we can represent it as $y = Mz$ for some $z \in \mathbb{R}^n$, consequently

$$Mx_p = MM^+Mz = Mz = y$$

i.e., $x_p = M^+y$ is a particular solution of $Mx = y$. Since the solution set of $Mx = y$ is equal to $\{x_p\} + \ker M$ the claim is shown. □

Finally, we provide the following lemma about intersections of (affine) subspaces, which is a generalization of [15, Lem. 3.4].

Lemma 2.3. *Consider three subspaces $\mathcal{U}, \mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, then $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is a singleton for all $u \in \mathcal{U}$ if, and only if, $\mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$. In that case*

$$\mathcal{V} \cap (\{u\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} u\}, \tag{2}$$

where $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} \oplus \mathcal{W} \to \mathcal{V}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$.

Proof. Step 1: We show that $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is nonempty for all $u \in \mathcal{U}$ if, and only if, $\mathcal{U} \subseteq \mathcal{V} + \mathcal{W}$.
Step 1a: Necessity.
Seeking a contradiction, assume $\mathcal{U} \nsubseteq \mathcal{V} + \mathcal{W}$, i.e. there exists $u \in \mathcal{U}$ which is not in $\mathcal{V} + \mathcal{W}$. Choose $v \in \mathcal{V} \cap (\{u\} + \mathcal{W})$, then there is $w \in \mathcal{W}$ with $v = u + w$, i.e. $u = v - w \in \mathcal{V} + \mathcal{W}$ which contradicts the choice of u.
Step 1b: Sufficiency.
Let $u \in \mathcal{U} \subseteq \mathcal{V} + \mathcal{W}$ and choose $v \in \mathcal{V}$, $w \in \mathcal{W}$ such that $u = v + w$, then $v = u - w \in \{u\} + \mathcal{W}$ and hence $v \in \mathcal{V} \cap \{u\} + \mathcal{W}$, i.e. the latter intersection is not empty.

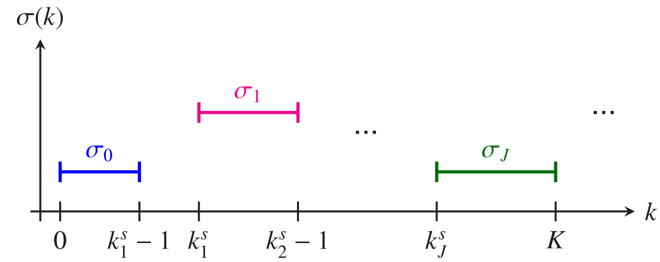


Fig. 1. Mode sequence (3).

Step 2: If $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is non-empty for at least one $u \in \mathcal{U}$ then $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is empty or a singleton for each $u \in \mathcal{U}$, if, and only if, $\mathcal{V} \cap \mathcal{W} = \{0\}$.
Step 2a: Necessity.
Seeking a contradiction assume that $\mathcal{V} \cap \mathcal{W} \neq \{0\}$ and choose $0 \neq p \in \mathcal{V} \cap \mathcal{W}$. Choose some $u \in \mathcal{U}$ for which $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is non-empty and choose $v \in \mathcal{V} \cap (\{u\} + \mathcal{W})$. Then there is $w \in \mathcal{W}$ with $v = u + w$. Since $v + p = u + w + p$ and $v + p \in \mathcal{V}$ as well as $w + p \in \mathcal{W}$ we arrive at $v + p \in \mathcal{V} \cap (\{u\} + \mathcal{W})$ and since $v \neq v + p$, the set $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is not a singleton (and also not empty).
Step 2b: Sufficiency.
For some $u \in \mathcal{U}$ for which $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is non-empty, let $v_1, v_2 \in \mathcal{V} \cap (\{u\} + \mathcal{W})$, then there exists $w_1, w_2 \in \mathcal{W}$ with $v_1 = u + w_1$ and $v_2 = u + w_2$. Consequently $\mathcal{V} \ni v_1 - v_2 = w_1 - w_2 \in \mathcal{W}$, i.e. $v_1 - v_2 \in \mathcal{V} \cap \mathcal{W} = \{0\}$, which implies that $v_1 = v_2$.
Step 3: We show (2).
Let $u \in \mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$ and choose (unique) $v \in \mathcal{V}$, $w \in \mathcal{W}$ such that $u = v + w$, then $\Pi_{\mathcal{V}}^{\mathcal{W}} u = v \in \mathcal{V}$ and $\Pi_{\mathcal{V}}^{\mathcal{W}} u = u + \Pi_{\mathcal{V}}^{\mathcal{W}} u - u = u + v - u = u - w \in \{u\} + \mathcal{W}$. Hence $\Pi_{\mathcal{V}}^{\mathcal{W}} u \in \mathcal{V} \cap (\{u\} + \mathcal{W})$ which together with Step 2 concludes the proof. □

Lemma 2.3 tells us that for any $u \in \mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$, there exists a unique $v \in \mathcal{V}$ for which there exists $w \in \mathcal{W}$ with $v = u + w$ and this vector is given by $v = \Pi_{\mathcal{V}}^{\mathcal{W}} u$.

3. Solution theory

3.1. Class of switching signals and solution notion

In addition to considering arbitrary switching signals, we put special focus on individual switching signals as well as switching signals with known switching sequences (but arbitrary mode durations). In particular, for a given mode sequence $(\sigma_k) = (\sigma_0, \sigma_1, \dots) \subseteq \{0, 1, \dots, \mathrm{s}\}^{\mathbb{N}}$ and a strictly increasing sequence of switching times $(0 = k_0^s, k_1^s, k_2^s, \dots) \in \mathbb{N}^{\mathbb{N}}$ we consider switching signals of the form[1] (see also Fig. 1 for an illustration)

$$(\sigma_k), \quad \sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s),\ j = \{0, 1, 2, \dots\}. \tag{3}$$

We refer to mode σ_0 as the initial mode. Furthermore, we will often consider a finite time interval $[0, K] \subseteq \mathbb{N}$ for (1) on which only $J \in \mathbb{N}$ (finitely many) switches occur.

Note that each switching signal σ is uniquely determined by its mode sequence $(\sigma_0, \sigma_1, \dots)$ and the sequence of mode durations $(k_{j+1}^s - k_j^s)_{j=0,1,\dots}$ (where for a finite time interval the last index is $j = J$ and we define $k_{J+1}^s := K + 1$).

In the following we put special focus on local solutions of (1a) on a *finite* time interval $[k_0, k_1]$; where such a solution is a sequence $x(k_0), x(k_0 + 1), \dots, x(k_1) \in \mathbb{R}^n$ which satisfies (1a) for $k = k_0, k_0 + 1, \dots, k_1 - 1$ and furthermore for which another (not necessarily unique)

[1] We use the standard interval notation also for natural numbers, in particular, $[k, \ell) := \{k, k+1, \dots, \ell - 1\}$ for any $k, \ell \in \mathbb{N}$ with $k < \ell$.

value $x(k_1+1) \in \mathbb{R}^n$ exists such that (1a) also holds for $k = k_1$. Note that this last requirement is different to non-singular systems, where $x(k_1)$ is already uniquely determined by the systems equations considered up to $k_1 - 1$. However, due to the singularity of $E_{\sigma(k_1-1)}$ the value of $x(k_1)$ is not yet fully fixed by the information of (1a) at $k = k_1 - 1$ and it is necessary to incorporate the additional information which can be concluded for $x(k_1)$ from (1a) evaluated at $k = k_1$.

3.2. Solvability under arbitrary switching signals

Before presenting our results concerning solvability for constrained switching signals, we first recall the known results for the case of arbitrary switching signals. Towards this goal, we recall first that a matrix pair (E, A) is called regular if $\det(sE - A)$ is not the zero polynomial and the following crucial definition.

Definition 3.1 (*Jointly Index-1, cf. [7,15,24–26]*)**.** The family $\{(E_i, A_i)\}_{i=0}^{\mathrm{s}}$ of regular matrix pairs is called *jointly index-1* if, and only if,

$$\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^n \ \forall i, j \in \{0, 1, 2, \ldots, \mathrm{s}\}, \tag{4}$$

where $\mathcal{S}_i := A_i^{-1}(\operatorname{im} E_i)$.

Remark 3.2 (*Index-1 for Individual Modes*)**.** It is well known (see e.g. [27]) that every regular matrix pair is equivalent to a matrix pair in *quasi-Weierstrass form (QWF)*, i.e. there exist invertible matrices $S, T \in \mathbb{R}^n$ such that

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{5}$$

where $N \in \mathbb{R}^{n_2 \times n_2}$ is nilpotent, $J \in \mathbb{R}^{n_1 \times n_1}$, $n_1 + n_2 = n$ and the identity matrices have corresponding sizes. The *index* of a matrix pair (E, A) is then defined as the nilpotency index of N, i.e. the smallest $\nu \in \mathbb{N}$ such that $N^\nu = 0$. In particular, (E, A) has index-1 if, and only if, $N = 0$ in the QWF. It can be shown (see e.g. [15, Lemma 2.3]) that a matrix pair (E, A) is (regular and) index-1 if, and only if,

$$\ker E \oplus \mathcal{S} = \mathbb{R}^n, \qquad \text{with } \mathcal{S} := A^{-1}(\operatorname{im} E),$$

which for regular (E, A) is in fact equivalent to $\ker E \cap \mathcal{S} = \{0\}$ (from Wong sequences' dimension formula in [27, Lemma 2.3]). Consequently, for a family of matrix pairs to be jointly index-1 it is necessary that each individual matrix pair is index-1 ($i = j$ in (4)), but the latter is *not sufficient* for jointly index-1 in general, see [15, Ex. 1.1].

Remark 3.3. One should distinguish the term *index* that corresponds to a matrix pair (E, A) to the index of a single matrix M. The latter is defined to be the smallest nonnegative integer k such that $\operatorname{rank} M^k = \operatorname{rank} M^{k+1}$. Clearly, the index of M is only positive if M is singular, in that case, the index of M is the maximal grade of 0-vectors of M [28]. Furthermore, it is easily seen that the index of M is equal to the index of the matrix pair (M, I).

