Averaging for switched impulsive systems with pulse width modulation ⋆

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Abstract

Linear switched impulsive systems (SIS) are characterized by ordinary differential equations as modes dynamics and state jumps at the switching time instants. The presence of possible jumps in the state makes nontrivial the application of classical averaging techniques. In this paper we consider SIS with pulse width modulation (PWM) and we propose an averaged model whose solution approximates the moving average of the SIS solution with an error which decreases with the multiple of the switching period and by decreasing the PWM period. The averaging result requires milder assumptions on the system matrices with respect to those needed by the previous averaging techniques for SIS. The interest of the proposed model is strengthened by the fact that it reduces to the classical averaged model for PWM systems when there are no jumps in the state. The theoretical results are verified through simulations obtained by considering a switched capacitor electrical circuit.

Key words: Averaging, switched impulsive systems, pulse width modulation, differential algebraic equations.

1 Introduction

Switching represents the natural behavior of many systems of practical interest, e.g., mechanical systems [20], electronic circuits [25], piecewise affine systems [9,8,1]. In particular, switched systems with pulse width modulation (PWM) are characterized by a sequence of modes which repeats periodically in time [21]. The "fast" switching behaviour determines oscillations, i.e., the so called *ripple*, of the state variables around a smooth trajectory whose dynamics are typically much slower than the switching period. The main goal of the averaging theory consists of obtaining a smooth model whose solution is able to capture the averaged behaviour of the switched system. The corresponding theoretical objective consists of proving that the error between the solutions of the switched and the averaged systems is of order of the switching period.

Averaging theory has been extensively studied for PWM systems with Lipschitz continuous solutions, see among others [5,19,23,26–28]. Recently a new approach for periodic averaging based on time delays has been proposed for fastly varying system [2,4]. However, the class of solutions considered therein is absolutely continuous. Indeed, the model structure considered therein does not allow the presence of state jumps at the switching time instants. On the other hand, there exist practical PWM systems, such as switched capacitor DC/DC converters, which exhibit state jumps at the switching time instants and they still present a sort of averaging behaviour [18,11]. These circuits can be modeled within the class of linear switched impulsive systems (SIS) where each mode is characterized by a set of linear ordinary differential equations and algebraic constraints which determine the rule of the state jumps at the switching time instants [22]. In this paper we study the application of averaging theory to SIS with PWM.

The presence of state discontinuities makes nontrivial the formal study of switched systems [3] and two aspects are specifically critical for the averaging analysis of SIS. The first issue is related to the fact that the amplitudes of the state discontinuities usually do not reduce by decreasing the switching period. The approach we propose

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for overcoming this obstacle consists of comparing the averaged solution with the moving average of the SIS solution. Another theoretical challenge is due to the dependence of the SIS solution on the matrices which characterize the state jumps which one would then expect should be included in the averaged model too. This dependence introduces several problems for the analysis which requires nontrivial theoretical arguments in order to be solved.

The averaging analysis for switched systems with state jumps is still at its infancy. An averaged model for homogeneous SIS with two modes was presented in [7] where strict algebraic conditions (commutativity) on the matrices characterizing the state jumps and those describing the modes dynamics were required. These conditions are not assumed in the analysis of this paper. The averaging result in [7] was extended to more than two modes in [6], to the non-autonomous case in [12] and to partial averaging in [13], however the corresponding theoretical findings were still based on the algebraic assumptions on the SIS matrices introduced in [7]. The commutativity condition was relaxed in [14] by using conditions on the kernel and the image of the matrices of the modes. However, there exist practical SIS for which these conditions are not satisfied [17,18].

In this paper we propose a continuous-time averaged model for SIS under milder assumptions with respect to those formerly used. The averaging property was conjectured by the authors in [15] without providing any formal proof and by taking inspiration from the application of theoretical findings in [16] applied to discrete-time models. In this paper we provide a formal proof for the averaging result by showing that the error between the solution of the averaged model and the moving average of the solution of the SIS decreases exponentially with the number of switching periods and linearly with respect to the period duration. The proposed averaged model is a generalization of the classical averaged model adopted for PWM systems with Lipschitz solution, in the sense that if there are no state jumps the matrices of the proposed model reduce to those of the classical one. A switched capacitor electrical circuit is considered as a motivating practical example and numerical simulations validate the effectiveness of the proposed model.

The rest of the paper is organized as follows. In Section 2 some preliminary definitions and properties of SIS are recalled. Motivating examples for the proposed analysis are presented in Section 3. Section 4 describes the structure of the proposed averaged model and Section 5 our main theoretical result (all proofs are reported in the Appendix). In Section 6 numerical verification of the theoretical results is proposed. The synthesis in Section 7 summarizes conclusions and future work.

2 Switched impulsive systems

In this section we present some preliminaries on notation, the definition of the class of SIS of interest and a resume of the existing results on averaging for SIS.

2.1 Notation

The following notation is adopted throughout the paper: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ ($\mathbb{R}^+_0$) is the set of positive (nonnegative) real numbers, $\mathbb{R}^n$ is the set of n-dimensional vectors of real numbers, $\mathbb{C}$ is the set of complex numbers, $F \in \mathbb{R}^{m\times n}$ indicate a real matrix with m rows and n columns, $\mathbb{N}_0$ ($\mathbb{N}$) is the set of (positive) natural numbers; $\|\cdot\|$ indicates the Euclidean norm on $\mathbb{R}^n$ and also the corresponding induced matrix norm; $\lfloor x \rfloor$ is the largest integer less or equal than $x \in \mathbb{R}$. A matrix $F \in \mathbb{R}^{n\times n}$ is *idempotent* if $F^k = F$ for any $k \in \mathbb{N}$; it is *Schur* if all its eigenvalues have magnitude smaller than 1. A pair of matrices $F_i, F_j \in \mathbb{R}^{n\times n}$ is *commutative* if $F_iF_j = F_jF_i$ with $i, j \in \mathbb{N}$. The product of q matrices F_i, $i = 1, \ldots, \mathtt{q}$ is defined as (note the order) $\prod_{i=1}^{\mathtt{q}} F_i = F_{\mathtt{q}}F_{\mathtt{q}-1}\cdots F_2F_1$. The following notation is used: $G_i(\xi) = e^{F_i\xi}$ for all $\xi \in \mathbb{R}$ and $G_{i,p} = G_i(d_ip) = e^{F_id_ip}$ for some $d_i \in D = [0, 1)$, $\Sigma = \{1, \ldots \mathtt{q}\}$ with $\mathtt{q} \in \mathbb{N}$. A function $u : \mathbb{R}^+_0 \to \mathbb{R}^n$ is a *Bohl function* if it is a linear combination of terms of the form $t^ke^{\lambda t}$ where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. A matrix function $G_p : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is said to be an $\mathrm{O}(p^r)$ function as $p \to 0$ for any $r \in \mathbb{N}_0$, ($G_p = \mathrm{O}(p^r)$ for short), if there exist constants $\alpha \in \mathbb{R}^+$ and $\bar{p} \in \mathbb{R}^+$ such that $\|G_p\| \leqslant \alpha p^r$ for all $p \in (0, \bar{p}]$.

2.2 SIS with pulse width modulation

The class of SIS considered in our analysis is now introduced. It is characterized by a PWM with $\mathtt{q} \in \mathbb{N}$ modes and a switching period $p \in \mathbb{R}^+$. The sequence of modes is assumed to be fixed. At each $t_k = kp$, $k \in \mathbb{N}_0$, the mode $i = 1$ is activated and it remains active since $t_k + d_1p$ where $d_1 \in D$ is the duty cycle of the first mode. Then the system commutes from the mode $(i-1)$-th to the mode i-th, $i = 2, \ldots, \mathtt{q}$, at the time instants $s_{k,i} := t_k + \sum_{j=1}^{i-1} d_jp$, $k \in \mathbb{N}_0$ where $d_i \in D$, is the duty cycle of the i-th mode; in particular, $\sum_{i=1}^{\mathtt{q}} d_i = 1$, see Fig. 1.

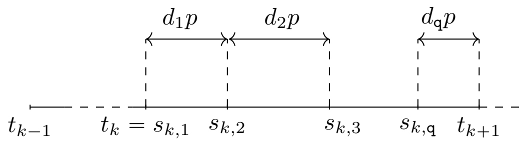

Fig. 1. Illustration of the switching times notation for $k \geqslant 1$.

The continuous-time switched impulsive system can be represented as follows

$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{1a}$$
$$\dot{x}(t) = F_i x(t), \quad t \in (s_{k,i}, s_{k,i+1}) \tag{1b}$$

with $x(0^-) = x_0 \in \mathbb{R}^n$ initial condition, for $k \in \mathbb{N}_0$, $i \in \Sigma$, where $s_{k,q+1} := t_{k+1} = s_{k+1,1}$, the state variable is the same for each mode and $x(s_{k,i}^-)$ $(x(s_{k,i}^+))$ is the state at the end (beginning) of the $(i-1)$-th (i-th) mode at the k-th period. The nonzero flow matrix $F_i \in \mathbb{R}^{n\times n}$, $i \in \Sigma$, characterizes the dynamics of the i-th mode and the jump matrix $\Pi_i \in \mathbb{R}^{n\times n}$, $i \in \Sigma$, (called consistency projector in the differential algebraic equations terminology) determines the possible jumps of the state variables at the switching time instants. Note that in contrast to earlier works, we do not assume that Π_i is a project (i.e. an idempotent matrix). The switched impulsive system (1) includes several practical systems and, among them, switched descriptor systems which can be represented in the form of homogeneous switched differential algebraic equations with regular matrix pairs [16].

