Nonlinear singular switched systems in discrete-time: solution theory and incremental stability under restricted switching signals

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***Abstract*— In this article the solvability analysis of discrete-time nonlinear singular switched systems with restricted switching signals is studied. We provide necessary and sufficient conditions for the solvability analysis under fixed switching signals and fixed mode sequences. The so-called surrogate systems (ordinary systems that have the equivalent behavior to the original switched systems) are introduced for solvable switched systems. Incremental stability, which ensures that all solution trajectories converge with each other, is then studied by utilizing these surrogate systems. Sufficient (and necessary) conditions are provided for this stability analysis using single and switched Lyapunov function approaches.**

I. Introduction

Singular systems can describe the behavior of many physical systems such as electrical circuits [1], [2], industrial processes [3], constrained mechanical systems [4], robotics [5], [6], economic systems [7], and neural networks [8], among others. In literature, this system class has appeared under many different names, e.g. systems with algebraic constraints [9], descriptor systems [10], semi-state systems [11], implicit systems [12], or difference-algebraic equations [13], [14], [15].

In the present letter, we consider a class of switched systems where each mode is a discrete-time nonlinear singular system of the form

$$E_{\sigma(k)}x(k+1) = F_{\sigma(k)}(x(k)) \tag{1}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $\sigma : \mathbb{Z}_{\geq 0} \to \{0, 1, 2, ..., \mathtt{p}\} =: \mathtt{P}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k; for each $i \in \mathtt{P}$, $F_i : \mathbb{R}^n \to \mathbb{R}^n$ is some nonlinearity and $E_i \in \mathbb{R}^{n\times n}$ is singular. We refer to the pair (E_i, F_i) as mode i.

We consider switching signals of the following form (see also Fig. 1)

$$\sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s),\ j \in \{0, 1, 2, ...\}, \tag{2}$$

where $k_j^s \in \mathbb{Z}_{\geq 0}$ denotes the switching time with $k_0^s = 0$ and $\sigma_j \in \{0, 1, ..., \mathtt{p}\}$. In this study, we assume that we either know the switching signal completely (we know the switching times $(k_j^s)_{j\in\mathbb{N}}$ *and* the mode sequence $(\sigma_j)_{j\in\mathbb{N}}$) or just the mode sequence $(\sigma_j)_{j\in\mathbb{N}}$ (the switching times are unknown/arbitrary); the former is called a fixed switching signal whereas the latter is called a fixed mode sequence. In contrast to considering arbitrary switching signals, we therefore only consider a subclass of switching signals or, in other words, restricted switching signals.

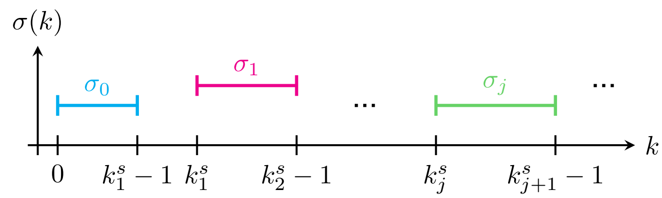

Fig. 1: Mode sequence (2)

If E_i are nonsingular i.e. system (1) being an ordinary system, no further assumption on F_i is needed for ensuring solvability with any initial value. However, if E_i for some i is singular, the unique solvability is not guaranteed; this is a well-known challenge in nonlinear singular systems, see e.g. [16] and [17, Example 1.1]. Studies about the solvability of system (1) have been extensively carried out; however, in existing studies, the system has a linear term, and the nonlinear term was considered only as a disturbance-like term, and it is assumed that the solvability theory for linear systems can still be applied (see e.g. [18], [19]). A preliminary study with F_i considered as part of the state's dynamics is available in [17] under a strong assumption of solvability for arbitrary switching signal. However, the solvability condition under arbitrary switching signals is not necessary for a particular (fixed) switching signal which has already arisen in singular linear switched systems [20]. We fill this gap in this paper by studying the solvability of system (1) under restricted switching signals, which is our first main contribution.

As our second main contribution we present the incremental stability analysis of (1); this stability notion has recently been studied in the literature that extends the classical notion of stability of an equilibrium point. Generally speaking, this stability notion is related to the asymptotic convergence behavior of the solutions to each other or to a particular steady-state trajectory [21]. In literature, incremental stability has been extensively studied in both continuous and discrete-time domains, see e.g. [22], [23], [24], [25], [26] and references therein. However, the existing studies that deal with systems in the discrete-time domain lack incremental stability analysis for singular nonlinear (switched) systems.