It can be shown, that the property of jointly index-1 is necessary and sufficient for the existence and uniqueness of solutions of the SLSS (1) under arbitrary switching:

Proposition 3.4 (*[15]*)**.** *The SLSS (1) has a unique solution for any switching signal $\sigma : [0, \infty) \to \{0, 1, \ldots, \mathrm{s}\}$ and any consistent initial condition $x_0 \in \mathcal{S}_{\sigma(0)}$ if, and only if, the corresponding family of matrix pairs $\{(E_i, A_i)\}$ is jointly index-1. Furthermore, each solution satisfies*

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k), \quad k \in \mathbb{N},$$

where $\Phi_{i,j}$ is the one-step-map from mode j to mode i, given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} \Phi_{(E_j, A_j)}, \tag{6}$$

where $\Pi_{\mathcal{S}_i}^{\ker E_i}$ is the unique projector onto $\mathcal{S}_i$ along $\ker E_i$ and $\Phi_{(E_j,A_j)}$ is the one step of mode j, given in terms of the corresponding QWF $(S_j E_j T_j, S_j A_j T_j) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} J_j & 0 \\ 0 & I \end{smallmatrix}\right] \right)$ as $\Phi_{(E_j,A_j)} := T_j \left[\begin{smallmatrix} J_j & 0 \\ 0 & 0 \end{smallmatrix}\right] T_j^{-1}$.

For motivation of the following, we highlight the key arguments of the proof of Proposition 3.4: The ability to uniquely determine $x(k+1)$ in terms of $x(k)$ relies on the fact that $x(k+1)$ has to satisfy the following two equations:

$$E_{\sigma(k)} x(k+1) = A_{\sigma(k)} x(k),$$
$$E_{\sigma(k+1)} x(k+2) = A_{\sigma(k+1)} x(k+1),$$

where $x(k)$ is already fixed, while $x(k+2)$ is free. Consequently, $x(k+1) \in E_{\sigma(k)}^{-1}\{A_{\sigma(k)} x(k)\} = \{E_{\sigma(k)}^{+} A_{\sigma(k)} x(k)\} + \ker E_{\sigma(k)}$ as well as $x(k+1) \in A_{\sigma(k+1)}^{-1}(\operatorname{im} E_{\sigma(k+1)}) = \mathcal{S}_{\sigma(k+1)}$ which, in view of Lemma 2.3 results in a unique value for $x(k+1)$ if, and only if, $\{E_{\sigma(k)}^{+} A_{\sigma(k)} x(k)\} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}$. This shows that $\ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} = \mathbb{R}^n$ is indeed sufficient for existence and uniqueness of solutions. Necessity follows from observing that the above condition must hold for arbitrary values of $\sigma(k)$ and $\sigma(k+1)$, in particular, $\ker E_j \cap \mathcal{S}_i = \{0\}$ for all $i, j \in \{0, 1, \ldots, \mathrm{s}\}$ needs to hold. In view of Remark 3.2 and some simple dimensional argument, the latter is equivalent to $\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^n$ for all i, j, hence the necessity of $\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^n$ for existence and uniqueness of solutions under arbitrary switching is shown.

3.3. Solvability for constrained switching signals

Based on the intuition behind the proof of Proposition 3.4 we will now propose two relaxations of jointly index-1 and provide corresponding solvability characterizations afterwards.

Definition 3.5. A family of regular matrix pairs $\{(E_0, A_0), (E_1, A_1), \ldots, (E_{\mathrm{s}}, A_{\mathrm{s}})\}$ (and the corresponding SLSS (1)) is called

- *sequentially index-1* w.r.t. a fixed mode sequence $(\sigma_0, \sigma_1, \sigma_2, \ldots)$ if

$$\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^n \quad \text{for } i = 0, 1, 2, \ldots, \mathrm{s} \tag{7a}$$

$$E_{\sigma_j}^{+}(\operatorname{im} E_{\sigma_j} \cap \operatorname{im} A_{\sigma_j}) \subseteq \ker E_{\sigma_j} \oplus \mathcal{S}_{\sigma_{j+1}} \quad \text{for } j = 0, 1, 2, \ldots \tag{7b}$$

- *switched index-1* w.r.t. a fixed and known switching signal $\sigma : \mathbb{N} \to \{0, 1, \ldots, \mathrm{s}\}$ if, for $k = 0, 1, 2, \ldots$,

$$E_{\sigma(k)}^{+}(\operatorname{im} E_{\sigma(k)} \cap \operatorname{im} A_{\sigma(k)}) \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)}. \tag{8}$$

Obviously, jointly index-1 implies sequential index-1 w.r.t. any mode sequence, which in turn implies switched index-1, and both jointly index-1 as well as sequential index-1 imply index-1 of each mode; however, the converse is not true in general and, furthermore, neither does index-1 for each mode imply switched index-1 nor the other way around. These observations are summarized in the left part of Fig. 2. Note furthermore, that although the pseudo-inverse $E_{\sigma(k)}^{+}$ in (8) is not unique, the validity of (8) does not depend on the specific choice of the pseudo-inverse (cf. the discussion after Definition 2.1).

Before providing the solvability results (already indicated in Fig. 2) we would like to provide some examples to illustrate the "non-implication" mentioned above. The fact, that index-1 for each mode is not sufficient for jointly index-1 was already illustrated in [15, Ex. 1.1].

The following example illustrates that sequential index-1 does not imply jointly index-1 in general and that also index-1 for each mode does not imply sequentially/switched index-1 in general.

Example 3.6. Consider the two matrix pairs

$$(E_0, A_0) = \left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right] \right),$$
$$(E_1, A_1) = \left(\left[\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right] \right).$$

Simple computations provide that

$$\ker E_0 = \operatorname{span}\{(0, 0, 1)^\top\},$$
$$\mathcal{S}_0 = \operatorname{span}\{(1, 0, 0)^\top, (0, 1, 0)^\top\},$$

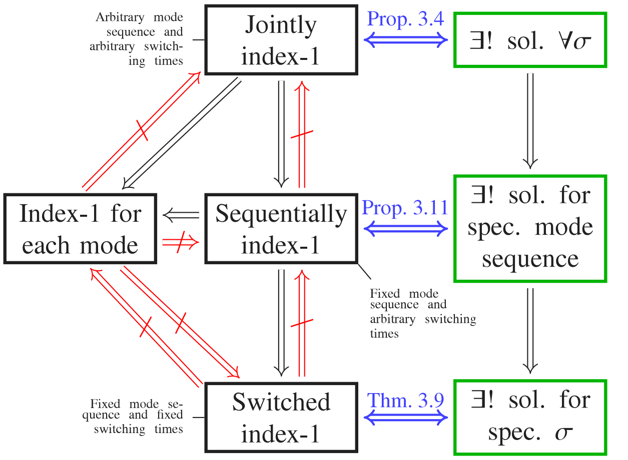


Fig. 2. Relationship between jointly index-1, sequentially index-1, and switched index-1.

$$\ker E_1 = \operatorname{span}\{(0,1,0)^\top\},$$
$$\mathcal{S}_1 = \operatorname{span}\{(1,0,-1)^\top,(0,1,-1)^\top\}.$$

Consequently, we can conclude the following:

- Each individual matrix pair/mode is index-1, because $\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^3, i = 0,1$.
- The two matrix pairs are sequentially index-1 with respect to the mode sequence $(0,1)$, because additionally to each mode being index-1, also $\ker E_0 \oplus \mathcal{S}_1 = \mathbb{R}^3$.
- The two matrix pairs are not jointly index-1, because $\ker E_1 \cap \mathcal{S}_0 = \operatorname{span}\{(0,1,0)^\top\} \neq \{0\}$.
- The two matrix pairs are not sequentially index-1 with respect to the mode sequence $(1,0)$, which also follows from $\ker E_1 \cap \mathcal{S}_0 \neq \{0\}$.
- The two matrix pairs are switched index-1 w.r.t. to any switching signal with mode sequence $(0,1)$ and arbitrary duration times, because they are already sequentially index-1 w.r.t to $(0,1)$.
- The two matrix pairs are not switched index-1 w.r.t. to any switching signal with mode sequence $(1,0)$ with a switch at $k = k_s$, because $\ker E_{\sigma(k_s)} \cap \mathcal{S}_{\sigma(k_s+1)} = \ker E_1 \cap \mathcal{S}_0 \neq \{0\}$.

These observations verify three of the non-implications in Fig. 2, namely that sequentially index-1 does not imply jointly index-1 and that index-1 for each mode does not imply sequentially and switched index-1.

The next example shows that the property of switched index-1 does not imply in general that each mode is index-1 (and consequently this example can also not be sequentially index-1).

Example 3.7. Consider the matrix pairs

$$(E_0,A_0) = \left(\begin{bmatrix} 0&0&1\\1&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1/4&0\\1/2&3/4&0\\1/2&1&2 \end{bmatrix} \right),$$
$$(E_1,A_1) = \left(\begin{bmatrix} 1&0&0\\0&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 2&0&0\\0&2&0\\0&0&1 \end{bmatrix} \right),$$
$$(E_2,A_2) = \left(\begin{bmatrix} 0&0&0\\1&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 1&0&0\\0&2&1\\0&0&1 \end{bmatrix} \right).$$

Simple computations provide that

$$\ker E_0 = \operatorname{span}\{(0,1,0)^\top\},$$
$$\mathcal{S}_0 = \operatorname{span}\{(4,0,-1)^\top,(0,2,-1)^\top\},$$
$$\ker E_1 = \operatorname{span}\{(0,0,1)^\top\},$$
$$\mathcal{S}_1 = \operatorname{span}\{(1,0,0)^\top,(0,0,1)^\top\},$$
$$\ker E_2 = \operatorname{span}\{(0,0,1)^\top\},$$
$$\mathcal{S}_2 = \operatorname{span}\{(0,1,0)^\top,(0,0,1)^\top\}.$$

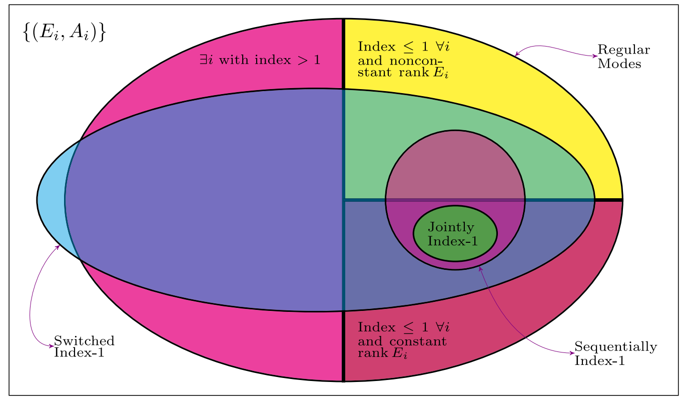


Fig. 3. Position of jointly, sequential and switched index-1 with respect to the possible classes of SLSSs.