The solution of (1) can be written by cascading the solutions of the different modes and by considering the jumps at the switching time instants. In particular, at the switching time instants one can write

$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{2a}$$
$$x(s_{k,i+1}^-) = G_{i,p} x(s_{k,i}^+), \tag{2b}$$

where $G_{i,p} = e^{F_i d_i p}$, for $k \in \mathbb{N}_0$, $i \in \Sigma$. By combining (2), one obtains that the left solution of (1) at the time instants multiple of the switching period must satisfy the following iterative equation

$$x_{k+1}^- = \Theta_p x_k^- \tag{3}$$

for all $k \in \mathbb{N}_0$ where $x_k^- := x(t_k^-)$, $x_0^- = x_0$ and

$$\Theta_p = \prod_{j=1}^{\mathtt{q}} G_{j,p} \Pi_j. \tag{4}$$

By iteratively applying (3), the left solution of (1) at the time instants multiple of p can be written as

$$x_k^- = \Theta_p^k x_0^- \tag{5}$$

for all $k \in \mathbb{N}_0$.

Remark 1 *The model* (1) *is autonomous, however the analysis presented below can be easily applied to the case of non-autonomous systems whose inputs are Bohl functions by extending the state space.*

2.3 Basics on averaging for SIS

Averaging theory has been already studied for SIS. In what follows we briefly recall the existing theoretical results in order to motivate the proposed analysis and to highlight the novelties of our results.

In [7] a SIS model (1) with $\mathtt{q} = 2$ was considered and the following averaged model

$$\dot{\xi}(t) = A_{\mathrm{av}} \xi(t), \quad t \in \mathbb{R}_0^+ \tag{6}$$

with $\xi(0) = \Pi x_0$, $A_{\mathrm{av}} = \Pi(F_1 d_1 + F_2 d_2)\Pi$, $\Pi = \Pi_2 \Pi_1$, was introduced. In particular, it was proved that if the matrices Π_1 and Π_2 are commutative and idempotent, and the conditions

$$\Pi_i F_i = F_i \Pi_i = F_i \tag{7}$$

hold for all $i \in \Sigma$ then for any finite $\bar{t} \in \mathbb{R}^+$ the error between the solution of (1) and that of (6) is decreasing with the same order of the switching period, i.e.

$$x(t) - \xi(t) = \mathrm{O}(p) \tag{8}$$

for all $t \in (0, \bar{t}]$. This result was extended to more than two modes in [6], to the non autonomous case in [12] and to partial averaging in [13].

The commutativity condition was relaxed in [14] by introducing the following conditions on the kernel and the image of the matrices of the system (1):

$$\operatorname{im} \Pi \subseteq \operatorname{im} \Pi_i, \tag{9a}$$
$$\ker \Pi \supseteq \ker \Pi_i, \tag{9b}$$

for all $i \in \Sigma$, where the matrix $\Pi \in \mathbb{R}^{n\times n}$ is given by

$$\Pi = \prod_{i=1}^{\mathtt{q}} \Pi_i. \tag{10}$$

Note that in the case of SIS with two modes with Π idempotent, the averaging result (8) holds even if condition (9b) does not hold. Condition (7) and the assumption that Π_i, $i \in \Sigma$, are idempotent were still required in order to obtain the averaging result in [14]. It should be noticed that the commutativity conditions imply (9) also if Π_i, $i \in \Sigma$, are not idempotent. In general, if the matrices Π_i, $i \in \Sigma$, are idempotent, the matrix Π may not be. However, if (9) hold and Π_i, $i \in \Sigma$, are idempotent then Π is idempotent.

Unfortunately, it arises that several practical electrical circuits do not satisfy (9), even if they present a sort of averaging behaviour [18]. The averaging result presented in this paper considers SIS (1) where the matrices Π_i, $i \in \Sigma$, do not commute, are not necessarily idempotent and the conditions (7) and (9) are not required.

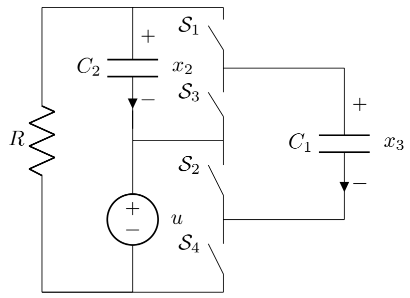

Fig. 2. Elementary cell of a ladder step-up switched capacitor converter.

3 Motivations

In this section we motivate the interest of our study by considering two examples of switched systems. Firstly, the SIS model of a switched capacitor circuit is described by highlighting the significance and relevance of state jumps occurring at switching time-instants. Then we show the applicability of the proposed approach to switched systems which can be viewed as periodic singularly perturbed systems with fast and slow modes.

3.1 A circuital example

Let us consider the switched capacitor electrical circuit shown in Fig. 2. The circuit represents the typical elementary cell of a ladder step-up switched capacitor and it consists of two capacitors and four electronic switches that are controlled in a complementary way. Then the modes of the system are two. It is assumed $i = 1$ in (1) when the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ are turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off and $i = 2$ in (1) for the reverse conduction of the switches pairs. By considering as input a constant voltage source $u = x_1$, the circuit can be modeled with x_2 and x_3 being the state variables corresponding to the voltages on the capacitors C_1 and C_2, respectively. Then the matrices pairs of (1) are:

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix}, \quad F_1 = -\frac{\rho}{R}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{11a}$$

$$\Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_2 = -\frac{1}{RC_2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11b}$$

where $\rho = \frac{1}{C_1+C_2}$.

It is easy to verify that Π_1 and Π_2 are not commutative and also (9) are not satisfied by (11).

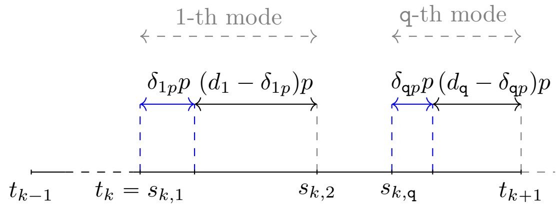

Fig. 3. Illustration of $\mathtt{q}$ modes with fast and slow dynamics.

In this paper we propose a continuous-time averaged model for the switched impulsive system (1) under milder assumptions with respect to (9). The averaging property was conjectured by the authors in [15] without providing any formal proof and by taking inspiration from the application of theoretical findings in [16] applied to discrete-time models. In next section we provide a formal proof for the averaging result based on new conditions on the system matrices which can be easily checked and are satisfied by (11).

3.2 A singularly perturbed system

The proposed averaging approach can be also applied to periodic continuous-time switched systems which do not present state discontinuities at the switching time instants but exhibit an alternate sequence of fast and slow modes. Since there are no jumps, the system can be modeled as a SIS in the form (1) where all the Π_i are identity matrices. The resulting singularly perturbed switched system can be described by the following equations

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = \frac{1}{\eta_p} A_i \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix}, \; t \in [s_{k,i}, s_{k,i} + \delta_{ip}] \tag{12a}$$

$$\begin{bmatrix} \dot{x}_s(t) \\ \dot{x}_f(t) \end{bmatrix} = F_i \begin{bmatrix} x_s(t) \\ x_f(t) \end{bmatrix}, \quad t \in [s_{k,i} + \delta_{ip}, s_{k,i+1}] \tag{12b}$$

where η_p is a small parameter that depends on the switching period p and it characterizes the time scale separation between the slow dynamics of x_s and the fast dynamics of x_f, A_i and F_i are the slow and fast dynamic matrices of the i-th mode which can be expressed in the following block forms

$$A_i = \begin{bmatrix} 0 & 0 \\ A_{i,1} & A_{i,2} \end{bmatrix} \qquad F_i = \begin{bmatrix} F_{i,1} & F_{i,2} \\ 0 & 0 \end{bmatrix} \tag{13}$$

with $A_{i,2}$ a non singular matrix. The fast i-th mode is active during the interval $(s_{k,i}, s_{k,i} + \delta_{1p}p]$, $\delta_{ip} \ll d_i$, $i \in \Sigma$, and at the time instant $s_{k,i} + \delta_{ip}$ the slow mode is activated and it remains active until $s_{k,i+1}$, see Fig. 3. Under the hypothesis that the parameter η_p goes to zero faster than p, we can approximate the fast dynamics of the i-th mode by a matrix with an error of order p^2, as described by the following equation

$$e^{A_i \frac{\delta_{ip}p}{\eta_p}} = \Pi_i + \mathrm{O}(p^2) \tag{14}$$

with

$$\Pi_i = \begin{bmatrix} I & 0 \\ A_{i,2}^{-1} A_{i,1} & 0. \end{bmatrix}. \tag{15}$$

Then the singularly perturbed switched system (12) can be approximated with a SIS in the form (1) where the matrices Π_i, $i \in \Sigma$ are given by (15).