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As a summary, solvability theory for system (1) is studied in this paper under restricted switching signals. Two types of restricted switching signals are considered: fixed switching signals and fixed mode sequences. For solvable systems, we are able to establish surrogate systems, i.e. ordinary systems that have equivalent behavior to the original singular systems. Moreover, by utilizing these surrogate systems, stability analysis is studied in terms of incremental stability.

II. Preliminaries

A. (Nonswitched) Nonlinear Singular Systems

To make the paper self-contained, we revisit in this section the solution theory of the nonswitched case of system (1), i.e. of the (nonswitched) Nonlinear Singular System (NSS)

$$Ex(k+1) = F(x(k)),\ k = 0, 1, ... \tag{3}$$

where E is singular with $\operatorname{rank} E = r < n$. We will make the following assumption for (3) (and later also for each mode of (1)):

Assumption 2.1: The set $\mathcal{S} := \{x \in \mathbb{R}^n : F(x) \in \operatorname{im} E\}$ of (3) is a (linear) subspace in $\mathbb{R}^n$.

We call (3) *locally uniquely solvable* (for short just *solvable*) if, for all $k \in \mathbb{Z}_{\geq 0}$ and for all $x_0 \in \mathcal{S}$ there exists a unique solution on $[0, k]$ of (3) considered on $[0, k]$ with $x(0) = x_0$. It has been pointed out in [17] that system (3) is solvable if, and only if $\mathcal{T} \subseteq \mathcal{S} \oplus \ker E$ where $\mathcal{T} = \{\ E^+F(\varsigma)\ |\ \varsigma \in \mathcal{S}\ \}$ and E^+ is a generalized inverse[1] of E. Furthermore, if solvable, its solution at any $k \in \mathbb{Z}_{\geq 0} = \{0, 1, ...\}$ with $x(0) = x_0 \in \mathcal{S}$ satisfies

$$x(k+1) = \Pi_{\mathcal{S}}^{\ker E} E^+ F(x(k)) =: \Phi(x(k)) \tag{4}$$

where $\Pi_{\mathcal{S}}^{\ker E}$ is the canonical projector from $\mathcal{S} \oplus \ker E$ to $\mathcal{S}$ (see e.g. [28] for technical details). In other words, any solution of (4) with $x(0) \in \mathcal{S}$ also solves (3). We call (4) the *surrogate (ordinary) system* of (3).

Remark 2.2: The set $\mathcal{S}$ is a positively invariant set of the solvable NSSs (3), i.e. $x(k) \in \mathcal{S}$ for all $k \in \mathbb{Z}_{\geq 0}$. This is a direct consequence of the solvability in which a solution at any time instant k, $x(k)$, satisfies $E\xi = F(x(k))$ for some $\xi \in \mathbb{R}^n$, i.e. $x(k) \in \mathcal{S}$.

By utilizing the surrogate ordinary system (4), we study the incremental stability of the NSS (3) in the following subsection.

B. Incremental Stability

Let $x(k; x_0)$ denote the solution of (3) via (4) at time instant $k \in \mathbb{Z}_{\geq 0}$ with the initial condition $x(0) = x_0 \in \mathcal{S}$. Throughout the paper, we use the standard notations for function classes[2] $\mathcal{K}, \mathcal{L}, \mathcal{K}_\infty$, and $\mathcal{KL}$, see e.g. [29]. Moreover, the norm $||\cdot||$ stands for the standard Euclidean norm, and $\mathbb{R}_{\geq 0}$ denotes the set of all nonnegative real numbers.

Definition 2.3 (c.f. Def. 1 in [21]): The NSS (3) is called *asymptotically incrementally stable* on a positively invariant $\mathcal{X} \subseteq \mathcal{S} \subsetneq \mathbb{R}^n$ if there exists $\beta \in \mathcal{KL}$ such that

$$||x(k; x_0') - x(k; x_0'')|| \leq \beta(||x_0' - x_0''||, k) \tag{5}$$

for all $x_0', x_0'' \in \mathcal{X}$, $k \in \mathbb{Z}_{\geq 0}$ and where $x(k; x_0)$ denotes the solution of (3) with initial values x_0.

Compared to existing definitions such as [21, Definition 1], which is also defined globally on $\mathbb{R}^n$, our definition above is only considered on the subspace $\mathcal{S}$ which is a strict subspace of $\mathbb{R}^n$ for any solvable NSS (3) with singular E (because $\mathcal{S} \cap \ker E = \{0\}$ is necessary for solvability). The following proposition provides a sufficient and necessary condition for incremental stability, which is inspired by the condition for ordinary nonlinear systems.