Consequently, we can conclude the following:

- As an individual system, mode 0 is index-1 ($\ker E_0 \oplus \mathcal{S}_0 = \mathbb{R}^3$) whereas both mode 1 and mode 2 are not index-1 ($\ker E_i \cap \mathcal{S}_i \neq \{0\}, i = 1,2$).
- In view of modes 1 and 2 not being index-1, the family $\{\, (E_i,A_i) \mid i = 0,1,2 \,\}$ cannot be jointly index-1, and also not sequentially index-1 for all mode sequences containing either mode 1 or 2.
- It is easily verified that the (sufficient) condition $\ker E_j \oplus \mathcal{S}_i = \mathbb{R}^3$ actually holds for some index pairs, namely all $(j,i) \in \{(0,0),(0,1),(1,0),(2,0)\}$; hence any switching signal which is only composed of these mode transitions (from mode-j to mode-i) leads to the property of switched index-1, an example for such a switching signal is given by

k	0	1	2	3	4	5	6	7	8	…
$\sigma(k)$	2	0	0	0	1	0	0	1	0	…

  where for $k > 8$ mode 1 is only active for one time-step each (because the mode sequence cannot contain $(1,1)$).

Hence we have that for a specific switching signal the considered family of regular matrix pairs is switched index-1 while the individual modes are not all index-1 (and hence sequentially index-1 can also not hold).

Before formulating our main results concerning the solvability of SLSS (1), we would like to highlight that the different classes of SLSS (1) can be categorized along two "dimensions": (1) Properties of the family of matrix pairs concerning the regularity and their index, and (2) The considered class of switching signals (completely arbitrary, mode sequence fixed, switching times and mode sequence fixed). This categorization and the position of the different index-1-notions are illustrated in Fig. 3.

In order to formulate our main solvability result for switched index-1 systems, we first need to establish, what notion of solvability of the SLSS (1) we actually consider.

Definition 3.8. For a given family of matrix pairs $\{(E_i,A_i)\}_{i=0}^{\mathrm{s}}$ and a given switching signal σ we call (1) *locally uniquely solvable* (for short just *solvable* in the following) if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$ and for all $x_{k_0} \in \mathcal{S}_{\sigma(k_0)}$ there exists a unique solution of (1) considered on $[k_0,k_1]$ with $x(k_0) = x_{k_0}$.

Note that our definition of solvability requires the ability to consider the SLSS (1) starting at an arbitrary initial time and an arbitrary consistent (at this initial time) initial value. Furthermore, it is required to uniquely solve the SLSS on any finite interval $[k_0,k_1]$, which in particular means that the uniqueness of the final value at k_1 does not depend on the values $x(k)$ for $k > k_1$. This notion is indeed stronger compared to just requiring unique solvability of (1) on $[0, \infty)$, but for the latter a simple characterization for solvability does not exist, see the forthcoming discussion in Remark 3.13

Theorem 3.9. *Consider the SLSS (1) with a corresponding family of regular matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{s}}$ and a given switching signal σ of the form (3). This SLSS is solvable in the sense of Definition 3.8 if, and only if, $\{(E_i, A_i)\}_{i=0}^{\mathrm{s}}$ is switched index-1 w.r.t. σ. Furthermore, if it is solvable, then its solution satisfies*

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k), \ \ \forall k \in \mathbb{N} \tag{9}$$

where $\Phi_{i,j}$ is the one-step map from mode j to mode i given by

$$\Phi_{i,j} := \Pi_{\mathcal{S}_i}^{\ker E_j} E_j^+ A_j \tag{10}$$

where E_j^+ is a generalized inverse of E_j (see Definition 2.1) and $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is the canonical projector from $\ker E_j \oplus \mathcal{S}_i$ to $\mathcal{S}_i$.

Proof. *Necessity:* Let $k_0 \in \mathbb{N}$, $k_1 := k_0 + 1$ and consider the SLSS (1) on $[k_0, k_1]$, i.e. $x_0 := x(k_0)$, $x_1 := x(k_0 + 1)$, $x_2 := x(k_0 + 2)$ have to satisfy

$$E_{\sigma(k_0)} x_1 = A_{\sigma(k_0)} x_0 \tag{11a}$$

$$E_{\sigma(k_1)} x_2 = A_{\sigma(k_1)} x_1. \tag{11b}$$

Solvability of (1) implies that all elements of the solution set $\{(x_1, x_2)\}$ of the system of linear Eqs. (11) for any given $x_0 \in \mathcal{S}_{\sigma(k_0)}$ have a unique first component x_1. Equivalently, $E_{\sigma(k_0)}^{-1}\{A_{\sigma(k_0)} x_0\} \cap A_{\sigma(k_1)}^{-1}(\operatorname{im} E_{\sigma(k_1)})$ must be a singleton. Using Lemma 2.2, the latter can be rewritten as $(\{E_{\sigma(k_0)}^+ A_{\sigma(k_0)} x_0\} + \ker E_{\sigma(k_0)}) \cap \mathcal{S}_{\sigma(k_1)}$. Using Lemma 2.3 for $\mathcal{V} = \mathcal{S}_{\sigma(k_1)}$, $\mathcal{W} = \ker E_{\sigma(k_0)}$ and $\mathcal{U} = E_{\sigma(k_0)}^+ A_{\sigma(k_0)} \mathcal{S}_{\sigma(k_0)}$, we see that unique solvability of (11) is equivalent to

$$E_{\sigma(k_0)}^+ A_{\sigma(k_0)} \mathcal{S}_{\sigma(k_0)} \subseteq \mathcal{S}_{\sigma(k_1)} \oplus \ker E_{\sigma(k_0)}.$$

From $A_{\sigma(k_0)} \mathcal{S}_{\sigma(k_0)} = A_{\sigma(k_0)} A_{\sigma(k_0)}^{-1}(\operatorname{im} E_{\sigma(k_0)}) = \operatorname{im} E_{\sigma(k_0)} \cap \operatorname{im} A_{\sigma(k_0)}$ we see that (8) for $k = k_0$ is indeed necessary for solvability of (1) on $[k_0, k_0+1]$.
Sufficiency: We show, that for each $k_0, k_1 \in \mathbb{N}$ with $k_1 > k_0$ and for each $x_0 \in \mathcal{S}_{\sigma(k_0)}$ the sequence $x : [k_0, k_1] \to \mathbb{R}^n$ given by $x(k_0) = x_0$ and $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k)$ is the unique solution of SLSS (1) with $x(k_0) = x_0$ on $[k_0, k_1]$. Inductively, we assume for $k \geq k_0$ that we have already shown that $x(k) \in \mathcal{S}_{\sigma(k)}$ and that x is the unique solution on $[k_0, k]$ with $x(k_0) = x_0$ (which is satisfied for $k = k_0$). We will now show that x is the unique solution on $[k_0, k+1]$, which by an induction argument then concludes the proof. In order to show the former, we just have to show that $x(k+1) = \Phi_{\sigma(k+1),\sigma(k)} x(k)$ satisfies

$$\begin{aligned} E_{\sigma(k)} x(k+1) &= A_{\sigma(k)} x(k), \\ E_{\sigma(k+1)} x_2 &= A_{\sigma(k+1)} x(k+1), \end{aligned} \tag{12}$$

for some $x_2 \in \mathbb{R}^n$ and that there is no other possible value for $x(k+1)$ satisfying this equation. Using the same arguments as in the necessity part of this proof, we can conclude that for any $x(k) \in \mathcal{S}_{\sigma(k)}$ there is a unique value for $x(k+1)$ satisfying (12) if (8) is satisfied at k. Furthermore, by Lemma 2.3, this value is uniquely given by

$$x(k+1) = \Pi_{\mathcal{S}_{\sigma(k+1)}}^{\ker E_{\sigma(k)}} E_{\sigma(k)}^+ A_{\sigma(k)} x(k) = \Phi_{\sigma(k+1),\sigma(k)} x(k).$$

Furthermore, $x(k+1) \in \operatorname{im} \Pi_{\mathcal{S}_{\sigma(k+1)}}^{\ker E_{\sigma(k)}} = \mathcal{S}_{\sigma(k+1)}$, which concludes the proof. □

Remark 3.10 (*Well-Definedness of One-Step Map*)**.** The one-step map matrix $\Phi_{i,j}$, is in general *not unique* because of the nonuniqueness of the generalized inverse E_j^+ chosen in the calculation; furthermore, the projector $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is only defined on a subspace of $\mathbb{R}^n$, so its $n \times n$ matrix representation is also not unique. However, the pseudo-inverse E_j^+ is only applied to vectors from the subspace $A_i \mathcal{S}_i = \operatorname{im} E_i \cap \operatorname{im} A_i \subseteq \operatorname{im} E_i$ which implies (cf. the discussion after Definition 2.1) that indeed the action of E_j^+ is unique when restricted to the relevant subspace. In particular, for calculations, the well-known Moore–Penrose inverse can be used, for which efficient algorithms are available in the literature, e.g. by using a singular value decomposition [29]. Furthermore, the restriction to the subspace $E_j^+ A_j \mathcal{S}_j = E_j^+(\ker E_j \cap \operatorname{im} A_j) \subseteq \ker E_j \oplus \mathcal{S}_i$ implies that also the action of $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is well defined. In particular, the projector $\Pi_{\mathcal{S}_i}^{\ker E_j}$ can arbitrarily be extended to a projector defined on the whole of $\mathbb{R}^n$ without changing the effect of the one-step map $\Phi_{i,j}$. Altogether, the above discussion shows that the one-step map $\Phi_{i,j}$ is in fact a well-defined map from $\mathcal{S}_j$ to $\mathcal{S}_i$.

Based on the result of Theorem 3.9 it is now not difficult to characterize solvability for a switched system with fixed mode sequence.