4 Continuous-time averaged model

In this section we first introduce the proposed averaged model by motivating the structure of its matrices. Then the main averaging result is claimed and the assumptions required for its proof are discussed.

4.1 Averaged model

The proposed continuous-time averaged model has the following structure

$$\dot{\xi}(t) = A_p \xi(t), \quad t \in \mathbb{R}_0^+ \tag{16a}$$
$$\mu(t) = \Gamma \xi(t) \tag{16b}$$

with $\xi(0) = x_0 \in \mathbb{R}^n$ initial condition, the dynamic matrix function $A_p : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ is given by

$$A_p = \tfrac{1}{p} \left(\Phi_p - I\right) \tag{17}$$

with the matrix function $\Phi_p : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$ and the matrix $\Gamma \in \mathbb{R}^{n\times n}$ given by

$$\Phi_p = \Pi + \Lambda p \tag{18a}$$
$$\Gamma = \sum_{j=1}^{\mathtt{q}} \left(\prod_{h=1}^{j} \Pi_h \right) d_j \tag{18b}$$

where $\Pi \in \mathbb{R}^{n\times n}$ is given by (10) and the matrix $\Lambda \in \mathbb{R}^{n\times n}$ given by

$$\Lambda = \sum_{j=1}^{\mathtt{q}} \left(\prod_{h=j+1}^{\mathtt{q}} \Pi_h F_j \prod_{h=1}^{j} \Pi_h \right) d_j. \tag{19}$$

where $\prod_{h=j+1}^{\mathtt{q}} \Pi_h$ for $j = \mathtt{q}$ is assumed to be the identity matrix.

The output $\mu \in \mathbb{R}^n$ of the model (16) is intended to be an approximation of the moving average of the solution of the impulsive systems (1). The dependence of (17) on the switching period is a crucial aspect in order to obtain a good approximation [16], which is an analogous dependence used in the well known result for the classical averaging technique applied to switched systems with modes represented by ordinary differential equations, i.e., by excluding jumps in the state [10].

It should be noticed that in the case of a switched ordinary differential equations, the matrices Π_i, $i \in \Sigma$, are equal to the identity matrix and the matrix Λ reduces to the dynamic matrices of the classical continuous-time averaged model of pulse width modulated systems with $\mathtt{q}$ modes, i.e. $\sum_{j=1}^{\mathtt{q}} F_j d_j$.

A further motivation for the choice of the matrices in (16) can be obtained by discretizing the model (16) with the forward Euler method and a sampling period p. By indicating with z_k the state variable at the time-step $k \in \mathbb{N}_0$ of the resulting discrete-time state-space model, from (16a) one obtains $z_{k+1} = z_k + pA_p z_k$. Then, by using (17) the following discrete-time model can be written

$$z_{k+1} = \Phi_p z_k, \quad k \in \mathbb{N}_0 \tag{20a}$$
$$\mu_k = \Gamma z_k \tag{20b}$$

with $z_0 = x_0$. The solution of (20) can be written as

$$z_k = \Phi_p^k z_0 \tag{21}$$

for all $k \in \mathbb{N}_0$. In the sequel we will show that $x_k^- = z_k + \mathrm{O}(p)$ for any k, which motivates the choice (17) with (18a).

The choice of the matrix Γ in the output equation (16b) can be motivated by considering the continuous-time moving average of the solution of (1), which is defined as

$$m(t) = \frac{1}{p} \int_t^{t+p} x(\tau) \mathrm{d}\tau \tag{22}$$

for any $t \in (0, \bar{t} - p]$ with $\bar{t} > p$, where $x(t)$ is the solution of (1). We will show that the $m(t_k) = \mu_k + \mathrm{O}(p)$ which motivates the choice (16b) with (18b).

In next section it is proved that, under some assumptions which are discussed in the sequel, there exist constants $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}_0^+$, $\varepsilon \in (0, 1)$, $\bar{p}_\varepsilon \in \mathbb{R}^+$ such that the following condition

$$\boxed{\|m(t) - \mu(t)\| \leqslant \alpha p + \beta \varepsilon^k} \tag{23}$$

with $k = \lfloor t/p \rfloor$, holds for all $p \in (0, \bar{p}_\varepsilon]$ and $t \in (0, \bar{t} - p]$, for any $\bar{t} \in \mathbb{R}^+$. In (23) the moving average $m(t)$ is given by (22) with $x(t)$ being a solution of the SIS (1), and $\mu(t)$ is the output of the averaged model (16).

4.2 Assumptions

The main result is proved starting from two basic assumptions. The first one can be expressed as follows.

Assumption 1 *Given the matrix function Φ_p expressed by* (18a)*, there exists a constant $\alpha \in \mathbb{R}_0^+$ and an induced matrix norm $\||\cdot\||$ such that*

$$\||\Phi_p\|| \leqslant 1 + \alpha p. \tag{24}$$

Assumption 1 can be verified through the feasibility of a suitable set of linear matrix inequalities [15, Lemma 4]. Note that Assumption 1 may be satisfied, while (24) is *not* satisfied w.r.t. to the standard (induced) Euclidian norm. As an example consider $\Phi_p = \left[\begin{smallmatrix} 1+p & 0.5 \\ 0 & 0.5+p \end{smallmatrix}\right]$. It is easily seen that for the norm $\||\cdot\|| : \mathbb{R}^2 \to \mathbb{R}_{\geqslant 0}$ given by $\||x\|| := \|Tx\|$ for $T = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$ we have that $\||\Phi_p\|| = \|T\Phi_p T^{-1}\| = 1 + p$, i.e. (24) is satisfied for $\alpha = 1$. However, $\|\Phi_0\| \approx 1.14 > 0$, so that (24) does not hold w.r.t. the Euclidian norm.

Remark 2 *Under the situation that all Π_i, $i \in \Sigma$, are idempotent, conditions* (9) *imply that Π is idempotent and then Assumption 1 holds. On the other hand, Assumption 1 is also verified if all powers of the matrix Π given by* (10) *are bounded, i.e. there exists a constant $M > 0$ such that $\|\Pi_i^k\| \leqslant M$ for all k, without requiring that Π is idempotent. This fact can be easily proved by using the Barabanov norm [24]. Therefore, the results presented in next section which are based only on Assumption 1 are proved under milder conditions with respect to the former averaging results which start from* (9).

An important result related to Assumption 1 is the following lemma which has been proved in [15].

Lemma 3 *Consider a Lipschitz continuous matrix function $p \mapsto \Phi_p \in \mathbb{R}^{n \times n}$. Assume there exists a constant $\alpha \in \mathbb{R}_0^+$ such that* (24) *holds. Then, for any Lipschitz continuous matrix function $p \mapsto M_p \in \mathbb{R}^{n \times n}$ such that $M_p = \mathrm{O}(p^2)$, it is*

$$\Phi_p^k = \mathrm{O}(1) \tag{25a}$$

$$(\Phi_p + M_p)^k = \Phi_p^k + \mathrm{O}(p). \tag{25b}$$

for all $k \in \{0, \dots, \ell_p\}$ with $\ell_p = \lfloor \bar{t}/p \rfloor$ and any finite $\bar{t} \in \mathbb{R}^+$.

Note that the asymptotic bounds (25) are valid no matter which matrix norm is used, because all matrix norms are equivalent and hence the norm-choice only effects the constants within the big-O notation. In the following we will only utilize Assumption 1 via Lemma 3 and therefore we can use in the remainder of this work always the standard Euclidian norm when bounding errors; in particular, knowledge of the specific (non-standard) norm satisfying (24) is not required.

A further technical assumption required in order to obtain our averaging results is the following.

Assumption 2 *Given the matrices Π and Λ expressed by* (10) *and* (19)*, respectively, there exists a coordinate transformation $T \in \mathbb{R}^{n \times n}$ such that*

$$T\Pi T^{-1} = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \tag{26a}$$

$$T\Lambda T^{-1} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_3 & \Lambda_2 \end{bmatrix} \tag{26b}$$

where V is Schur, with V and Λ_2 square matrices of the same dimension.

Consider the case that Π_i, $i \in \Sigma$, are idempotent. It can be easily shown that (9) implies (26a), with $V = 0$ but the opposite is not true in general. Indeed, (11) do not satisfy (9) but Assumption 2 holds for these matrices as it can be verified by considering, for instance, the following coordinate transformation

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{C_1+C_2}{C_2} & 1 & \frac{C_1}{C_2} \\ -1 & 0 & 1 \end{bmatrix}. \tag{27}$$

Note that (26a) together with the Schur condition of matrix V, also if Π_i, $i \in \Sigma$, are not idempotent, implies that $\lim_{k\to\infty}[(T\Pi T^{-1})^{k+1} - (T\Pi T^{-1})^k] = 0$ which means that the transformed matrix $T\Pi T^{-1}$ converges to an idempotent matrix when k goes to infinity.