Proposition 2.4: Consider the NSS (3) under Assumption 2.1 via its surrogate system (4). Then, this system is *asymptotically incrementally stable* on $\mathcal{S}$ if, and only if, there exist a continuous function $V : \mathcal{S} \times \mathcal{S} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that

$$\alpha_1(||x' - x''||) \leq V(x', x'') \leq \alpha_2(||x' - x''||), \tag{6a}$$

$$V(\Phi(x'), \Phi(x'')) - V(x', x'') \leq -\alpha_3(||x' - x''||) \tag{6b}$$

hold for all $x', x'' \in \mathcal{S}$.

Proof: The proof is similar to the proof of incremental stability analysis for ordinary systems in [21, Theorem 5] by considering the *augmented system*

$$z(k+1) = \widetilde{\Phi}(z(k)) \tag{7}$$

with $z = \left[\begin{smallmatrix} x' \\ x'' \end{smallmatrix}\right] \in \mathcal{S} \times \mathcal{S}$ and $\widetilde{\Phi} = \left[\begin{smallmatrix} \Phi(x'(k)) \\ \Phi(x''(k)) \end{smallmatrix}\right]$ where Φ is as in (4) and then stability of (7) w.r.t. the diagonal set $\Delta := \{\ \left[\begin{smallmatrix} x \\ x \end{smallmatrix}\right] \in \mathcal{S} \times \mathcal{S}\ |\ x \in \mathcal{S}\ \}$ is shown. The complete proof is omitted due to length limitations. ■

Such function V satisfying (6) is called an incremental Lyapunov function for (3). A similar idea will also be applied to the forthcoming incremental stability analysis for switched systems.

III. Solution Theory

Recall the NSSS (1). Define for each mode $i \in \{0, 1, ..., \mathtt{p}\}$ the set $\mathcal{S}_i := \{x \in \mathbb{R}^n : F_i(x) \in \operatorname{im} E_i\}$. The solvability notion used in this study is described as follows.

Definition 3.1 (c.f. Def. 4.2 in [17]): The NSSS (1) is called *locally uniquely solvable* (for short just *solvable*) w.r.t. a fixed and known switching signal σ of the form (2) if, for all $k_0, k_1 \in \mathbb{Z}_{\geq 0}$, $k_1 > k_0$ and for all $x_{k_0} \in \mathcal{S}_{\sigma(k_0)}$ there exists a unique solution of (1) under σ considered on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$.

For the given switching signal, this solvability notion requires the existence of a unique solution considered on any time interval with any arbitrary initial time instant and, furthermore, for any consistent initial value at that initial time instant. The solvability characterization for system (1) with respect to the solvability notion above is studied under the following assumption on the set $\mathcal{S}_i$:

[1] A generalized inverse of $M \in \mathbb{R}^{m \times n}$ is a matrix $M^+ \in \mathbb{R}^{n \times m}$ that satisfies $MM^+M = M$ [27].

[2] A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{K}$ if it is continuous, zero at zero, and strictly increasing. If it is also unbounded, then α belongs to class-$\mathcal{K}_\infty$. Meanwhile, a function $\beta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{L}$ if it is continuous, strictly decreasing, and $\lim_{t\to\infty} \beta(t) = 0$. A function $\gamma : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ belongs to class-$\mathcal{KL}$ if it belongs to class-$\mathcal{K}$ in its first argument and class-$\mathcal{L}$ in its second argument.

Assumption 3.2: For each $i \in \{0, 1, ..., \mathtt{p}\}$, $\mathcal{S}_i$ is a linear subspace in $\mathbb{R}^n$.

Two types of restricted switching signals are considered in this study. The first one is the case with a fixed and known mode sequence denoted by $(\sigma_0, \sigma_1, ...)$. This case covers any switching signals that have the given mode sequence with arbitrary switching times. The second one is the case of a fixed and known switching signal σ where both its mode sequence and switching times are fixed and known. Based on the solvability notion in Definition 3.1, the NSSS (1) is solvable w.r.t. $(\sigma_0, \sigma_1, ...)$ if it is solvable w.r.t. all switching signals with the mode sequence $(\sigma_0, \sigma_1, ...)$ and with arbitrary switching times. We first present the solvability condition in the following theorem for system (1) under a fixed and known switching signal.