Proposition 3.11. *Consider the SLSS (1) with a corresponding family of regular matrix pairs $\{(E_i, A_i)\}_{i=0}^{\mathrm{s}}$ and a given mode sequence $(\sigma_0, \sigma_1, \sigma_2, \ldots)$. The SLSS (1) is solvable (in the sense of Definition 3.8) for any switching signal with this given mode sequence if, and only if, $\{(E_i, A_i)\}_{i=0}^{\mathrm{s}}$ is sequentially index-1 w.r.t. $(\sigma_0, \sigma_1, \sigma_2, \ldots)$. Furthermore, in case of solvability, the solution is given by the one-step map $\Phi_{i,j}$ as in (10) via (9), which takes the form (6) if $E_j^+ = T_j \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] S_j$ where T_j and S_j are given as in (5).*

Proof. Sufficiency is clear from the fact that (7) implies (8) for any switching signal with the given mode sequence. For the necessity, we first observe that due to Theorem 3.9 the condition (8) needs to hold for all possible switching signals with the given mode sequence. In particular, for all $k, j \in \mathbb{N}$ and all switching signals with $\sigma(k) = \sigma(k+1) = \sigma_j$ the necessary condition (7) implies $\ker E_{\sigma_j} \cap \mathcal{S}_{\sigma_j} = \{0\}$. The latter implies that the matrix pair $(E_{\sigma_j}, A_{\sigma_j})$ must be index-1 (see e.g. [30, App. A, Thm. 13]), i.e. (7a) must hold. Furthermore, (8) must hold at any switch from mode σ_j to σ_{j+1} for $j = 0, 1, \ldots$. Thus, (7b) must hold, which completes the proof. □

Remark 3.12 (*Index-1 of Individual Modes*)**.** From Propositions 3.4 and 3.11, switched systems that are solvable for all switching signals or fixed mode sequences with arbitrary switching times must be composed of index-1 modes. In contrast, from Proposition Theorem 3.9, a solvable switched system for a fixed switching signal may contain modes with higher indexes (more than one). However, these higher index modes can only be active for one isolated time instant, because for each mode i which is active for at least two consecutive time-steps, condition (8) implies $\ker E_i \cap \mathcal{S}_i = \{0\}$ which in turn implies index-1 for mode i; see also Example 3.7 and the forthcoming Example 3.17 for more explanations with illustrations.

Remark 3.13 (*Discussion of Solvability Definition*)**.** One may wonder, why we consider *local* solvability in Definition 3.8 instead of just requiring that there exists a unique solution on $[0, \infty)$ for every consistent initial value $x(0)$. The following switched system illustrates the fundamental difference between both approaches:

$$\begin{array}{l|l|l} k=0: & k=1,2: & k\geq 3: \\ 0 = x(k) & \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = x(k) & 0 = x(k) \end{array}$$

It is easily seen that, $x(k) = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$, $k \geq 0$, is the only (and hence unique) solution on $[0, \infty)$ with consistency space $\mathcal{S}_0 = \{0\}$. However, if we consider the switched system only on the interval $[1, 2]$ then any solution $x(1) = \left[\begin{smallmatrix} x_{11} \\ x_{12} \end{smallmatrix}\right]$ and $x(2) = \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right]$ needs to satisfy

$$\begin{aligned} k=1: &\quad \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_{11} \\ x_{21} \end{smallmatrix}\right], \\ k=2: &\quad \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} \alpha \\ \beta \end{smallmatrix}\right] = \left[\begin{smallmatrix} x_{21} \\ x_{22} \end{smallmatrix}\right], \end{aligned}$$

for some $\alpha, \beta \in \mathbb{R}$. First observe that any solution must satisfy $x_{11} \overset{k=1}{=} x_{22} \overset{k=2}{=} 0$, however $\mathcal{S}_1 = \operatorname{im} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$, which means that *not* for all $x(1) \in \mathcal{S}_1$

a solution on $[1,2]$ exists. Secondly, we see that $x_{22} = \beta$ is not uniquely determined (without taking the behavior of the switched system at $k = 3$ into account). This shows that a switched system that is globally uniquely solvable for all consistent initial values is not necessarily locally uniquely solvable (while the converse is of course true). In fact, the example illustrates that for an only globally solvable system the consistency of the state value $x(k)$ is in general not only determined by the active mode k ($k = 1$ in the example), but also depends on future modes. Furthermore, the state $x(k+1)$ in general cannot uniquely be determined from the knowledge of $x(k)$ together with the knowledge of modes k and $k+1$ ($k+1 = 2$ in the example). So in both cases, knowledge of the future behavior of the switched system is necessary to conclude existence and/or uniqueness which in most cases is not desirable. This is in fact related to the concept of causality with respect to the switching signal, cf. [15]. It should also be noted that the time duration which is needed to look ahead to decide about existence and uniqueness grows with the index of the corresponding matrix pairs involved, in general, if a mode has index ν and this mode is also active for at least ν time steps, then one needs to look ahead $\nu - 1$ steps to conclude existence and uniqueness (in the example the index was two and it was necessary to look one step ahead). $\diamond$

Remark 3.14 (*Regularity of Individual Modes*)**.** It is well known that for unswitched systems, regularity is necessary for the existence and uniqueness of a solution, see e.g. [4,31,32]. Thus, when considering arbitrary switching times, each mode considered on its activation interval can be seen as an unswitched system. From this point of view, the regularity of each mode is then necessary for the existence and uniqueness of solutions. However, when considering a fixed switching signal, regularity is in fact not necessary anymore. This is shown by the system where the second mode is not regular, however, the whole switched system has the unique solution $(x_0, x_0, 0, 0, \dots)$.

Inspired by the above remark we observe that the switched index-1 condition (8) (without the additional regularity assumption) is in fact a necessary and sufficient condition for the solvability of general time-varying singular systems of the form $E(k)x(k+1) = A(k)x(k)$. This is formally stated in the following corollary.

Corollary 3.15. *The general time-varying singular linear system*

$$E(k)x(k+1) = A(k)x(k), \;\; k = 0, 1, \dots$$

is solvable in the sense of Definition 3.8 (with switching signal $\sigma(k) = k$) if, and only if, for $k = 0, 1, \dots$

$$E(k)^+ \left(\operatorname{im} E(k) \cap \operatorname{im} A(k)\right) \subseteq \ker E(k) \oplus S(k+1).$$

In the case of solvability, its corresponding (time-varying) one-step map is given by $\Phi(k) := \Pi^{\ker E(k)}_{S(k+1)} E(k)^+ A(k)$.

The following examples illustrate solutions of SLSSs which were calculated by using the one-step map formula introduced in Theorem 3.9.

Example 3.16. Consider the SLSS (1) composed by modes as in Example 3.6, which is sequential index-1 w.r.t. the mode sequence $(\sigma_k) = (0, 1)$. Employing the QWF (5) and the generalized inverse formula in Proposition 3.11 provides $S_0 = T_0 = I$,

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \; T_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ -1 & -1 & 0 \end{bmatrix},$$

$$E_0^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; E_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix},$$

and the one-step map formula (10) yields the matrices $\Phi_{i,j}$ that map mode j to mode i as follows

$$\Phi_{0,0} = \Phi_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \Phi_{1,1} = \Phi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -2 & 0 & -1 \end{bmatrix},$$

$$\Phi_{1,0} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & 0 \end{bmatrix}.$$

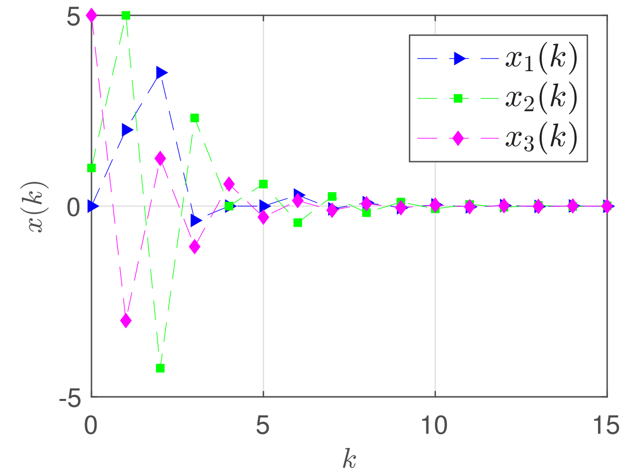


Fig. 4. Solution of Example 3.17.

Its explicit solution under the switching signal $\sigma(k) = 0$ for $k < k^s$ and 1 for $k \geq k^s$ is

$$x(k) = \begin{cases} \Phi_{0,0}^k x(0) & k < k^s \\ \Phi_{1,0} \Phi_0^{k^s - 1} x(0) & k = k^s \\ \Phi_1^{k - k^s} \Phi_{1,0} \Phi_0^{k^s - 1} x(0) & k > k^s. \end{cases} \quad \diamond$$

Example 3.17. Consider the SLSS from Example 3.7 where any switching signal composed of mode transitions from mode-j to mode-i $(j, i) \in \{(0,0), (0,1), (1,0), (2,0)\}$ leads to the property of switched index-1 as long as mode-1 and mode-2 are active only for one time step as discussed in Example 3.7. By choosing

$$E_0^+ = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; E_1^+ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; E_2^+ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

the one-step map matrices from mode j to mode i, $\Phi_{i,j}$, are given by

$$\Phi_{0,2} = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -\frac{1}{2} & -\frac{3}{4} \end{bmatrix}, \qquad \Phi_{0,0} = \Phi_0 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{4} & -\frac{3}{4} & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix},$$

$$\Phi_{1,0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 \end{bmatrix}, \qquad \Phi_{0,1} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}.$$

Clearly, it is switched index-1 w.r.t. the mode sequence $(\sigma_k) = (2, 0, 1, 0)$ under switching times $k_1^s = 1$, $k_2^s = 4$ and $k_3^s = 5$. Its solution with the initial value $x(0) = (0, 1, 5)^\top$ is shown in Fig. 4. $\diamond$

3.4. Transition matrices

By using the one-step map matrix given in Theorem 3.9, we can derive the solution at a switching time k_j^s by using the following corollary.

Corollary 3.18 (*Solutions at Switching Times*)**.** *The solution of a switched index-1 SLSS (1) w.r.t. the switching signal (3) at every switching time k_j^s is given by*

$$x(k_j^s) = \Psi_\sigma(j, 0) x(0) \tag{13}$$

where for $j = 0, 1, \dots$

$$\Psi_\sigma(j, 0) = \Phi_{\sigma_j, \sigma_{j-1}} \Phi_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s - 1} \cdots \Phi_{\sigma_1, \sigma_0} \Phi_{\sigma_0}^{k_1^s - k_0^s - 1}. \tag{14}$$

Matrix $\Psi_\sigma(j, 0)$ above can be rewritten in a recursive form as follows

$$\Psi_\sigma(j, 0) = \Phi_{\sigma_j, \sigma_{j-1}} \Phi_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s - 1} \Psi_\sigma(j-1, 0) \tag{15}$$

with $\Psi_\sigma(0, 0) = I_n$. This is easier and more efficient to compute since it does not contain repetitive calculations as in (14). Matrix $\Psi_\sigma(j, 0)$ above maps the initial value to the solution at a switching time that is useful in observability characterization later. Moreover, we also define in the following corollary a state transition matrix that maps the state at a certain switching time k_j^s to the state at the final time $K > k_J^s$.