Remark 4 *Assumption 2 allows one to obtain a useful transformation for the matrix function Φ_p. Indeed, by using Assumption 2 one can write*

$$T\Phi_p T^{-1} = \begin{bmatrix} I + \Lambda_1 p & 0 \\ \Lambda_3 p & V + \Lambda_2 p \end{bmatrix}. \tag{28}$$

For sufficiently small p the matrices $I + \Lambda_1 p$ and $V + \Lambda_2 p$ have no common eigenvalues, hence there is a unique solution R_p of the Sylvester equation

$$R_p(I + \Lambda_1 p) - (V + \Lambda_2 p)R_p = -\Lambda_3 p \tag{29}$$

such that

$$T_p \Phi_p T_p^{-1} = \begin{bmatrix} I + \Lambda_1 p & 0 \\ 0 & V + \Lambda_2 p \end{bmatrix} \tag{30}$$

with

$$T_p := \begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} T. \tag{31}$$

Note that $R_p = 0$ is the solution of (29) for $p = 0$ and (29) can be written as

$$(M + p\,\mathcal{M})\mathrm{vec}(R_p) = -p\,\mathrm{vec}(\Lambda_3)$$

where $\mathrm{vec}(\cdot) : \mathbb{R}^{r\times r} \to \mathbb{R}^{r^2}$ *is the standard vectorization operator,* $M := I \otimes V - I \otimes I$ *and* $\mathcal{M} := I \otimes \Lambda_2 - \Lambda_1^\top \otimes I$. *Hence a standard perturbation analysis shows that*

$$\|R_p\| \leqslant \frac{\|M^{-1}\|\|\Lambda_3\|}{1 - \|M^{-1}\|\|\mathcal{M}\|p}p = O(p).$$

Remark 4 will be used for obtaining the main result of the paper which shows that if Assumptions 1 and 2 hold then (23) is satisfied.

It is interesting to compare (8) and (23) in the light of the required assumptions. First of all the approximation result (8) involves the solution $x(t)$ of the impulsive system while in (23) the corresponding moving average $m(t)$ is considered. The variables $\xi(t)$ and $\mu(t)$ do not present jumps, so as $m(t)$. The reason why $x(t)$ can be used in (8), is that the amplitudes of the state jumps converge to zero with p if (9) holds, which is not assumed in our main averaging result. Instead, if Assumptions 1 and 2 hold it is still possible to have nontrivial jumps when p decreases. On the other hand, the inequality (23) says that the error $m(t) - \mu(t)$ decreases with the multiple of the switching period and by decreasing the PWM period.

5 Averaging results

In this section the averaging result (23) is proved. To this aim, some preliminary steps are required. We first prove that the difference between the solution of the SIS (1) evaluated at the multiple of the switching period and the solution of the discrete-time system (20), is of order of the switching period.

Lemma 5 *Consider the continuous-time SIS* (1) *with initial condition x_0, over a time interval $t \in [0, \bar{t}]$ with some $\bar{t} \in \mathbb{R}^+$ and the discrete-time model* (20) *with $k = \lfloor t/p \rfloor$ and initial condition $z_0 = x_0$. If Assumption 1 is satisfied, then the following condition*

$$x_k^- = z_k + \mathrm{O}(p) \tag{32}$$

where x_k^- is given by (5) *and z_k is given by* (21)*, holds for all $k \in \{0, \ldots, \ell_p\}$, $\ell_p = \lfloor \bar{t}/p \rfloor$.*

By using Lemma 5 one can prove that the difference between the moving average (22) evaluated at the multiples of the switching period and the output of the discrete-time model (20), is of order of the switching period. As for Lemma 5, also the following result requires Assumption 1 but not Assumption 2.

Lemma 6 *Consider the continuous-time SIS* (1) *with initial condition x_0, over a time interval $t \in [0, \bar{t}]$ with some $\bar{t} \in \mathbb{R}^+$, the moving average of its solution given by* (22) *evaluated at t_k for $k \in \{0, \ldots, \ell_p - 1\}$, $\ell_p = \lfloor \bar{t}/p \rfloor$, and the discrete time model* (20) *with $k = \lfloor t/p \rfloor$ and initial condition $z_0 = x_0$. If Assumption 1 is satisfied then the following condition*

$$m(t_k) = \mu_k + \mathrm{O}(p) \tag{33}$$

holds for all $k \in \{0, \ldots, \ell_p - 1\}$.

A further step towards the proof of our main result consists of considering the error between the moving average $m(t)$ expressed by (22) and the values obtained by sampling $m(t)$ at the multiple of p, i.e., $m(t_k)$ where $t_k = kp$ and $k = \lfloor t/p \rfloor$. In particular, by using Assumption 2 one can prove the following result.

Lemma 7 *Consider the continuous-time SIS* (1) *with initial condition x_0, over a time interval $t \in [0, \bar{t}]$ with some $\bar{t} \in \mathbb{R}^+$, the moving average $m(t)$ of its solution given by* (22)*. If Assumptions 1 and 2 are satisfied then there exist constants $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}_0^+$, $\varepsilon \in (0, 1)$ and $\bar{p}_\varepsilon \in \mathbb{R}^+$ such that the following condition*

$$\|m(t) - m(t_k)\| \leqslant \alpha p + \beta \varepsilon^k \tag{34}$$

with $t_k = kp$, $k = \lfloor t/p \rfloor$, holds for any $t \in (0, \bar{t} - p]$ and any $p \in (0, \bar{p}_\varepsilon]$.

Lemma 7 allows one to conclude that the approximation result is valid for any backward Δ-shifted version of (22) defined as

$$m_\Delta(t) = \frac{1}{p}\int_{t-\Delta}^{t-\Delta+p} x(\tau)\mathtt{d}\tau \tag{35}$$

with $\Delta \in [0, p)$, where $x(t)$ is the solution of (1). Indeed, since (22) is defined as a p-forward moving average, it is easy to verify that a condition similar to (34) holds for m_Δ, i.e., $\|m_\Delta(t) - m(t_k)\| \leqslant \alpha p + \beta \varepsilon^k$ for any $\Delta \in [0, p)$, $t \in (\Delta, \bar{t} + \Delta - p]$ with $\bar{t} > p$. In the following for the sake of simplicity we consider the case $\Delta = 0$.

By using the lemmas above, we can prove the following theorem which synthesizes our main result.

Theorem 8 *Consider the continuous-time SIS* (1) *with initial condition x_0, over a time interval $t \in [0, \bar{t}]$ with some $\bar{t} \in \mathbb{R}^+$, the corresponding moving average $m(t)$ given by* (22) *and the output $\mu(t)$ of the continuous-time model* (16) *with initial condition $\xi(0) = x_0$. If Assumptions 1 and 2 hold, then there exist constants $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}_0^+$, $\varepsilon \in (0, 1)$, and $\bar{p}_\varepsilon \in \mathbb{R}^+$ such that* (23) *with $k = \lfloor t/p \rfloor$ holds for all $p \in (0, \bar{p}_\varepsilon]$ and $t \in (0, \bar{t} - p]$.*

The averaging approximation expressed by (23) and proved in Theorem 8 shows that the error between the moving average $m(t)$ of the SIS solution and the output of the averaged model depends on p and k too. In other words, it is not enough to let the switching period going to zero in order to reduce the error of the averaging process, but some periods must elapse too. This is due to the fact that the algebraic conditions on the modes matrices have been relaxed. The following remark shows that under more restrictive conditions on the modes matrices, one can recover the classical $\mathrm{O}(p)$ averaging result.

Remark 9 *It is easy to show by checking the proof of Theorem 8 that, under Assumptions 1 and 2, if all* Π_i, $i \in \Sigma$, *are idempotent and* (9) *hold than there exist constants* $\alpha_m, \alpha_\mu \in \mathbb{R}^+$ *and* $\overline{p} \in \mathbb{R}^+$ *such that*

$$\|m(t) - m(t_k)\| \leqslant \alpha_m p, \tag{36a}$$
$$\|m(t) - \mu(t)\| \leqslant \alpha_\mu p, \tag{36b}$$

with $t_k = kp$, $k = \lfloor t/p \rfloor$ *holds for all* $p \in (0, \overline{p}]$ *and* $t \in (0, \overline{t} - p]$. *It should be noticed that in the case of a SIS with two modes under Assumptions 1 and 2, if all* Π_i, $i = 1, 2$ *and the product* Π *are idempotent then* (36) *is still valid even if* (9a) *holds and* (9b) *doesn't.*

6 Simulation results

In this section three examples with their respective simulations are analyzed to validate the effectiveness of the results presented in the previous section. The first example describes the proposed averaged model (16) and its effectiveness for the electronic circuit shown in Fig. 2. The second example compares the results obtained with the proposed averaged model and the classical one (6) in a case where the jump matrices are projectors and satisfy conditions (9), in particular, it is highlighted that in general the average models differ. The last example shows the application of our technique for an unstable system and it also illustrates that in some cases the newly proposed average model coincides with the previously proposed averaged model.