Theorem 3.3: The NSSS (1) under Assumption 3.2 is solvable w.r.t. a fixed and known switching signal $\sigma : \mathbb{Z}_{\geq 0} \to \{0, 1, ..., \mathtt{p}\}$ in the sense of Definition 3.1 if, and only if,

$$\mathcal{T}_{\sigma(k)} \subseteq \ker E_{\sigma(k)} \oplus \mathcal{S}_{\sigma(k+1)} \quad \text{for } k = 0, 1, 2, \dots; \tag{8}$$

where $\mathcal{T}_i = \{E_i^+ F_i(\varsigma) | \varsigma \in \mathcal{S}_i\}$. Furthermore, if it is solvable, its solution at any time instant $k \in \mathbb{Z}_{\geq 0}$ satisfies the following surrogate ordinary system

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}(x(k)), \ x(0) \in \mathcal{S}_{\sigma(0)} \tag{9}$$

where $\Phi_{i,j}$ is called the one-step map from mode-j to mode-i given by

$$\Phi_{i,j}(x(k)) := \Pi_{\mathcal{S}_i}^{\ker E_j} E_j^+ F_j(x(k)), \tag{10}$$

the matrix E_j^+ is a generalized inverse of E_j and $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is the canonical projector from $\mathcal{S}_i \oplus \ker E_j$ to $\mathcal{S}_i$. //

Proof: Due to Assumption 3.2, the proof is similar to the proof of the solvability under arbitrary switching signals in [17]. Therefore, the complete proof is omitted here, however, in the following discussion, we highlight the main idea of the proof with a fixed and known switching signal to provide some additional insights.

On every time interval $[k, k+1]$ with a given $x(k)$, the solution $x(k+1)$ must satisfy

$$\begin{cases} E_{\sigma(k)} x(k+1) = F_{\sigma(k)}(x(k)) \\ \quad E_{\sigma(k+1)} \xi = F_{\sigma(k+1)}(x(k+1)) \end{cases}$$

for some $\xi \in \mathbb{R}^n$, which is equivalent to

$$\begin{cases} x(k+1) \in \{E_{\sigma(k)}^+ F_{\sigma(k)}(x(k))\} + \ker E_{\sigma(k)} \\ x(k+1) \in \mathcal{S}_{\sigma(k+1)} \end{cases}$$

(by the preimage formula[3]). The solvability condition (8) is necessary and sufficient by applying the projector lemma in [17, Lemma 2.3][4] with $\mathcal{U} = \mathcal{T}_{\sigma(k)}$, $\mathcal{V} = \mathcal{S}_{\sigma(k+1)}$ and $\mathcal{W} = \ker E_{\sigma(k)}$, from which also the surrogate system (9) follows. ■

Remark 3.4: One may wonder how really necessary the solvability notion from Definition 3.1 is for switched systems compared to nonswitched systems. Note that solvability for individual modes is not always sufficient nor necessary for particular switching signals and, furthermore, solvability is dependent upon the mode sequence; this already happens in linear systems, see [20], the justification for nonlinear systems is similar and thus omitted. Furthermore, it is not always possible to derive surrogate systems (9) without the solvability notion in Definition 3.1, see Remark 3.8 in [17] for a counter-example.

Now, by applying the solvability condition (8) to all switching signals that belong to a given mode sequence, we are able to establish a necessary and sufficient condition for solvability under a fixed mode sequence.

Proposition 3.5: The NSSS (1) under Assumption 3.2 is solvable w.r.t. the fixed and known mode sequence $(\sigma_0, \sigma_1, \dots)$ if, and only if,

$$\mathcal{T}_i \subseteq \mathcal{S}_i \oplus \ker E_i \quad \text{for all } i \in \{0, 1, \dots, \mathtt{p}\} \tag{12a}$$

$$\mathcal{T}_{\sigma_j} \subseteq \mathcal{S}_{\sigma_{j+1}} \oplus \ker E_{\sigma_j} \quad \text{for } j = 0, 1, 2, \dots. \tag{12b}$$

Furthermore, if solvable, the surrogate ordinary system (9) is valid for every switching signal with the given mode sequence.

Proof: The sufficiency is obvious since (12) implies that (8) is satisfied by all switching signals with the given mode sequence. For the necessity, solvability w.r.t. the given mode sequence $(\sigma_0, \sigma_1, \dots)$ implies solvability w.r.t. any arbitrary switching signal with the given mode sequence. Thus, for all $k, i \in \mathbb{Z}_{\geq 0}$ and all switching signals with $\sigma(k) = \sigma(k+1) = \sigma_i$, $\mathcal{T}_i \subseteq \mathcal{S}_i \oplus \ker E_i$. Furthermore, at all switches from mode σ_j to σ_{j+1}, the condition (8) is also satisfied, which implies $\mathcal{T}_{\sigma_j} \subseteq \mathcal{S}_{\sigma_{j+1}} \oplus \ker E_{\sigma_j}$ for $j = 0, 1, ...$. ■

We close this section with the following remark which reveals that in fact, the set $\mathcal{S}_{\sigma(k)}$ is a positively invariant set of the mode that is active at time instant k.