Corollary 3.19 (*Final State Transition Matrix*). *The solution of a switched index-1 SLSS (1) w.r.t. switching signal (3) satisfies*

$$x(K) = \Psi_\sigma^*(K, j)x(k_j^s) \tag{16}$$

where for $j = J, J-1, \dots, 0$ and with $\Phi_{\sigma_0,\sigma_{-1}} = I_n$

$$\begin{aligned}\Psi_\sigma^*(K, j) = {}& \Phi_{\sigma_J}^{K-k_J^s}\Phi_{\sigma_J,\sigma_{J-1}}\Phi_{\sigma_{J-1}}^{k_J^s-k_{J-1}^s-1}\\ &\Phi_{\sigma_{J-1},\sigma_{J-1}}\Phi_{\sigma_{J-2}}^{k_{J-1}^s-k_{J-2}^s-1}\cdots\Phi_{\sigma_j,\sigma_{j-1}}\Phi_j^{k_{j+1}^s-k_j^s-1}.\end{aligned} \tag{17}$$

We call the matrix $\Psi_\sigma^*(K, j)$ as the final state transition matrix from switching times, and will use it later in the determinability characterization. Furthermore, it can be rewritten in the following recursive form which is more computationally friendly

$$\Psi_\sigma^*(K, j) = \Psi_\sigma^*(K, j+1)\Phi_{\sigma_j,\sigma_{j-1}}\Phi_j^{k_{j+1}^s-k_j^s-1} \tag{18}$$

with $\Psi_\sigma^*(K, J) = \Phi_{\sigma_J}^{K-k_J^s}$.

4. Observability

In this section we consider the SLSS (1) only on the finite time interval $[0, K]$, $K = k_{J+1}^s - 1$. Note that for $k = K$ Eq. (1) involves $x(K+1)$ whose existence we need to assume, but otherwise the solution is only considered for $k = 0, 1, \dots, K$.

Note that due to linearity it should be possible to extend the forthcoming observability results easily to the inhomogeneous case, where an input is added to (1). However, so far only for the jointly index-1 case a solution theory is available for the inhomogeneous case [7], and extending the solution theory for the inhomogeneous case to the sequential and in particular switched index-1 case is outside the scope of the manuscript and hence we restrict our attention to the homogeneous case.

4.1. Observability definitions

Observability of a system means that knowledge of the external signal (the output) implies full knowledge of the internal signals (the state). The formal definition is as follows.

Definition 4.1 (*Observability*). The SLSS (1) is called *observable* on $[0, K]$ w.r.t. a fixed switching signal given by (3) if for all solutions (x^1, y^1) and (x^2, y^2) of (1) on $[0, K]$ the following implication holds:

$$y_{[0,K]}^1 = y_{[0,K]}^2 \implies x_{[0,K]}^1 = x_{[0,K]}^2. \tag{19}$$

Utilizing linearity, the observability definition can be simplified as follows.

Proposition 4.2 (*Zero Observability*). *The SLSS (1) is observable on $[0, K]$ w.r.t. a fixed switching signal given by (3) if, and only if, for all solutions (x, y) of (1) on $[0, K]$ the following implication holds:*

$$y_{[0,K]} = 0 \implies x_{[0,K]} = 0. \tag{20}$$

Proof. *Necessity:* This is obvious by considering the trivial solution $(x^2, y^2) = (0, 0)$.

Sufficiency: Consider two solutions (x^1, y^1) and (x^2, y^2) of (1) on $[0, K]$ with $y_{[0,K]}^1 = y_{[0,K]}^2$. By linearity, $x := x^1 - x^2$ is also a solution on $[0, K]$ with output $y = y^1 - y^2 = 0$. Hence by assumption $x_{[0,K]} = 0$, i.e. $x_{[0,K]}^1 = x_{[0,K]}^2$. □

Under a unique solvability assumption of (1) it follows that $x_{[0,K]} = 0$ if, and only if, $x(0) = 0$. Thus, the observability condition for (1) reduces to

$$y(k) = 0 \ \forall k \in [0, K] \implies x(0) = 0. \tag{21}$$

In other words, observability is concerned with recovering the value of the state in the *past* from the measured output values. For some applications (e.g. designing feedback rules based on observers) it may however be more relevant to recover the present state from the already measured output. This ability is called determinability and is formally defined as follows:

Definition 4.3 (*Determinability*). The SLSS (1) is called *determinable* on $[0, K]$ w.r.t. a fixed switching signal of the form (3) if, and only if, the following implication holds for all solutions $(x^1, y^1), (x^2, y^2)$ of (1):

$$y_{[0,K]}^1 = y_{[0,K]}^2 \implies x^1(K) = x^2(K) \tag{22}$$

Similar to observability, we can also simplify the determinability definition as follows:

Proposition 4.4 (*Zero Determinability*). *The SLSS (1) is determinable on $[0, K]$ w.r.t. a fixed switching signal of the form (3) if, and only if, the following implication holds for all solutions (x, y) of (1):*

$$y_{[0,K]} = 0 \implies x(K) = 0. \tag{23}$$

The proof is very similar to the proof of Proposition 4.2 and therefore omitted.

4.2. Observability characterizations

Using the notation from Corollary 3.18 we can formulate the following characterization of observability.

Theorem 4.5. *Consider the SLSS (1) and assume it is of switched index-1 w.r.t. a fixed switching signal given by (3). Then, this system is observable on $[0, K]$ if, and only if,*

$$S_{\sigma_0} \cap \bigcap_{j=0}^{J} \Psi_\sigma(j, 0)^{-1}(\mathcal{O}_{\sigma_j}^{k_{j+1}^s-k_j^s-1}) = \{0\} \tag{24}$$

where $\Psi_\sigma(j, 0)$ is given by (15) (which is not assumed to be invertible, in particular, $\Psi_\sigma(j, 0)^{-1}$ stands for the preimage) and, for $k \in \mathbb{N}$,

$$\mathcal{O}_{\sigma_j}^k = \ker \underbrace{[C_{\sigma_j}^\top, (C_{\sigma_j}\Phi_{\sigma_j})^\top, \dots, (C_{\sigma_j}\Phi_{\sigma_j}^k)^\top]^\top}_{=:O_{\sigma_j}^k}. \tag{25}$$

Proof. The outputs over the time interval $[0, K]$ can be written as:

$$\begin{bmatrix} y(0) \\ y(1) \\ \vdots \\ y(k_1^s-1) \\ y(k_1^s) \\ y(k_1^s+1) \\ \vdots \\ y(k_2^s-1) \\ y(k_2^s) \\ y(k_2^s+1) \\ \vdots \\ y(k_3^s-1) \\ \vdots \\ y(K) \end{bmatrix} = \begin{bmatrix} C_{\sigma_0} \\ C_{\sigma_0}\Phi_{\sigma_0} \\ \vdots \\ C_{\sigma_0}\Phi_{\sigma_0}^{k_1^s-1} \\ C_{\sigma_1}\Psi_1(1,0) \\ C_{\sigma_1}\Phi_{\sigma_1}\Psi_1(1,0) \\ \vdots \\ C_{\sigma_1}\Phi_{\sigma_1}^{k_2^s-k_1^s-1}\Psi_1(1,0) \\ C_{\sigma_2}\Psi_2(2,0) \\ C_{\sigma_2}\Phi_{\sigma_2}\Psi_2(2,0) \\ \vdots \\ C_{\sigma_2}\Phi_{\sigma_2}^{k_3^s-k_2^s-1}\Psi_2(2,0) \\ \vdots \\ C_{\sigma_J}\Phi_{\sigma_J}^{K-k_J^s-1}\Psi_\sigma(J,0) \end{bmatrix} x_0$$

$$= \underbrace{\begin{bmatrix} O_{\sigma_0}^{k_1^s-1} \\ O_{\sigma_1}^{k_2^s-k_1^s-1}\Psi_1(1,0) \\ O_{\sigma_2}^{k_3^s-k_2^s-1}\Psi_2(2,0) \\ \cdots \\ O_{\sigma_J}^{K-k_J^s-1}\Psi_\sigma(J,0) \end{bmatrix}}_{O_K} x_0 = O_{[0,K]}x_0.$$

By using the fact that $\ker(O\Phi) = \Phi^{-1}(\ker O)$ for any matrices O and Φ of appropriate size, we have

$$\ker O_{[0,K]} = \mathcal{O}_{\sigma_0}^{k_1^s-1} \cap \Psi_1(1,0)^{-1}(\mathcal{O}_{\sigma_1}^{k_2^s-k_1^s-1})$$
$$\cap \cdots \cap \Psi_\sigma(J,0)^{-1}(\mathcal{O}_{\sigma_J}^{K-k_J^s-1}) =: \mathcal{O}_{[0,K]}.$$

Sufficiency: Assume $0 \neq x_0 \in \mathcal{S}_{\sigma_0} \cap \mathcal{O}_{[0,K]}$. Then there exists a unique solution x of the SLSS (1) with $x(0) = x_0 \in \mathcal{S}_{\sigma_0}$. Since $x(0) \in \mathcal{O}_{[0,K]}$ it follows from above that $y(k) = 0$, $0 \leq k \leq K$. This means that there exists a non-trivial solution of x with zero output. Hence, (1) is not observable.

Necessity: Consider a solution of (1) then $x(0) \in \mathcal{S}_{\sigma_0}$. Furthermore, if $y(k) = 0$ for all $k \in [0,K]$, then $x(0) \in \mathcal{O}_{[0,K]}$. Hence $x(0) \in \mathcal{S}_{\sigma_0} \cap \mathcal{O}_{[0,K]} = \{0\}$. □

We call the subspace on the left hand side in (24) the unobservable space of the SLSS (1), and this system is observable if, and only if, the unobservable space is a singleton set with the zero vector.

Example 4.6. Recall Example 3.17. With $C_0 = (\frac{1}{4}, \frac{2}{4}, 1)$, $C_1 = (0,1,-1)$ and $C_2 = (0,1,0)$, the switched system under the same switching signal is not observable on $[0,7]$ since the unobservable space in (24) is $\text{span}\{(0,0,1)^\top\}$. ⋄

By exploiting the Cayley–Hamilton Theorem, it is possible to simplify (25) when the observation time is long enough and each mode is active for at least n time instants. This is summarized by the following corollary.

Corollary 4.7. *Consider a switched index-1 SLSS (1) w.r.t. the switching signal (3) and each mode is long enough active i.e. $k_{j+1}^s - k_j^s \geq n-1$, for all j. Then, the system (1) is observable on $[0,K]$ if, and only if,*

$$\mathcal{S}_{\sigma_0} \cap \bigcap_{j=0}^{J} \Psi_\sigma(j,0)^{-1}(\ker O_{\sigma_j}) = \{0\} \tag{26}$$

where $O_{\sigma_j} := O_{\sigma_j}^{n-1}$ for $j = 0,1,\ldots,J$. In particular, if $K \geq k_J^s + n - 1$, then observability does not depend on the total length of the observation interval $[0,K]$.