Example 10 *Let us go back to the motivating example of our analysis whose equivalent circuit is shown in Fig. 2. By considering* (11) *it follows that*

$$\Pi = \Pi_2 \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 1 & 0 & 0 \end{bmatrix} \tag{37}$$

being $\rho = \frac{1}{C_1 + C_2}$. *It can be easily verified that the matrix* Π *is product bounded, i.e.* Π^k *is bounded for all* k, *and then Assumption 1 holds independently of the circuit parameters. Moreover, Assumption 2 holds by considering*

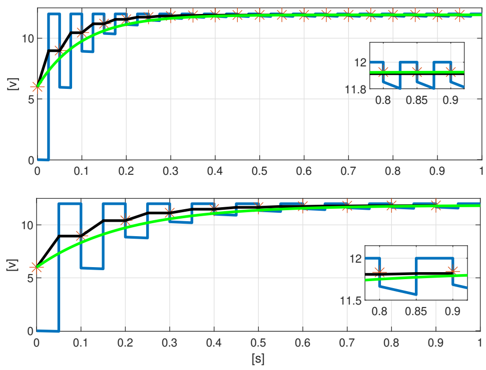

Fig. 4. Time evolution of the state variable x_2 of Example 10 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (16) (green lines), discrete-time model (20) (red stars), moving average (22) (black lines).

the transformation matrix (27). *The matrix* (19) *can be written as*

$$\Lambda = \Pi_2 F_1 \Pi_1 d_1 + F_2 \Pi_2 \Pi_1 d_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{RC_2} d_2 & -\frac{\rho}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and the matrix (18b) *is given by*

$$\Gamma = \Pi_1 d_1 + \Pi_2 \Pi_1 d_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ d_2 & C_2\rho d_1 & C_1\rho d_1 \end{bmatrix}$$

where we used the condition $d_1 + d_2 = 1$.

We now compare the solutions of the SIS (1), *the averaged model* (16) *proposed in this paper and the discrete-time model* (20) *together with the moving average* (22). *Let us consider* $C_1 = C_2 = 120\,\mu\mathrm{F}$, $R = 10\,\mathrm{k}\Omega$, $u = 12\,\mathrm{V}$, $d_1 = d_2 = 0.5$ *and null initial conditions. Fig. 4 and Fig. 5 show the dynamics of the state variables* x_2 *and* x_3, *respectively, for different values of the switching period, over a time interval of* $1\,\mathrm{s}$.

Figures 6 and 7 show the left hand side of (23) *computed for the state variable* x_3 *in logarithmic scale as a function of time and as a function of the multiple of the switching period (steps), respectively, for different values of the switching period.*

Clearly, the intersections among the different curves in Fig. 6 show that the initial error is not of order p, *but it quickly decreases with an increasing number of steps and from Fig. 7 it is visible that the rate of convergence with respect to the step counter* k *is independent of* p, *which is related to the* $\beta\varepsilon^k$ *term in* (23) *and is due to the*

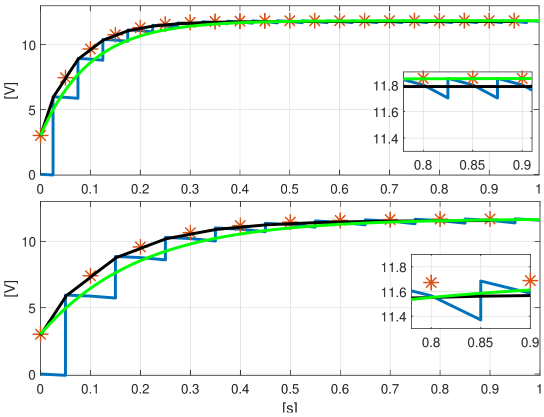

Fig. 5. Time evolution of the state variable x_3 of Example 10 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (16) (green lines), discrete-time model (20) (red stars), moving average (22) (black lines).

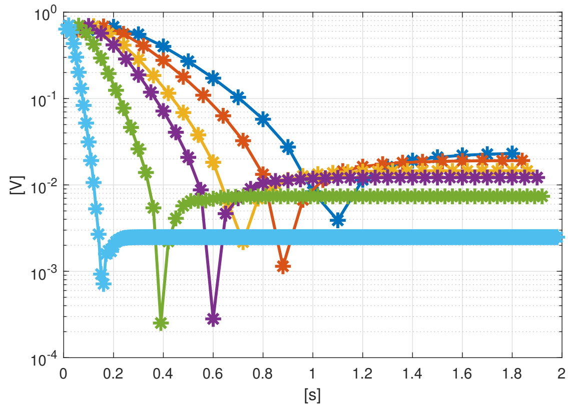

Fig. 6. Time evolution of the error $\|m(t_k) - \mu(t_k)\|$ computed for the state variable x_3 of Example 10 (the vertical axis is in logarithmic scale) for different values of the switching period: $p = 0.5\,\mathrm{s}$ (blue line), $p = 0.25\,\mathrm{s}$ (orange line), $p = 0.1\,\mathrm{s}$ (green line), $p = 0.05\,\mathrm{s}$ (purple line), $p = 0.01\,\mathrm{s}$ (red line) and $p = 0.005\,\mathrm{s}$ (cyan line).

fact that matrices Π_i, $i \in \Sigma$, are neither idempotent nor commutative.

Note that the matrix Π^k, with Π given by (37), converges to the idempotent matrix $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ when k goes to infinity. This property, together with the stable averaged matrix A_p allows one to motivate the behavior shown in Fig. 4 in the sense that the jumps do not influence the stability of the slow dynamics captured by the trajectories of the averaged model. After a sufficient number of steps ($k \geqslant 17$), the error settles to a value which is of order p, which is related to the αp term in (23). Note that the intermediate lower values for the error are due to the intersections of the solution of the averaged model and the moving average in this specific example (visible in Fig. 5, where the green curve is initially below the black curve, but later is above).

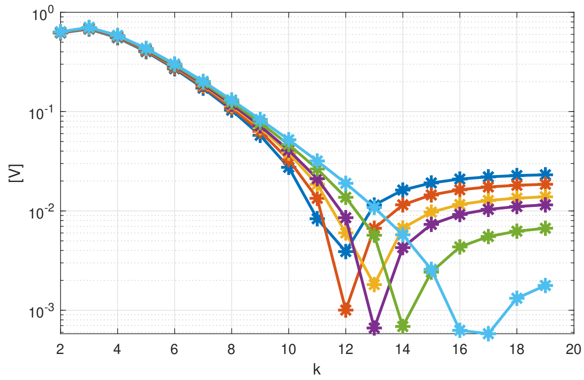

Fig. 7. The error $\|m(t_k) - \mu(t_k)\|$ computed for the state variable x_3 of Example 10 (the vertical axis is in logarithmic scale) versus the multiples of p for different values of the switching period: $p = 0.5\,\mathrm{s}$ (blue line), $p = 0.25\,\mathrm{s}$ (orange line), $p = 0.1\,\mathrm{s}$ (green line), $p = 0.05\,\mathrm{s}$ (purple line), $p = 0.01\,\mathrm{s}$ (red line) and $p = 0.005\,\mathrm{s}$ (cyan line).

The error of the left hand side of (23) computed for the whole state has an analogous behaviour as the one plotted in Fig.s 6 and 7 and a corresponding plot is therefore omitted.

Example 11 *Let us consider the system (1) with $\mathsf{q} = 2$ and the following matrices*

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} 0 & -2 & 0 \\ 1 & -3 & 0 \\ -1 & 3 & 0 \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

Note that for this example the matrices Π_1 and Π_2 are idempotent and satisfy the condition (9), hence $\Pi := \Pi_2\Pi_1$ is idempotent as well. Nevertheless, it is easy to verify that the proposed averaged model (16) is different from (6). In Fig. 8 the time evolution of the state variables x_1 and x_2 for $p = 0.1\,\mathrm{s}$ are reported by showing the solutions of the SIS (1), the averaged model (16), and other solutions of interest.

A significant difference is clearly the choice of the initial conditions. The averaged model (6) exploits the fact that Π is a projector and chooses an initial value in the subspace $\operatorname{im}\Pi$, thereby matching very well the moving average after one period. The reason is that after one period the solution of the switched systems has distance of order p from the subspace $\operatorname{im}\Pi$. Our newly proposed averaged model doesn't assume this property and instead chooses

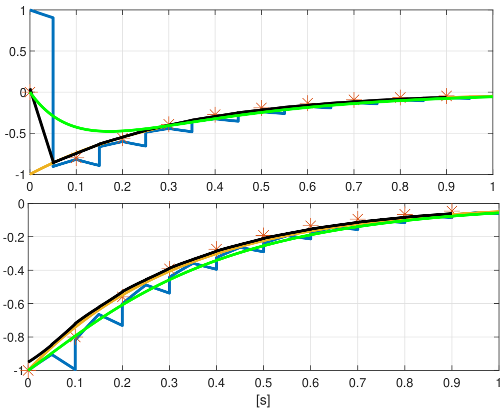

Fig. 8. Time evolution of the state variable x_1 (top) and x_2 (bottom) of Example 11 with $p = 0.1$ s: SIS (1) (blue lines), averaged model (6) (orange lines), averaged model (16) (green lines), discrete-time model (20) (red stars), moving average (22) (black lines).