Remark 3.6: From the proof of Theorem 3.3, for any time instant $k \in \mathbb{Z}_{\geq 0}$, $x(k) \in \mathcal{S}_{\sigma(k)}$. This is a direct consequence of the solution $x(k)$ satisfying $E_{\sigma(k)} \xi = F_{\sigma(k)}(x(k))$ for some $\xi \in \mathbb{R}^n$.

IV. Incremental Stability

Consider the solvable NSSS (1) which can be analyzed via its surrogate ordinary switched system (9). In this case, the surrogate ordinary switched system (9) can be seen as a time-varying system, where incremental stability characterization and contraction analysis for time-varying (ordinary) systems can be applicable [23]. However, the existing conditions are required to be checked for every time step, which is not necessary for (9) since for some time intervals, the system stays at a certain mode, and thus it can be characterized by mode-wise approach. Furthermore, in the existing studies for time-varying systems, the characterizations for incremental

[3] For any matrix $M \in \mathbb{R}^{n \times n}$ and $x \in \operatorname{im} M$, $M^{-1}\{x\} = \{M^+ x\} + \ker M$ where M^+ is a generalized inverse of M and $M^{-1}\mathcal{X}$ is the preimage of $M \in \mathbb{R}^{n \times n}$ over a set $\mathcal{X}$ i.e. $M^{-1}\mathcal{X} = \{\xi \in \mathbb{R}^n : M\xi \in \mathcal{X}\}$.

[4] Consider a set $\mathcal{U} \subseteq \mathbb{R}^n$ and two subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, then $\mathcal{V} \cap (\{u\} + \mathcal{W})$ is a singleton for all $u \in \mathcal{U}$ if, and only if, $\mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$. In that case

$$\mathcal{V} \cap (\{u\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} u\}, \tag{11}$$

where $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} \oplus \mathcal{W} \to \mathcal{V}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$.

stability were considered in a positively invariant set $\mathbb{X}$, which also serves as the consistency set of the system that is defined globally, i.e. on $[k_0, \infty)$. We note in general that the consistency set of NSSS (1) which corresponds to each mode may be different i.e. it is not necessary to have $\mathcal{S}_i = \mathcal{S}_j,\ i \neq j$. Therefore, in this study, we define new incremental stability notions with respect to a time-dependent set.

A. Single Lyapunov Function Approach

Consider the time-dependent set $\widehat{\mathcal{S}}(k)$ defined by $\widehat{\mathcal{S}} : \mathbb{Z}_{\geq 0} \to \{\mathcal{S}_0, \mathcal{S}_1, ..., \mathcal{S}_{\mathrm{p}}\}$ with $\widehat{\mathcal{S}}(k) = \mathcal{S}_{\sigma(k)}$. Following Definition 1 in [30] for a time-dependent positively invariant set w.r.t. a dynamical system, by Remark 3.6, the time-dependent set $\widehat{\mathcal{S}}(k)$ is a time-dependent positively invariant set w.r.t. system (1). Throughout the rest of the paper, $||x(k)||_{\mathcal{X}(k)}$ denotes the distance of vector $x(k)$ to set $\mathcal{X}(k)$ defined by $||x(k)||_{\mathcal{X}(k)} = \inf_{\xi \in \mathcal{X}(k)} ||x(k) - \xi||$.

Definition 4.1 (c.f. Def. 1 in [23]): The NSSS (1) is called *asymptotically incrementally stable* w.r.t. a fixed switching signal σ on a time-dependent positively invariant set $\mathcal{X}(k)$ if there exists $\beta \in \mathcal{KL}$ such that

$$||x'(k; x_0') - x''(k; x_0'')|| \leq \beta(||x_0' - x_0''||, k) \tag{13}$$

for all $x_0', x_0'' \in \mathcal{X}(0)$ and all $k \in \mathbb{Z}_{\geq 0}$.

Compared to Definition 1 in [23] which is defined on a constant positive invariant set and is defined also globally on $\mathbb{R}^n$, our Definition 4.1 is defined on a time-dependent positive invariant set, and furthermore, it cannot be defined globally on $\mathbb{R}^n$ since $\mathcal{S}_i \subsetneq \mathbb{R}^n$ for all i. Moreover, the incremental stability notion above is defined nonuniformly w.r.t. time since we are only interested in initial conditions $x(0) = x_0 \in \mathcal{S}_{\sigma(0)}$.