Remark 4.8. In (24) the switching times explicitly occur in $\mathcal{O}_{\sigma_j}^{k_{j+1}^s-k_j^s-1}$ and implicitly in $\Psi_\sigma(j,0)$, which indicate that, in general, observability depends on the switching times (and not only on the mode sequence) and changing the switching times may produce different observability result. Furthermore, assuming that each mode is active long enough, the dependence of observability on the switching times is only partially removed in (26) because $\Psi_\sigma(j,0)$ still depends on them.

The following example demonstrates the dependence of observability on switching times.

Example 4.9. Consider the SLSS (1) composed by two modes given by

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 1&0&0\\1&1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\1&0&1\\1&1&1 \end{bmatrix}, [\,1\ 0\ 1\,]\right),$$
$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 0&0&1\\1&0&1\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\1&1&0\\1&1&1 \end{bmatrix}, [\,0\ 0\ 0\,]\right).$$

It is easily seen that this SLSS is sequentially index-1 for the mode sequence $(\sigma_k) = (0,1,0)$, and thus it is switched index-1 w.r.t. the same mode sequence with any arbitrary switching times k_1^s (from the first mode to the second mode) and k_2^s (from the second mode to the third mode), and any final time K with $k_1^s < k_2^s < K$. With respect to this mode sequence, with switching times from the range $3 \leq k_1^s$, $k_1^s + 2 < k_2^s \leq 10$, $K = k_2^2 + 3$ (i.e. satisfying the dwell time condition from Corollary 4.7) the observability property is shown in Fig. 5. It can be seen that, indeed, the observability of this system depends on the switching times; in fact, the SLSS is observable if, and only if, the mode duration $k_2^s - k_2^s$ of mode 1 is odd. The state and output trajectories for the specific switching times $k_1^s = 3$ and $k_2^s = 6$ (odd mode duration of mode 1) as well as for $k_1^s = 3$ and $k_2^s = 7$ (even mode duration of mode 1) are shown in Figs. 6(a) and 6(b), respectively. ⋄

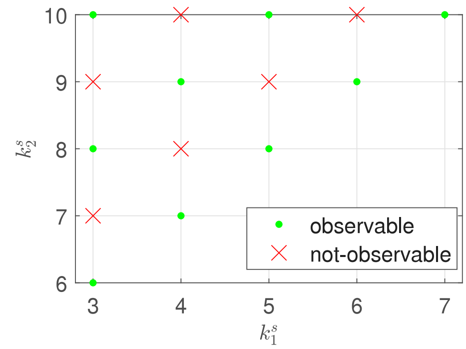


Fig. 5. Switching times vs observability of Example 4.9.

The dependence of observability on the switching times also occurs in continuous time (see [21, Th. 12]). However, in contrast to the continuous time case, this dependence on the switching time also occurs for the single-switch case as the following example shows.

Example 4.10. Consider the SLSS (1) observed on the time interval $[0,K]$ with $K = 10$, and composed by two modes with matrices

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 0&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&0&0&0\\1&0&0&0\\1&1&0&1\\0&0&1&0 \end{bmatrix}, \begin{bmatrix} 0\\1\\1\\1 \end{bmatrix}^\top\right)$$
$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&1&1\\0&0&0&1\\1&0&0&1\\1&0&1&1 \end{bmatrix}, \begin{bmatrix} 1\\1\\0\\1 \end{bmatrix}^\top\right).$$

Simple computations provide

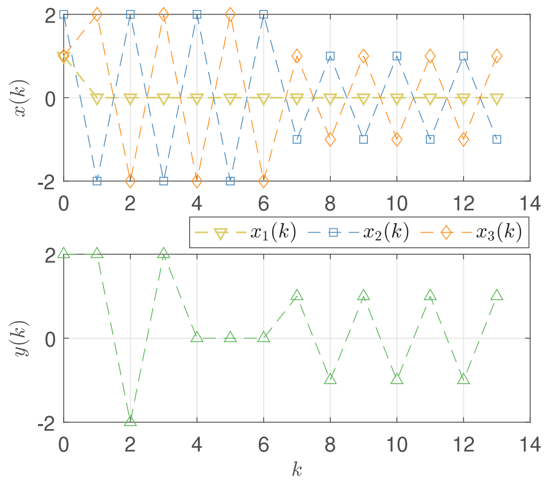
$$\mathcal{S}_0 = \text{span}\{(1,0,0,-1)^\top, (0,1,0,-1)^\top\},$$
$$\mathcal{S}_1 = \text{span}\{(1,0,0,-1)^\top, (0,1,0,0)^\top\},$$
$$\ker E_0 = \ker E_1 = \text{span}\{(0,0,1,0)^\top, (0,0,0,1)^\top\}.$$

Both mode 0 and mode 1 as individual systems are not-observable on $[0,K]$ since

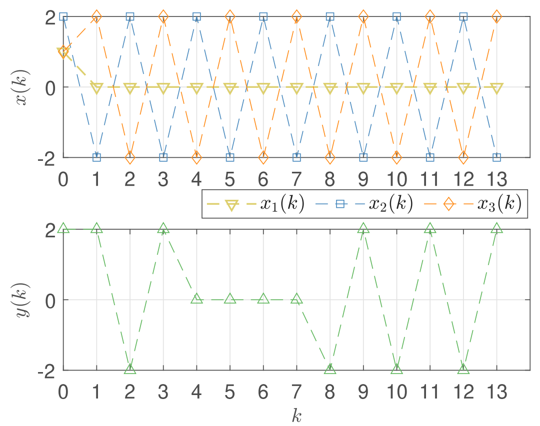
$$\mathcal{S}_0 \cap \mathcal{O}_0 = \text{span}\{(0,1,0,-1)^\top\} \neq \{0\}, \text{ and}$$
$$\mathcal{S}_1 \cap \mathcal{O}_1 = \text{span}\{(1,0,0,-1)^\top\} \neq \{0\}.$$

We consider now the switched systems with a single switch and mode sequences $(\sigma_k) = (0,1)$ and $(1,0)$. With respect to both those mode sequences, the system (1) is sequentially index-1, and thus it is switched index-1 w.r.t. the same mode sequence with any arbitrary switching time k^s.

The dependence of observability on various switching times $k^s \in [1,K]$ is illustrated in Fig. 7. While for the mode sequence $(0,1)$ the switched system is unobservable for all possible switching times, the observability for mode sequence $(1,0)$ depends on the switching time (for $k_s = 1$ or $k_s = 10$ the switched system is not observable, while it is observable for all other k_s). It should be noted however that when restricting to the case of a minimal dwell-time as in Corollary 4.7, observability becomes independent from the switching times for this example. ⋄



(a) $k_1^s = 3, k_2^s = 6$, observable



(b) $k_1^s = 3, k_2^s = 7$, not-observable

Fig. 6. State and output for Example 4.9.

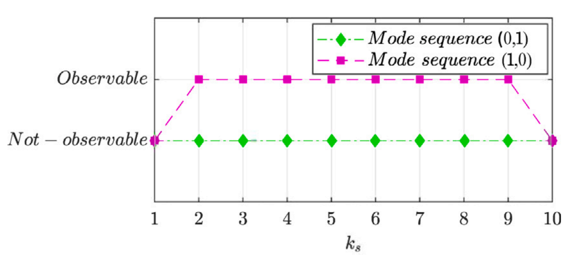


Fig. 7. Switching time vs observability of Example 4.10.

It is an important property of a switched system with a given mode sequence whether the observability property does or does not depend on the switching times and we have already discussed some special cases in [16]. However, there seems to be no easy general characterization and this is a topic of future research.

4.3. Determinability characterization

For a SLSS (1) which is switched index-1 w.r.t. to a switching signal given by (3) we define the following sequence of subspaces, which will play a crucial role in characterizing determinability:

$$\mathcal{Q}^0 = \ker C_{\sigma_0} \cap \mathcal{S}_{\sigma_0} \tag{27a}$$

$$\mathcal{Q}^k = \ker C_{\sigma(k)} \cap \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)}\mathcal{Q}^{k-1}, k = 1, 2, \dots, K \tag{27b}$$

Lemma 4.11. *Consider a switched index-1 SLSS with corresponding subspace sequence (27). For every $k \in [0, K]$, we have that $x_k \in \mathcal{Q}^k$ if, and only if, there exists a solution of (1) with $x(k) = x_k$ and $y(i) = 0$ for all $i \in [0, k]$.*

Proof. *Sufficiency*: For $k = 0$ the claim is clear, because $x_0 \in \mathcal{S}_{\sigma_0}$ implies existence of a solution with $x(0) = x_0$ and $x_0 \in \ker C_{\sigma_0}$ implies that $y(0) = C_{\sigma_0} x_0 = 0$. For $k > 0$ we proceed inductively, i.e., we assume the claim holds for $k-1$. From $x_k \in \mathcal{Q}^k \subseteq \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)}\mathcal{Q}^{k-1}$ it follows the existence of a $x_{k-1} \in \mathcal{Q}^{k-1}$ with $x_k = \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)} x_{k-1}$. By inductive assumption, there exists a solution x of (1) on $[0, k-1]$ with $x(k-1) = x_{k-1}$ and $y(i) = 0$ for all $i \in [0, k-1]$. By Theorem 3.9, setting $x(k) = x_k = \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)} x_{k-1}$ yields a solution on $[0, k]$ and from $x_k \in \mathcal{Q}^k \subseteq \ker C_{\sigma(k)}$ it follows that also $y(k) = C_{\sigma_k} x_k = 0$ which concludes the sufficiency part of the proof.
Necessity: For $k = 0$ the claim is clearly true, because every solution x of SLSS (1) needs to satisfy $x(0) \in \mathcal{S}_{\sigma_0}$ and $y(0) = 0$ implies $x(0) \in \ker C_{\sigma_0}$. For $k > 0$ we again proceed inductively. Therefore, consider a solution x of SLSS with $y(i) = 0$ for all $i \in [0, k]$. This implies $x(k) \in \ker C_{\sigma(k)}$ and $x(k) = \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)} x(k-1)$. Using the inductivity assumption, we know that $x(k-1) \in \mathcal{Q}^{k-1}$, because $y(i) = 0$ for all $i \in [0, k-1]$. Hence $x(k) \in \ker C_{\sigma(k)} \cap \boldsymbol{\Phi}_{\sigma(k),\sigma(k-1)}\mathcal{Q}^{k-1} = \mathcal{Q}^k$ as desired. □

From Lemma 4.11 we can directly obtain the following determinability characterization:

Corollary 4.12. *Consider a SLSS (1) which is switched index-1 w.r.t. a switching signal σ given by (3). Then, (1) is determinable on $[0, K]$ w.r.t. σ if, and only if,*

$$\mathcal{Q}^K = \{0\} \tag{28}$$

where $\mathcal{Q}^K$ is recursively given by (27).