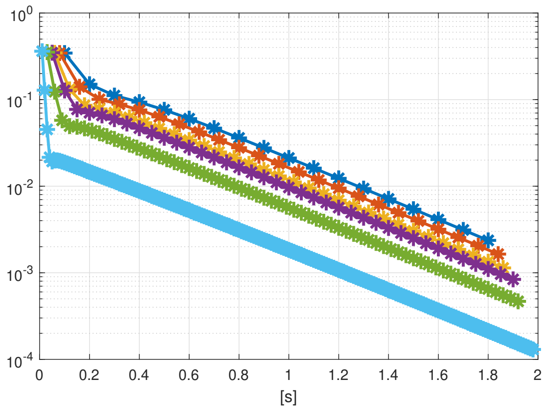

Fig. 9. Time evolution of the error $\|m(t_k) - \mu(t_k)\|$ of Example 11 (the vertical axis is in logarithmic scale) for different values of the switching period: $p = 0.1$ s (blue line), $p = 0.08$ s (red line), $p = 0.06$ s (yellow line), $p = 0.05$ s (purple line), $p = 0.03$ s (green line) and $p = 0.01$ s (cyan line).

an initial value which is consistent with the moving average over the first interval which is still very far away from the subspace $\operatorname{im}\Pi$. *Furthermore, the dynamics of our newly proposed averaged model approximate the jump towards the common consistency space by introducing an eigenvalue* $-1/p$ *in the matrix* A_p *and it takes some steps until the initial error vanishes. This is also clearly visible in Fig. 9 which shows the error of the proposed averaged model, i.e., the left hand side of* (23)*, for different switching periods. For each p the error decreases over time and for any time instant the error decreases with decreasing p. The continued exponential decay is due to the fact, that all solutions (switched and averaged) converge exponentially to zero and hence trivially the error also converges exponentially to zero.*

Note that the matrix F_2 *is not Hurwitz but the dynamic matrix* A_p *of the resulting averaged model is Hurwitz for all* $p > 0$. *Looking at Fig. 8 the interpretation is that the fast dynamics characterized by the idempotent jump matrices allow the trajectory of the SIS to get closer to the trajectory of the averaged model when the number of elapsed periods increases.*

Example 12 *Let us consider the following numerical example where the matrices* F_i *and* Π_i*, with* $i \in \{1, 2\}$ *are given by*

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} -4 & -1 & -4 \\ -1 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$
$$\Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} -10 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The matrices Π_1 *and* Π_2 *do not satisfy conditions* (9b)*, however the products* $\Pi_2\Pi_1$ *and* $\Pi_1\Pi_2$ *are idempotent. Then according to Remark 9 the error between the moving average* $m(t)$ *of the solution of this system and its samples* $m(t_k)$ *is* $\mathrm{O}(p)$. *By considering* $d_1 = d_2 = 0.5$ *and the following matrices*

$$\Lambda = \begin{bmatrix} -7 & -1 & -7 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

the dynamic matrix (17) *is given by*

$$A_p = \begin{bmatrix} -7 & -1 & -(7p-1)/p \\ -1 & 2 & -1 \\ 0 & 0 & -1/p \end{bmatrix}$$

where p is the switching period. Let us compare the solutions of the SIS (1)*, the averaged model* (16) *and the discrete-time model* (20) *together with the moving average* (22)*. Figures 10 and 11 show the dynamics of the state variables* x_1 *and* x_2*, respectively, for different values of the switching period, over a time interval of* 0.5 s. *It is evident that the error between the output* $\mu(t)$ *and the moving average* $m(t)$ *is* $\mathrm{O}(p)$*, i.e., it is enough to let the switching period going to zero without needing some periods to elapse.*

It is remarkable to make a comparison between the averaged model (6) *presented in our previous studies and the proposed model* (16)*. Let us consider the averaged dynamic matrix of the continuous averaged model* (6) *which*

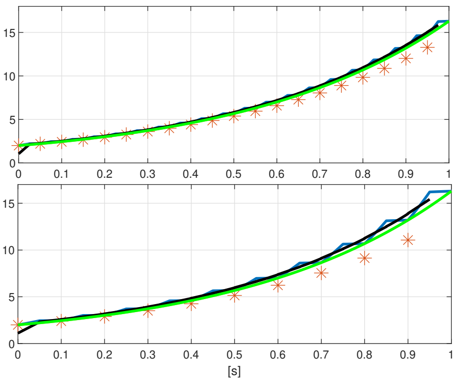

Fig. 10. Time evolution of the state variable x_1 of Example 12 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (16) (green lines), discrete-time model (20) (red stars), moving average (35) with $\delta = p/2$ (black lines).

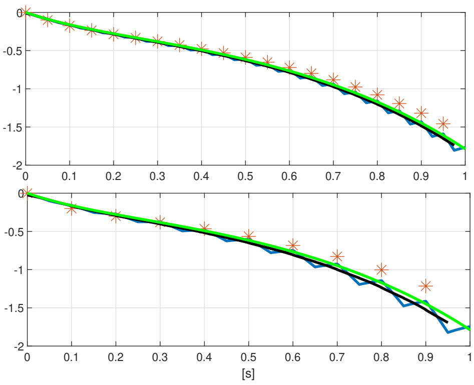

Fig. 11. Time evolution of the state variable x_2 of Example 12 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (16) (green lines), discrete-time model (20) (red stars), moving average (35) with $\delta = p/2$ (black lines).

is given by

$$A_{\mathrm{av}} = \Pi(F_1 d_1 + F_2 d_2)\Pi = \begin{bmatrix} -7 & -1 & -7 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

It is easy to see that $\Gamma A_p = A_{\mathrm{av}}$. *Moreover the initial condition for* (6) *and* (17) *are the same, indeed* $\Gamma x_0 = \Pi x_0$. *Then the solutions of* (6) *and* (17) *keep very close to each other.*

7 Conclusion

A new averaged model for SIS which exhibit state jumps at the switching time instants has been presented. The proposed model generalizes the classical averaged model widely adopted for the analysis of switched PWM systems with Lipschitz continuous solution. The averaging result requires milder assumptions on the system matrices with respect to previous averaging analyses for SIS. A switched capacitor electrical circuit has been used to validate the results and to motivate their practical usefulness.

Future work will be dedicated to the study of scenarios with time-varying and state-dependent duty cycles. Furthermore, other directions of future research are the application of the proposed averaging approach for the stability analysis of switched impulsive systems and singularly perturbed systems.

Appendix

7.1 Proof of Lemma 5

Proof. Consider (3)–(4). By using the Taylor approximation one can write

$$G_{j,p} = e^{F_j d_j p} = I + F_j d_j p + \mathrm{O}(p^2) = I + \mathrm{O}(p) \quad (38)$$

for all $j \in \Sigma$, where I is the identity matrix. By using (38) in (4) one obtains

$$\Theta_p = \prod_{j=1}^{\mathtt{q}} G_{j,p}\Pi_j = \Pi + \Lambda p + \mathrm{O}(p^2) = \Phi_p + \mathrm{O}(p^2) \quad (39)$$

where Π is given by (10), Λ by (19) and Φ_p by (18a). By applying Lemma 3 with Assumption 1, from (25b) it follows

$$\Theta_p^k = \Phi_p^k + \mathrm{O}(p) \quad (40)$$

for all $k \in \{0, \ldots, \ell_p\}$. By subtracting (21) to (5) one obtains

$$\begin{aligned} x_k^- &= z_k + \Theta_p^k x_0^- - \Phi_p^k z_0 \\ &\overset{a}{=} z_k + \Phi_p^k (x_0^- - z_0) + \mathrm{O}(p) \\ &= z_k + \mathrm{O}(p) \end{aligned} \quad (41)$$

where in $\overset{a}{=}$ we used (40). ■

7.2 Proof of Lemma 6

Proof. Consider (22). By solving (1) and by using (2) one can write

$$\begin{aligned} pm(t_k) &= \int_{kp}^{(k+1)p} x(t)\mathtt{d}t \\ &= \sum_{i=1}^{\mathtt{q}} \int_0^{d_i p} G_i(\psi)\Pi_i x(s_{k,i}^-)\mathtt{d}\psi \\ &= \sum_{i=1}^{\mathtt{q}} \int_0^{d_i p} G_i(\psi)\Pi_i \prod_{h=1}^{i-1} G_{h,p}\Pi_h x_k^- \mathtt{d}\psi \end{aligned} \quad (42)$$

for all $k \in \{0, \ldots, \ell_p - 1\}$. Then, from (42) by using (38) and by noticing the presence of the integral one can write:

$$\begin{aligned} pm(t_k) &= \sum_{i=1}^{\mathtt{q}} \Pi_i \prod_{h=1}^{i-1} \Pi_h x_k^- d_i p + \mathrm{O}(p^2) \\ &= \sum_{i=1}^{\mathtt{q}} \prod_{h=1}^{i} \Pi_h x_k^- d_i p + \mathrm{O}(p^2) \\ &= \Gamma p x_k^- + \mathrm{O}(p^2) \stackrel{a}{=} \Gamma p z_k + \mathrm{O}(p^2) \\ &= \mu_k p + \mathrm{O}(p^2) \end{aligned} \tag{43}$$

where Γ is given by (18b), in $\stackrel{a}{=}$ we used Lemma 5 with Assumption 1 and μ_k is defined by (20b). By dividing both sides of (43) by p it follows that (33) holds. ■

7.3 Proof of Lemma 7

Proof. By definition it is $m(t) = m(t_k)$ for any $t = t_k = kp$, $k \in \{0, \ldots, \ell_p - 1\}$ and then in the time instants multiple of the switching period the condition (34) is trivially satisfied.