Lemma 4.2 (Single Lyapunov function): Consider the NSSS (1) under Assumption 3.2 via its surrogate switched system (9). This system is *asymptotically incrementally stable* w.r.t. a fixed and known switching signal σ of the form (2) on the time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$ if, and only if, there exist a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for $k = 0, 1, ...$

$$\begin{aligned} \alpha_1(||x'(k) - x''(k)||) &\leq V(x'(k), x''(k)) \\ &\leq \alpha_2(||x'(k) - x''(k)||), \end{aligned} \tag{14a}$$

$$\begin{aligned} V(x'(k+1), x''(k+1)) &- V(x'(k), x''(k)) \\ &\leq -\alpha_3(||x'(k) - x''(k)||) \end{aligned} \tag{14b}$$

hold for all solutions x' and x'' of (1) with the given switching signal σ.

Proof: The proof is similar to the proof for time-varying systems in [23, Theorem 9] by considering the switched *augmented system*

$$z(k+1) = \widetilde{\Phi}_k(z(k)) \tag{15}$$

with $z(k) = \begin{bmatrix} x(k)' \\ x(k)'' \end{bmatrix} \in \mathcal{S}_{\sigma(k)} \times \mathcal{S}_{\sigma(k)}$ and $\widetilde{\Phi}_k = \begin{bmatrix} \Phi_{\sigma(k+1),\sigma(k)}(x(k)') \\ \Phi_{\sigma(k+1),\sigma(k)}(x(k)'') \end{bmatrix}$ where $\Phi_{i,j}$ is the one-step map as in (10). The claim then follows from showing that the switched diagonal set $\Delta(k) := \left\{ \begin{bmatrix} x \\ x \end{bmatrix} \in \mathcal{S}_{\sigma(k)} \times \mathcal{S}_{\sigma(k)} \;\middle|\; x \in \mathcal{S}_{\sigma(k)} \right\}$ is asymptotically stable for (15). Similar to [31, Theorem 1] and [32, Chapter 5], this stability is shown via the existence of a Lyapunov function $W : \mathbb{R}^{2n} \to \mathbb{R}_{\geq 0}$ and functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for $k = 0, 1, ...$

$$\begin{aligned} \alpha_1(||z(k; x_0)||_{\Delta(k)}) &\leq W(z(k; x_0)) \leq \alpha_2(||x(z; x_0)||_{\Delta(k)}), \\ W(\Phi_{\sigma(k+1),\sigma(k)}&(z(k; x_0))) - W(z(k; x_0)) \\ &\leq -\alpha_3(||z(k; x_0)||_{\Delta(k)}), \end{aligned}$$

for all $x_0 \in \mathcal{S}_{\sigma(0)}$. We omit details due to space limitations. ■

A function V that satisfies Lemma 4.2 is called an incremental Lyapunov function. Note that while Lemma 4.2 provided a *characterization* (i.e. necessary *and* sufficient) of incremental stability in terms of a Lyapunov function, one may argue that the condition (14) is not practical since it needs to be checked for all explicit solutions. Therefore, we provide a sufficient condition in the following corollary, which is more convenient to check by utilizing the one-step map introduced in Theorem 3.3, in particular, it doesn't require knowledge of the solutions.

Corollary 4.3: Consider the NSSS (1) under Assumption 3.2 with a fixed and known switching signal σ of the form (2) via its surrogate switched system (9) and a time-dependent positively invariant set $\widehat{\mathcal{S}}(k)$. If there exist a continuous function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, and $\alpha_3 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for $k = 0, 1, ...$

$$\alpha_1(||x' - x''||) \leq V(x', x'') \leq \alpha_2(||x' - x''||), \tag{16a}$$

$$\begin{aligned} V(\Phi_{\sigma(k+1),\sigma(k)}(x'), \Phi_{\sigma(k+1),\sigma(k)}(x'')) - V(x', x'') \\ \leq -\alpha_3(||x' - x''||) \end{aligned} \tag{16b}$$

hold for all $x', x'' \in \widehat{\mathcal{S}}(k)$ then this system is *asymptotically incrementally stable* w.r.t. σ on $\widehat{\mathcal{S}}(k)$.

B. Switched Lyapunov Function Approach

The conditions in Lemma 4.2 and Corollary 4.3 require a single incremental Lyapunov function. If every mode as an individual (non-switched) system is asymptotically incrementally stable on its consistency space, then we can utilize the corresponding incremental Lyapunov functions of all modes to formulate a switched incremental Lyapunov function for the switched system composed of those modes. This is provided in the following theorem.