In contrast to the characterization of observability given in Theorem 4.5 the dependence of determinability on the switching times is not so apparent. However, by introducing the family of maps $\Omega_{i,j}$ which map a subspace $\mathcal{S}$ to

$$\Omega_{i,j}\mathcal{S} := \ker C_i \cap \boldsymbol{\Phi}_{i,j}\mathcal{S},$$

we see that $\mathcal{Q}^k = \Omega_{\sigma(k),\sigma(k-1)}\mathcal{Q}^{k-1}$. In particular, for a switching signal given by (3), we can conclude that the undeterminable space $\mathcal{Q}^{k_{j+1}^s-1}$ can be expressed in terms of corresponding powers of the operator $\Omega_{\sigma_j,\sigma_j}$ applied to $\mathcal{Q}^{k_j^s}$, i.e.

$$\mathcal{Q}^{k_{j+1}^s-1} = \Omega_{\sigma_j,\sigma_j}^{k_{j+1}^s-k_j^s-1}\mathcal{Q}^{k_j^s} = \Omega_{\sigma_j,\sigma_j}^{k_{j+1}^s-k_j^s-1}\Omega_{\sigma_j,\sigma_{j-1}}\mathcal{Q}^{k_j^s-1}.$$

Consequently, we can see that $\mathcal{Q}^K$ can be calculated by the following nested formula:

$$\Omega_{\sigma_J,\sigma_J}^{K-k_J^s-1}\Omega_{\sigma_J,\sigma_{J-1}}(\Omega_{\sigma_{J-1},\sigma_{J-1}}^{k_J^s-k_{J-1}^s-1}\Omega_{\sigma_J,\sigma_{J-1}}(\cdots(\Omega_{\sigma_0,\sigma_0}^{k_1^s-1}\mathcal{Q}^0))),$$

which clearly shows the dependence on the switching times. However, it seems not possible to easily simplify this expression in case the mode durations are sufficiently large (as in Corollary 4.7).

The following example illustrates that determinability is indeed a weaker property than observability.

Example 4.13. Consider the SLSS (1) with the following system's matrices

$$(E_0, A_0, C_0) = \left(\begin{bmatrix} 0&1&1&0\\0&1&1&0\\1&1&1&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0&1\\0&0&1&0\\0&0&1&0\\0&1&1&1 \end{bmatrix}, \begin{bmatrix} 1\\0\\1\\1 \end{bmatrix}^\top \right),$$

$$(E_1, A_1, C_1) = \left(\begin{bmatrix} 1&1&0&0\\1&1&0&0\\1&0&1&0\\0&0&0&0 \end{bmatrix}, \begin{bmatrix} 1&1&1&0\\1&1&0&0\\1&0&1&0\\0&1&1&1 \end{bmatrix}, \begin{bmatrix} 1\\1\\0\\0 \end{bmatrix}^\top \right)$$

and with the mode sequence $(0, 1)$, which can be shown to be sequentially index-1 w.r.t. this mode sequence, and thus it is switched index-1 w.r.t. any (single switch) switching signal with the same mode sequence. Using Theorem 4.5 it can be shown that this SLSS is unobservable on $[0, 12]$ for any switching time k^s with $1 \le k^s \le 12$. On the other hand, only for $k^s = 1$ the SLSS is not determinable on $[0, 12]$, for all other switching times $k_s \ge 2$ the system is determinable. This shows that we can recover the final state although the initial state cannot be recovered from the same output measurement. ⋄

5. Summary

In this paper, we have fully characterized the solvability of switched linear singular systems for two types of constrained switching signals. These characterizations generalize the notion of "jointly index-1" as well as the one-step map previously reported in the literature. A somewhat surprising result compared to the continuous time case is that each mode being index-1 is neither sufficient nor necessary for the solvability of a SLSS with a given switching signal. A first application of this novel solution theory is the study of observability and determinability, and characterizations of these properties are given.

Another application not discussed here is the consideration of stability, however, in view of the existence of the one-step-map it is possible to rewrite the singular switched system (with fixed switching signal) as a non-singular switched system whose stability can be checked with available methods for linear time-varying systems. Nevertheless, simplifications on the stability criteria are possible since the system is not fully time-varying; this is one of our future works.

CRediT authorship contribution statement

Sutrisno Sutrisno: Conceptualization, Methodology, Investigation, Writing – original draft, Writing – review & editing, Visualization. **Stephan Trenn:** Methodology, Investigation, Validation, Formal analysis, Writing – review & editing, Supervision.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix. Some basic linear algebra results

Lemma A.1. *For any invertible matrix $A \in \mathbb{R}^{n\times n}$, and any $C \in \mathbb{R}^{m\times n}$, the following equation holds for every $k \in \mathbb{Z}$ and every $\hat{n} \ge n$*

$$\ker\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} = \ker\begin{bmatrix} CA^k \\ CA^{1+k} \\ \vdots \\ CA^{\hat{n}-1+k}\end{bmatrix}. \tag{A.1}$$

Proof. Consider first $k = 1$. By Cayley–Hamilton theorem, each of the rows of CA^k for $k > n-1$ are linearly dependent on the rows of the observability matrix $[C^\top, (CA)^\top, \dots, (CA^{n-1})^\top]^\top$, hence

$$\ker\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} = \ker\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \\ CA^n \\ \vdots \\ CA^{\hat{n}}\end{bmatrix} \subseteq \ker\begin{bmatrix} CA \\ \vdots \\ CA^{\hat{n}-1} \\ CA^{\hat{n}}\end{bmatrix}.$$

To show the converse subspace relation, let $x \in \ker\begin{bmatrix} CA \\ \vdots \\ CA^{\hat{n}-1} \\ CA^{\hat{n}}\end{bmatrix}$. Then the Cayley–Hamilton theorem ensures existence of $a_0, a_1, \dots, a_{n-1} \in \mathbb{R}$ such that

$$0 = CA^n x = C\sum_{i=0}^{n} a_i A^i x.$$

By assumption $CA^i x = 0$ for $i = 1, 2, \dots, n-1$, hence we can conclude that

$$0 = \sum_{i=0}^{n} a_i CA^i x = a_0 Cx.$$

Since A is invertible, the characteristic polynomial $\det(sI - A) = \sum_{i=0}^{n} a_i s$ cannot have the root $\lambda = 0$, i.e. $a_0 \ne 0$ and we can conclude that $Cx = 0$. Therefore, $x \in \ker C$ and thus $x \in \ker\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{\hat{n}-1}\end{bmatrix}$.

Now our claim is that for any $k \in \mathbb{Z}$,

$$\ker\begin{bmatrix} CA^k \\ CA^{1+k} \\ \vdots \\ CA^{\hat{n}-1+k}\end{bmatrix} = \ker\begin{bmatrix} CA^{k+1} \\ CA^{1+k+1} \\ \vdots \\ CA^{\hat{n}-1+k+1}\end{bmatrix}, \tag{A.2}$$

from which (A.1) follows inductively. Let $\hat{C} := CA^k$ (this is also applicable for $k < 0$ since A is invertible) then using the same arguments as in the case of $k = 1$, we have that

$$\ker\begin{bmatrix} \hat{C} \\ \hat{C}A \\ \vdots \\ \hat{C}A^{\hat{n}-1}\end{bmatrix} = \ker\begin{bmatrix} \hat{C}A \\ \hat{C}A^2 \\ \vdots \\ \hat{C}A^{\hat{n}}\end{bmatrix} \tag{A.3}$$

i.e. (A.2) holds. □

Lemma A.2. *For any subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ and matrix $A \in \mathbb{R}^{n\times n}$:*

(i) $A(\mathcal{V}\cap\mathcal{W}) = A\mathcal{V}\cap A\mathcal{W} \iff (\mathcal{V}+\mathcal{W})\cap\ker A = (\mathcal{V}\cap\ker A)+(\mathcal{W}\cap\ker A)$
(ii) $\mathcal{V}\cap A\mathcal{V}\cap\cdots\cap A^n\mathcal{V} = \mathcal{V}\cap A\mathcal{V}\cap\cdots\cap A^{n+\ell}\mathcal{V}$ *for all* $\ell \in \mathbb{N}$.

Proof. The proof for part (i) is available in [33, App. A]. To prove (ii), note first that it suffices to show that $\mathcal{V}\cap A\mathcal{V}\cap\cdots\cap A^n\mathcal{V} \subseteq A^{n+\ell}\mathcal{V}$ for all $\ell \in \mathbb{N}$.
Case 1: A is invertible.
We show inductively that if $x \in \mathcal{V}\cap A\mathcal{V}\cap\ldots\cap A^{n+\ell}\mathcal{V}$ then $x \in A^{n+\ell+1}\mathcal{V}$. Under the assumption that $x \in \mathcal{V}\cap\ldots\cap A^{n+\ell}\mathcal{V}$ we can choose for each $i = 0, \dots, n+\ell$ some $v_i \in \mathcal{V}$ such that $x = A^i v_i$. Note that by assumption $x = v_0 \in \mathcal{V}$ and hence $x = A^i v_i \in \mathcal{V}$ as well; furthermore, $A^{-k}x = v_k \in \mathcal{V}$, hence also $A^{i-k}v_i = x \in \mathcal{V}$ for all $k \in \{0, 1, \dots, i\}$. Let $\lambda_0 + \lambda_1 s + \cdots + \lambda_{n-1}s^{n-1} + s^n$ be the characteristic polynomial of A^{-1} and let $v_{n+\ell+1} := -\sum_{i=0}^{n-1} \lambda_i v_{i+\ell+1} \in \mathcal{V}$, then

$$\begin{aligned} A^{n+\ell+1}v_{n+\ell+1} &= -A^n\sum_{i=0}^{n-1}\lambda_i A^{\ell+1}v_{\ell+1+i} \\ &= A^n\sum_{i=0}^{n-1} -\lambda_i A^{-i}x = x, \end{aligned}$$

where $A^{-n} = \sum_{i=0}^{n-1} -\lambda_i A^{-i}$ (Cayley–Hamilton for A^{-1}) is used.

Case 2: A is singular.