Let us consider the moving average over a time interval of length p which starts in i-th mode. For any $t \in [s_{k,i}, s_{k,i+1}]$, $k \in \{0, \ldots, \ell_p - 1\}$, $\tau_i = t - s_{k,i}$, i.e. $\tau_i \in [0, d_i p]$, by substituting the solution of SIS (1) in (22) and by reminding that the duty cycles are constant, one can write

$$\begin{aligned} pm(t) = pm(s_{k,i} + \tau_i) &= \int_{\tau_i}^{d_i p} G_i(\psi)\Pi_i x(s_{k,i}^-)\mathtt{d}\psi \\ &+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j x(s_{k,j}^-)\mathtt{d}\psi \\ &+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j x(s_{k+1,j}^-)\mathtt{d}\psi \\ &+ \int_0^{\tau_i} G_i(\psi)\Pi_i x(s_{k+1,i}^-)\mathtt{d}\psi. \end{aligned}$$

By using (2)–(4) it follows

$$\begin{aligned} pm(t) &= \int_{\tau_i}^{d_i p} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\ &+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\ &+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_{k+1}^- \mathtt{d}\psi \\ &+ \int_0^{\tau_i} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_{k+1}^- \mathtt{d}\psi. \end{aligned} \tag{44}$$

Let us rewrite (42) as follows

$$\begin{aligned} pm(t_k) &= \sum_{j=1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\ &= \int_0^{d_i p} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\ &+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\ &+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi. \end{aligned} \tag{45}$$

By taking the difference between (45) and (44) one obtains

$$\begin{aligned} p(m(t) - m(t_k)) &= \int_0^{\tau_i} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w (x_{k+1}^- - x_k^-)\mathtt{d}\psi \\ &+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w (x_{k+1}^- - x_k^-)\mathtt{d}\psi. \end{aligned} \tag{46}$$

By using (3)–(4)

$$x_{k+1}^- = \Theta_p x_k^- = \prod_{i=1}^{\mathtt{q}} \Pi_i x_k^- + \mathrm{O}(p) = \Pi x_k^- + \mathrm{O}(p), \tag{47}$$

together with $G_i(\psi) = I + \mathrm{O}(p)$ and $G_{w,p} = I + \mathrm{O}(p)$, from (46) one can write

$$\begin{aligned} p(m(t) - m(t_k)) &= \tau_i \prod_{w=1}^{i} \Pi_w (\Pi - I) x_k^- \\ &+ \sum_{j=1}^{i-1} d_j p \prod_{w=1}^{j} \Pi_w (\Pi - I) x_k^- + \mathrm{O}(p^2). \end{aligned} \tag{48}$$

By using (5) and (40) in Lemma 5, the expression (48) can be rewritten as

$$p(m(t) - m(t_k)) = \left(\tau_i \prod_{w=1}^{i} \Pi_w + \sum_{j=1}^{i-1} d_j p \prod_{w=1}^{j} \Pi_w \right) (\Pi - I)\Phi_p^k x_0^- + \mathrm{O}(p^2). \tag{49}$$

where $\tau_i = t - s_{k,i}$ and $k = \lfloor t/p \rfloor$.

From Assumption 2 and Remark 4 there exists a matrix T_p such that (30) holds and then one has

$$
\begin{aligned}
T_p(\Pi - I)\Phi_p^k T_p^{-1} &= T_p(\Pi - I)T_p^{-1}T_p\Phi_p^k T_p^{-1} \\
&= \left(\begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} T\Pi T^{-1} \begin{bmatrix} I & 0 \\ -R_p & I \end{bmatrix} - T_pT_p^{-1}\right) T_p\Phi_p^k T_p^{-1} \\
&= \left(\begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I & 0 \\ -R_p & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}\right) T_p\Phi_p^k T_p^{-1} \\
&= \begin{bmatrix} 0 & 0 \\ & V - I \end{bmatrix} \begin{bmatrix} (I + \Lambda_1 p)^k & 0 \\ 0 & (V + \Lambda_2 p)^k \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (V - I)(V + \Lambda_2 p)^k \end{bmatrix}.
\end{aligned} \tag{50}
$$

Since V is Schur it follows there exist constants $\beta_1 \in \mathbb{R}_0^+$, $\varepsilon \in (0,1)$ and $\bar{p}_\varepsilon \in \mathbb{R}^+$ such that, by taking the norms on both side of (50) it is

$$
\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\| \leqslant \beta_1 \varepsilon^k \tag{51}
$$

for all $p \in (0, \bar{p}_\varepsilon]$. Moreover one can write

$$
\begin{aligned}
\|(\Pi - I)\Phi_p^k\| &= \|T_p^{-1}T_p(\Pi - I)\Phi_p^k T_p^{-1}T_p\| \\
&\leqslant \|T_p^{-1}\|\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\|\|T_p\| \\
&\leqslant \beta_0\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\| \leqslant \beta_0\beta_1\varepsilon^k,
\end{aligned} \tag{52}
$$

where $\beta_0 \in \mathbb{R}^+$ is such that

$$
\|T_p\|\|T_p^{-1}\| \leqslant \beta_0 \tag{53}
$$

which exists for sufficiently small p because R_p in (31) is $O(p)$.

Then, by dividing both sides of (49) by p, by considering that $\tau_i = \mathrm{O}(p)$, by taking the norms on both sides, given the initial condition x_0^- and by using (52), it follows that there exists an $\alpha_i \in \mathbb{R}^+$ such that the following condition

$$
\begin{aligned}
\|m(t) - m(t_k)\| \leqslant& \| \left(\frac{\tau_i}{p}\prod_{w=1}^{i}\Pi_w + \sum_{j=1}^{i-1} d_j \prod_{w=1}^{j}\Pi_w\right) \\
& (\Pi - I)\Phi_p^k x_0^-\| + \alpha_i p \\
\leqslant& \left(\|\prod_{w=1}^{i}\Pi_w\| + \sum_{j=1}^{i-1}\|\prod_{w=1}^{j}\Pi_w\|\right) \\
& \|(\Pi - I)\Phi_p^k\|\|x_0^-\| + \alpha_i p \\
\leqslant& \sum_{j=1}^{i}\|\prod_{w=1}^{j}\Pi_w\|\beta_0\beta_1\varepsilon^k\|x_0^-\| + \alpha_i p \\
\leqslant& \beta\varepsilon^k + \alpha_i p
\end{aligned} \tag{54}
$$

is satisfied for any $t \in [s_{k,i}, s_{k,i+1})$, $\tau_i = t - s_{k,i}$, for all $k \in \{0, \ldots, \ell_p - 1\}$ and $p \in (0, \bar{p}_\varepsilon]$, where

$$
\beta = \beta_0\beta_1\|x_0^-\| \sum_{j=1}^{\mathrm{q}} \|\prod_{w=1}^{j}\Pi_w\|.
$$

By considering (54) for all $i \in \Sigma$ it follows that (34) holds for all $t \in (0, \bar{t} - p]$ and any $p \in (0, \bar{p}_\varepsilon]$ with $\alpha = \max_{i\in\Sigma}\alpha_i$. ■

7.4 Proof of Theorem 8

Proof. Let us consider (16) and (22). By taking the norm of the difference one can write

$$
\begin{aligned}
\|m(t) - \mu(t)\| &= \|m(t) - m(t_k) + m(t_k) - \mu(t)\| \\
&\overset{(a)}{\leqslant} \alpha_1 p + \beta_1\varepsilon_1^k + \|m(t_k) - \mu_k + \mu_k - \mu(t)\| \\
&\overset{(b)}{\leqslant} \alpha_3 p + \beta_1\varepsilon_1^k + \|\mu_k - \mu(t)\| \\
&\overset{(c)}{\leqslant} \alpha_3 p + \beta_1\varepsilon_1^k + \|\Gamma\|\,\|z_k - \xi(t)\| \\
&\leqslant \alpha_3 p + \beta_1\varepsilon_1^k + \|\Gamma\|\,\|z_k - \xi(kp)\| \\
&+ \|\Gamma\|\,\|\xi(kp) - \xi(t)\|
\end{aligned} \tag{55}
$$

holds for all $p \in (0, \bar{p}_{\varepsilon_1}]$, $t \in (0, \bar{t}]$, $k = \lfloor t/p \rfloor$, where in (a) we used Lemma 7 with α called α_1, β called β_1 and ε called ε_1, in (b) we used Lemma 6 which allows one to write (33) as $\|m(t_k) - \mu_k\| \leqslant \alpha_2 p$ and we defined $\alpha_3 = 2\max\{\alpha_1, \alpha_2\}$, in (c) we used (20b) and (16b).