Theorem 4.4 (Switched Lyapunov function approach): Consider the solvable NSSS (1) under Assumption 3.2. Assume each mode i is asymptotically incrementally stable on $\mathcal{S}_i$ with the corresponding incremental Lyapunov function $V_i : \mathcal{S}_i \times \mathcal{S}_i \to \mathbb{R}_{\geq 0}$ and class-$\mathcal{K}_\infty$ functions α_{1i}, α_{2i} and α_{3i} satisfying Proposition 2.4. If the following two conditions hold: (1) For all $x, x', x'' \in \mathcal{S}_i \cap \mathcal{S}_j$ with $||x'|| = ||x''||$ and all $i, j \in \{0, 1, ..., \mathtt{p}\}$:

$$V_i(x', x'') = V_j(x', x'') \tag{17a}$$

$$\alpha_i(||x||) = \alpha_{1j}(||x||),\ \alpha_{2i}(||x||) = \alpha_{2j}(||x||),\\ \alpha_{3i}(||x||) = \alpha_{3j}(||x||) \tag{17b}$$

and (2) for $k = 0, 1, \ldots$:

$$V_{\sigma(k+1)}(\Phi_{\sigma(k+1),\sigma(k)}(x'), \Phi_{\sigma(k+1),\sigma(k)}(x''))\\ -V_{\sigma(k)}(x', x'') \le -\alpha_3(||x' - x''||)\ \forall x', x'' \in \mathcal{S}_{\sigma(k)} \tag{18}$$

with $\alpha_3 : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0},\ \alpha_3(||x||) = \alpha_{3i}(||x||)$ if $x \in \mathcal{S}_i$ and 0 otherwise, then system (1) is *asymptotically incrementally stable* w.r.t. the given fixed and known switching signal σ on $\widehat{\mathcal{S}}(k)$.

Proof: For the given switching signal, we construct the following incremental (switched) Lyapunov function from the incremental Lyapunov functions of individual modes as follows:

$$V : \mathbb{R}^{2n} \to \mathbb{R},\ V(x_1, x_2) = \begin{cases} V_i(x_1, x_2) & \text{if } x_1, x_2 \in \mathcal{S}_i \\ 0 & \text{otherwise.} \end{cases}$$

Condition (17a) ensures that V is well defined for all $x_1, x_2 \in \mathbb{R}^n$, i.e. it guarantees that $V(x)$ is unique for every $x_1, x_2 \in \mathbb{R}^n$. From the functions $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}$ of all individual modes, we also construct for the switched system the corresponding functions $\alpha_\ell : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0},\ \ell = 1, 2, 3$ defined by

$$\alpha_\ell(||x||) = \begin{cases} \alpha_{\ell i}(||x||) & \text{if } x \in \mathcal{S}_i \\ 0 & \text{otherwise.} \end{cases}$$

Since $\mathcal{S}_i$ are subspaces, $\{0\} \in \mathcal{S}_i$ and $\mathcal{S}_i \cap \mathcal{S}_j \supseteq \{0\}$. Thus, $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ under conditions (17b). Since each V_i and $\alpha_{\ell i},\ \ell = 1, 2, 3$ satisfy (6), the functions V and α_ℓ defined above satisfy

$$\alpha_1(||x' - x''||) \le V(x', x'') \le \alpha_2(||x' - x''||).$$

and together with condition (18) implies the incremental stability on $\widehat{\mathcal{S}}(k)$ w.r.t. the given switching signal σ. ■

Such a piecewise function V in the proof above is called a switched Lyapunov function; the term comes from the fact that V switches over the individual Lyapunov functions depending on in which consistency space the solution is in at a given time. Compared to the single Lyapunov function approach presented in Lemma 4.2, the switched Lyapunov function approach is simpler in terms of finding the Lyapunov function since it is formulated from the Lyapunov functions of the individual modes. However, stability for each mode is required here; this assumption is not required in the single Lyapunov function approach, i.e. the switched system may contain unstable modes. Nevertheless, the single Lyapunov approach requires a Lyapunov function that fits the whole switched system, which is intuitively more difficult to find.

Theorem 4.4 can be easily extended to characterize the incremental stability of the NSSS (1) with respect to a fixed and known mode sequence in which the switching times are arbitrary, or with respect to arbitrary switching signals (both mode sequence and switching times are arbitrary) by considering the condition in Theorem 4.4 to be satisfied by the involved switching signals. In particular, Lyapunov function construction methods in ordinary systems can be utilized, such as Yoshizawa method [33], least square optimization approach [34], collocation approaches [35], [36] and linear programming approach [37]. We close this part by providing an example illustrating the derived theoretical results.