In that case there exists a coordinate transformation such that the matrix representation of A becomes $\left[\begin{smallmatrix} J & 0 \\ 0 & N\end{smallmatrix}\right]$ where $J \in \mathbb{R}^{n_1\times n_1}$ nonsingular and $N \in \mathbb{R}^{n_2\times n_2}$ nilpotent with nilpotency index $n_0 \le n$. In these coordinates, let $\mathcal{V} = \operatorname{im}\left[\begin{smallmatrix} V_1 \\ V_2\end{smallmatrix}\right]$ for some suitable matrices V_1 and V_2 and $\mathcal{V}\cap A\mathcal{V}\cap\cdots\cap A^n\mathcal{V} = \operatorname{im}\left[\begin{smallmatrix} V_1 \\ V_2\end{smallmatrix}\right] \cap \operatorname{im}\left[\begin{smallmatrix} JV_1 \\ NV_2\end{smallmatrix}\right] \cap\ldots\cap \left[\begin{smallmatrix} J^{n_0-1}V_1 \\ N^{n_0-1}V_2\end{smallmatrix}\right] \cap \left[\begin{smallmatrix} J^{n_0}V_1 \\ 0\end{smallmatrix}\right] \cap\ldots\cap \left[\begin{smallmatrix} J^{n}V_1 \\ 0\end{smallmatrix}\right] \subseteq (\operatorname{im}V_1 \cap J\operatorname{im}V_1\cap\cdots\cap J^n\operatorname{im}V_1)\otimes\{0\}$, where the last subspace relation follows from the general fact, that for suitably sized matrices P, Q, R, S it always holds that $\operatorname{im}\left[\begin{smallmatrix} P \\ Q\end{smallmatrix}\right] \cap \operatorname{im}\left[\begin{smallmatrix} R \\ S\end{smallmatrix}\right] \subseteq (\operatorname{im}P \cap \operatorname{im}R)\otimes(\operatorname{im}Q\cap\operatorname{im}S)$. From Case 1 applied to the invertible matrix J, we know that $\operatorname{im} V_1 \cap \cdots \cap J^n \operatorname{im} V_1 \subseteq J^{n+\ell} \operatorname{im} V_1$ for any $\ell \in \mathbb{N}$ and hence $\mathcal{V} \cap A\mathcal{V} \cap \cdots \cap A^n \mathcal{V} \subseteq \operatorname{im} \begin{bmatrix} J^{n+\ell} V_1 \\ 0 \end{bmatrix} = A^{n+\ell}\mathcal{V}$ as desired. □

Remark A.3. The statement of Lemma A.2(ii) does *not* hold in general when n is replaced by $n-1$. As a counterexample consider $\mathcal{V} = \mathbb{R}^2$ and $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Nevertheless, the proof reveals that A being nilpotent with maximal possible nilpotency index n is the only case where n cannot be replaced by $n-1$.

References

[1] S.S. Ge, Z. Sun, T.H. Lee, Reachability and controllability of switched linear discrete-time systems, IEEE Trans. Automat. Control 46 (2001) 1–12, http://dx.doi.org/10.1109/ACC.2001.946016.

[2] M. Babaali, M. Egerstedt, Observability of switched linear systems, in: Hybrid Systems: Computation and Control, in: Lecture Notes in Computer Science, vol. 2993, Springer, 2004, pp. 48–63, http://dx.doi.org/10.1007/978-3-540-24743-2_4.

[3] M. Egerstedt, M. Babaali, On observability and reachability in a class of discrete-time switched linear systems, in: Proc. of the American Control Conference, 2005, pp. 1179–1180, http://dx.doi.org/10.1109/ACC.2005.1470121.

[4] D.G. Luenberger, Dynamic equations in descriptor form, IEEE Trans. Autom. Control 22 (1977) 312–321, http://dx.doi.org/10.1109/TAC.1977.1101502.

[5] J. Wei, X. Zhang, H. Zhi, X. Zhu, New finite-time stability conditions of linear discrete switched singular systems with finite-time unstable subsystems, J. Franklin Inst. B 357 (1) (2020) 279–293, http://dx.doi.org/10.1016/j.jfranklin.2019.03.045.

[6] B. Men, X. Yu, Q. Zhang, Research on stability for discrete-time switched linear singular systems, in: 2010 Chinese Control and Decision Conference, CCDC 2010, 2010, pp. 2459–2463, http://dx.doi.org/10.1109/CCDC.2010.5498784.

[7] P.K. Anh, P.T. Linh, D.D. Thuan, S. Trenn, Stability analysis for switched discrete-time linear singular systems, Automatica 119 (109100) (2020) http://dx.doi.org/10.1016/j.automatica.2020.109100.

[8] G. Zhai, X. Xu, A unified approach to stability analysis of switched linear descriptor systems under arbitrary switching, Int. J. Appl. Math. Comput. Sci. 20 (2) (2010) 249–259, http://dx.doi.org/10.2478/v10006-010-0018-2.

[9] P.T. Linh, Stability of arbitrarily switched discrete-time linear singular systems of index-1, VNU J. Sci.: Math. - Phys. 34 (4) (2018) 77–84, http://dx.doi.org/10.25073/2588-1124/vnumap.4312.

[10] G. Zhai, X. Xu, D.W. Ho, Stability of switched linear discrete-time descriptor systems: A new commutation condition, Internat. J. Control 85 (11) (2012) 1779–1788, http://dx.doi.org/10.1080/00207179.2012.703788.

[11] G. Zhai, R. Kou, J. Imae, T. Kobayashi, Stability analysis and design for switched descriptor systems, Int. J. Control Autom. Syst. 7 (3) (2009) 349–355, http://dx.doi.org/10.1007/s12555-009-0303-8.

[12] P.K. Anh, P.T. Linh, Stability of periodically switched discrete-time linear singular systems, J. Difference Equ. Appl. 23 (10) (2017) 1680–1693, http://dx.doi.org/10.1080/10236198.2017.1356293.

[13] P. Gu, S. Tian, Iterative learning control for discrete-time switched singular systems, J. Difference Equ. Appl. 24 (9) (2018) 1414–1428, http://dx.doi.org/10.1080/10236198.2018.1494165.

[14] D. Koenig, B. Marx, Hinf-filtering and state feedback control for discrete-time switched descriptor systems, IET Control Theory Appl. 3 (6) (2009) 661–670, http://dx.doi.org/10.1049/iet-cta.2008.0132.

[15] P.K. Anh, P.T. Linh, D.D. Thuan, S. Trenn, The one-step-map for switched singular systems in discrete-time, in: Proc. 58th IEEE Conf. Decision Control, (CDC) 2019, 2019, pp. 605–610, http://dx.doi.org/10.1109/CDC40024.2019.9030154.

[16] Sutrisno, S. Trenn, Observability of switched linear singular systems in discrete time: Single switch case, in: Proc. of European Control Conference, (ECC), 2021, pp. 287–292, 10.23919/ECC54610.2021.9654844.

[17] Sutrisno, S. Trenn, Observability and determinability characterizations for linear switched systems in discrete time, in: 2021 60th IEEE Conference on Decision and Control, (CDC), 2021, pp. 2474–2479, http://dx.doi.org/10.1109/CDC45484.2021.9682894.

[18] R. Vidal, A. Chiuso, S.S. Sastry, Observability and identifiability of jump linear systems, in: Proc. 41st IEEE Conf. Decis. Control, 2002, pp. 3614–3619, http://dx.doi.org/10.1109/CDC.2002.1184923.

[19] M. Witczak, V. Puig, D. Rotondo, P. Witczak, A necessary and sufficient condition for total observability of discrete-time linear time-varying systems, IFAC-PapersOnLine 50 (1) (2017) 729–734, http://dx.doi.org/10.1016/j.ifacol.2017.08.232, 20th IFAC World Congress.

[20] A. Tanwani, S. Trenn, On observability of switched differential-algebraic equations, in: Proc. 49th IEEE Conf. Decis. Control, Atlanta, USA, 2010, pp. 5656–5661, http://dx.doi.org/10.1109/CDC.2010.5717685.

[21] A. Tanwani, S. Trenn, Observability of switched differential-algebraic equations for general switching signals, in: Proc. 51st IEEE Conf. Decis. Control, Maui, USA, 2012, pp. 2648–2653, http://dx.doi.org/10.1109/CDC.2012.6427087.

[22] A. Tanwani, S. Trenn, On detectability of switched linear differential-algebraic equations, in: Proc. 54th IEEE Conf. Decis. Control, Osaka, Japan, 2015, pp. 2957–2962, http://dx.doi.org/10.1109/CDC.2015.7402666.

[23] R. Penrose, A generalized inverse for matrices, Proc. Camb. Phil. Soc. 51 (1955) 406–413, http://dx.doi.org/10.1017/S0305004100030401.

[24] L. Loi, N. Du, P. Anh, On linear implicit non-autonomous systems of difference equations, J. Difference Equ. Appl. 8 (12) (2002) 1085–1105, http://dx.doi.org/10.1080/1023619021000053962.

[25] P.K. Anh, H.T.N. Yen, Floquet theorem for linear implicit nonautonomous difference systems, J. Math. Anal. Appl. 321 (2) (2006) 921–929, http://dx.doi.org/10.1016/j.jmaa.2005.08.075.

[26] P. Ky Anh, N. Huu Du, L. Le Cong, Singular difference equations: An overview, Vietnam J. Math. 35 (2007) 339–372.

[27] T. Berger, A. Ilchmann, S. Trenn, The quasi-weierstraß form for regular matrix pencils, Linear Algebra Appl. 436 (10) (2012) 4052–4069, http://dx.doi.org/10.1016/j.laa.2009.12.036.

[28] A. Ben-Israel, T. Greville, Generalized Inverse, Springer-Verlag, New York, 2003, http://dx.doi.org/10.1007/b97366.

[29] G.H. Golub, C.F. Van Loan, Matrix Computations, Fourth, in: Johns Hopkins Studies in the Mathematical Sciences, The Johns Hopkins University Press, Baltimore, 2013.

[30] E. Griepentrog, R. März, Differential-Algebraic Equations and their Numerical Treatment, in: Teubner-Texte zur Mathematik, (88) Teubner, Leipzig, 1986.

[31] D.G. Luenberger, Time-invariant descriptor systems, Automatica 14 (1978) 473–480, http://dx.doi.org/10.1016/0005-1098(78)90006-7.

[32] L. Dai, Singular Control Systems, in: Lecture Notes in Control and Information Sciences, (118) Springer-Verlag, Berlin, 1989, http://dx.doi.org/10.1007/BFb0002475.

[33] A. Tanwani, H. Shim, D. Liberzon, Observability for switched linear systems: Characterization and observer design, IEEE Trans. Autom. Control 58 (4) (2013) 891–904, http://dx.doi.org/10.1109/TAC.2012.2224257.