Let us consider the term $\|z_k - \xi(kp)\|$ in (55). By solving (16a) and by using (21) one can write

$$
\xi(kp) - z_k = \left(e^{(\Phi_p - I)k} - \Phi_p^k\right) x_0. \tag{56}
$$

From (56) and by using arguments similar to (52) it follows

$$
\begin{aligned}
\|\xi(kp) - z_k\| &\leqslant \|\left(e^{(\Phi_p - I)k} - \Phi_p^k\right)\|\|x_0\| \\
&\leqslant \beta_0\|T_p\left(e^{(\Phi_p - I)k} - \Phi_p^k\right)T_p^{-1}\|\|x_0\|
\end{aligned} \tag{57}
$$

where T_p is given by (31) and we used (53). From Remark 4 one can write

$$
T_p(\Phi_p - I)T_p^{-1} = \begin{bmatrix} \Lambda_1 p & 0 \\ 0 & V - I + \Lambda_2 p \end{bmatrix}, \tag{58}
$$

and then

$$\begin{aligned}
&T_p\left(e^{(\Phi_p-I)k}-\Phi_p^k\right)T_p^{-1}\\
&=e^{T_p(\Phi_p-I)T_p^{-1}k}-(T_p\Phi_pT_p^{-1})^k\\
&=e^{\left(\left[\begin{smallmatrix}\Lambda_1 p & 0\\ 0 & V-I+\Lambda_2 p\end{smallmatrix}\right]\right)k}-\left[\begin{smallmatrix}I+\Lambda_1 p & 0\\ 0 & V+\Lambda_2 p\end{smallmatrix}\right]^k\\
&=\left[\begin{smallmatrix}(e^{\Lambda_1 p})^k & 0\\ 0 & (e^{V-I+\Lambda_2 p})^k\end{smallmatrix}\right]-\left[\begin{smallmatrix}(I+\Lambda_1 p)^k & 0\\ 0 & (V+\Lambda_2 p)^k\end{smallmatrix}\right].
\end{aligned} \tag{59}$$

Considering the Taylor expansion of the exponential function, we have $e^{\Lambda_1 p}=I+\Lambda_1 p+O(p^2)$ and hence being $k=\lfloor t/p\rfloor$,

$$(e^{\Lambda_1 p})^k=(I+\Lambda_1 p)^k+O(p). \tag{60}$$

By using (60) in (59) one has

$$T_p\left(e^{(\Phi_p-I)k}-\Phi_p^k\right)T_p^{-1}=\left[\begin{smallmatrix}\mathrm{O}(p) & 0\\ 0 & (e^{V-I+\Lambda_2 p})^k-(V+\Lambda_2 p)^k\end{smallmatrix}\right]. \tag{61}$$

Since the matrix V is Schur by hypothesis, for sufficiently small p the eigenvalues of $V+\Lambda_2 p$ have magnitude smaller than 1 and $V-I+\Lambda_2 p$ is Hurwitz (and hence the eigenvalues of $e^{V-I+\Lambda_2 p}$ also have magnitude smaller than 1). Consequently, there exist constants $\beta_2,\beta_3\in\mathbb{R}_0^+$, $\varepsilon_2\in(0,1)$ and $\bar{p}_{\varepsilon_2}\in\mathbb{R}^+$ such that

$$\|(e^{V-I+\Lambda_2 p})^k\|\leqslant\beta_2\varepsilon_2^k \tag{62a}$$
$$\|(V+\Lambda_2 p)\|^k\leqslant\beta_3\varepsilon_2^k \tag{62b}$$

for all $p\in(0,\bar{p}_{\varepsilon_2}]$. By taking the norms on both sides of (61) and by using (62) it follows that there exists a constant $\alpha_4\in\mathbb{R}^+$, such that

$$\|T_p\left(e^{(\Phi_p-I)k}-\Phi_p^k\right)T_p^{-1}\|\leqslant\alpha_4 p+\beta_4\varepsilon_2^k. \tag{63}$$

where $\beta_4=2\max\{\beta_2,\beta_3\}$. Then from (57) with (63) the following inequality

$$\|\xi(kp)-z_k\|\leqslant\alpha_5 p+\beta_5\varepsilon_2^k \tag{64}$$

with $\alpha_5=\alpha_4\beta_0\|x_0\|$, $\beta_5=\beta_0\beta_4\|x_0\|$, holds for all $p\in(0,\bar{p}_{\varepsilon_2}]$, $t\in(0,\bar{t}]$, $k=\lfloor t/p\rfloor$.

By substituting (64) in (55) it follows

$$\|m(t)-\mu(t)\|\leqslant\alpha_6 p+\beta_6\varepsilon_3^k+\|\Gamma\|\,\|\xi(kp)-\xi(t)\| \tag{65}$$

with $\alpha_6=2\max\{\alpha_3,\alpha_5\}$, $\beta_6=2\max\{\beta_1,\beta_5\}$ and $\varepsilon_3=\max\{\varepsilon_1,\varepsilon_2\}$ and for all $p\in(0,\bar{p}_{\varepsilon_3}]$ with $\bar{p}_{\varepsilon_3}=\min\{\bar{p}_{\varepsilon_1},\bar{p}_{\varepsilon_2}\}$.

By considering the last term in (65) and the solution of (16a), for any $t\in[kp,kp+p)$ one can write

$$\begin{aligned}
\xi(t)-\xi(kp)&=\left(e^{\frac{1}{p}(\Phi_p-I)(t-kp)}-I\right)\xi(kp)\\
&=\left(e^{(\Phi_p-I)(\frac{t}{p}-k)}-I\right)\xi(kp).
\end{aligned} \tag{66}$$

By using (53) in (66) it follows

$$\begin{aligned}
&\|\xi(t)-\xi(kp)\|=\|\left(e^{(\Phi_p-I)(\frac{t}{p}-k)}-I\right)e^{(\Phi_p-I)k}x_0\|\\
&=\|\left(e^{(\Phi_p-I)\frac{t}{p}}-e^{(\Phi_p-I)k}\right)x_0\|\\
&\leqslant\beta_0\|T_p\left(e^{(\Phi_p-I)\frac{t}{p}}-e^{(\Phi_p-I)k}\right)T_p^{-1}\|\|x_0\|\\
&\overset{(a)}{=}\beta_0\|\left[\begin{smallmatrix}(e^{\Lambda_1 p})^{\frac{t}{p}}-(e^{\Lambda_1 p})^k & 0\\ 0 & (e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\end{smallmatrix}\right]\|\,\|x_0\|\\
&=\beta_0\|\left[\begin{smallmatrix}e^{\Lambda_1 t}-e^{\Lambda_1 kp} & 0\\ 0 & (e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\end{smallmatrix}\right]\|\,\|x_0\|
\end{aligned} \tag{67}$$

where in (a) we used arguments similar to those used for (59). By taking the Taylor series one can write

$$e^{\Lambda_1 t}-e^{\Lambda_1 kp}=\Lambda_1(t-kp)+\mathrm{O}(p^2)=\mathrm{O}(p). \tag{68}$$

Since V is Schur then $V-I$ is Hurwitz and there exists a sufficiently small p such that $V-I+\Lambda_2 p$ is Hurwitz and $e^{V-I+\Lambda_2 p}$ is Schur. Then there exists a constant $\beta_7\in\mathbb{R}_0^+$ such that

$$\begin{aligned}
&\|(e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\|\\
&\qquad\leqslant\|(e^{V-I+\Lambda_2 p})\|^{\frac{t}{p}}+\|(e^{V-I+\Lambda_2 p})\|^k\\
&\qquad\leqslant\beta_7\varepsilon_2^k+\beta_2\varepsilon_2^k.
\end{aligned} \tag{69}$$

By using (68) and (69) in (67) it follows that there exists a constant $\alpha_7\in\mathbb{R}^+$ such that

$$\|\xi(t)-\xi(kp)\|\leqslant\alpha_7 p+\beta_8\varepsilon_2^k \tag{70}$$

with $\beta_8=2\beta_0\|x_0\|\max\{\beta_2,\beta_7\}$. By substituting (70) in (65), it follows that (23) holds with $\alpha=2\max\{\alpha_6,\alpha_7\}$, $\beta=2\max\{\beta_6,\beta_8\}$ and $\varepsilon=\max\{\varepsilon_2,\varepsilon_3\}$ and for all $p\in(0,\bar{p}_\varepsilon]$ with $\bar{p}_\varepsilon=\min\{\bar{p}_{\varepsilon_2},\bar{p}_{\varepsilon_3}\}$.

■

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