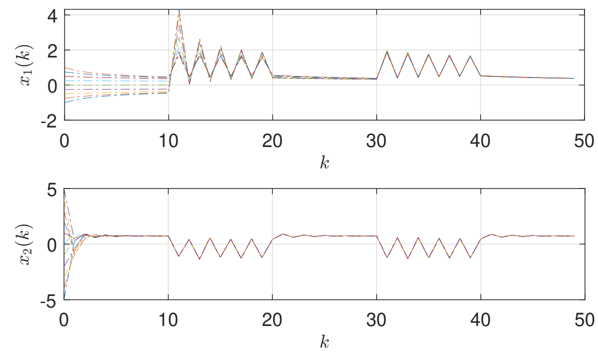

Fig. 2: Solution trajectories of the switched system in Example 4.5

Example 4.5: Consider system (1) composed of the following two modes:

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \sin(x_1) + \cos(x_2) \\ \sin(x_1) - \cos(x_2)x_2 + x_3^{\frac{1}{3}} \\ x_3^{\frac{1}{3}} \end{bmatrix} \right),$$

$$(E_1, F_1(x)) = \left(\begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} e^{1-x_1} + x_3^{\frac{1}{3}} \sin(x_2) \\ e^{1-x_1} - x_3^{\frac{1}{3}} \sin(x_2)x_2 + x_3^{\frac{1}{3}} e^{1-x_1} \\ x_3^{\frac{1}{3}} e^{1-x_1} \end{bmatrix} \right).$$

Geometric computations provide $\ker E_0 = \ker E_1 = \operatorname{span}\{(0, 0, 1)^\top\}$, $\mathcal{S}_0 = \mathcal{S}_1 = \operatorname{span}\{(1, 0, 0)^\top, (0, 1, 0)^\top\}$. Since $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n,\ \forall i, j \in \{0, 1\}$, the condition (8) holds independently of $\mathcal{T}_{\sigma(k)}$ i.e. the switched system is solvable w.r.t. any arbitrary switching signal (each mode as an individual system is also solvable). Choosing $E_0^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $E_1^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and with $\Pi_{\mathcal{S}_1}^{\ker E_0} = \Pi_{\mathcal{S}_0}^{\ker E_1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ provide

$$\Phi_0(x(k)) = \Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{bmatrix} \sin(x_1) \\ \cos(x_2) \\ 0 \end{bmatrix}$$

and

$$\Phi_1(x(k)) = \Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k))\\ = \begin{bmatrix} (1 + \frac{1}{2}x_3^{\frac{1}{3}})e^{1-x_1} \\ (-1 - x_3^{\frac{1}{3}})e^{1-x_1} + x_3^{\frac{1}{3}} \sin(x_2)x_2 \\ 0 \end{bmatrix}.$$

As an individual system, each mode is incrementally stable by considering the functions $\alpha_1(\xi) = \xi^2$, $\alpha_2(\xi) = \xi^2$, $\alpha_2(\xi) = 0$ for both modes and simple incremental Lyapunov function $V_i(x) = x_1^2 + x_2^2 + x_3^2,\ i = 0, 1$. Now, by considering the switched incremental Lyapunov function as in the proof of Theorem 4.4, the switched system is incrementally stable w.r.t. any switching signal. The trajectories of the solutions for x_1 under the periodic switching signal $\sigma(k) = 0$ if $k \in [0, 10) \cup [20, 30) \cup \cdots$ and $\sigma(k) = 1$ if $k \in [10, 20) \cup [30, 40) \cup \cdots$ is shown in Fig. 2, which illustrates incrementally stable trajectories (trajectories of x_3 are not shown since its solution is $x_3(k) = 0$ for $k = 1, 2, ...$ and therefore not exciting).

V. SUMMARY

A knowledge gap in the solvability of discrete-time nonlinear singular switched systems with fixed switching signals has been addressed in this paper. Surrogate ordinary systems, which have equivalent behaviors to the original singular systems, have been established by utilizing the one-step map from the current mode to the successive mode. Via these surrogate systems, we are able to establish sufficient (and necessary) conditions for incremental stability using single and switched Lyapunov function approaches.

The results with a fixed mode sequence but arbitrary switching times can also be used for switched systems with the switching rule triggered also by events as long as the mode sequence is known; if the mode sequence is not known, one may refer to studies with arbitrary switching signals such as [17]. Therefore, one possible extension for future work is designing state-feedback control algorithms for switched systems including ones with event-triggered switching rules. Other possible extensions could be observer designs and further studies for systems with inputs.

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