Model reduction for switched DAEs

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Model reduction setup

Given: Large scale switched DAE

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \qquad x(t_0^-) \in \mathcal{X}_0 \subseteq \mathbb{R}^n \\ y &= C_\sigma x + D_\sigma u \end{aligned}$$

- $\sigma : [t_0, t_f) \to \mathcal{Q} := \{0, 1, 2, \ldots, \mathtt{m}\}$ known switching signal
- $(E_k, A_k, B_k, C_k, D_k) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{p\times n} \times \mathbb{R}^{p\times m}$ mode $k \in \mathcal{Q}$
- $n \gg m, p$

Goal

Find reduced switched system with similar input-output behavior

Challenges

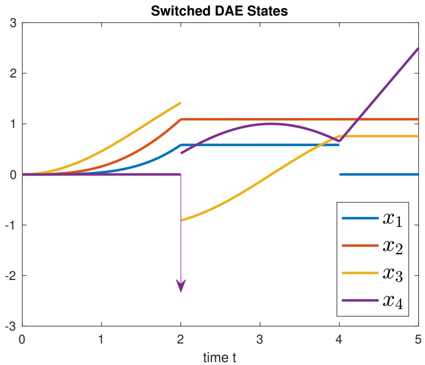
No restriction on index of each DAE mode
$\to$ Jumps and Dirac impulses must be preserved

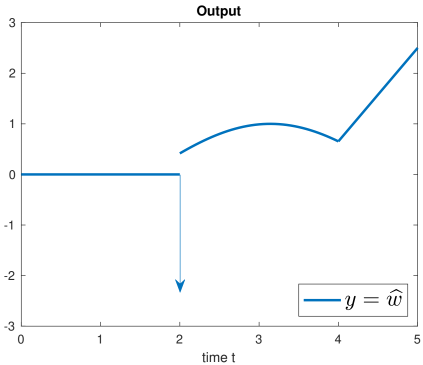
Example (Example 1 from Hossain & T. 2023 (preprint))

Switched DAE with $n = 4$, $m = p = 1$

$\sigma(t) = 0$ on $[0, 2)$, $\sigma(t) = 1$ on $[2, 4)$, $\sigma(t) = 2$ on $[4, 5)$

Simulation for $x_0 = 0$ and $u(t) = \sin(t)$:





Overview reduction method

Step 1

Rewrite swDAE as equivalent swODE with jumps and Diracs (same state dimension)

Step 2 (cf. Hossain & T. 2023, Automatica)

Remove unobservable and uncontrollable states (reduced realization)

Step 3

Decouple output Dirac inducing states (decoupling assumption)

Step 4 (cf. Hossain & T. 2024, IEEE TAC)

Midpoint balanced truncation on non-output-Dirac-inducing states

Step 1: swDAE $\to$ swODE with jumps and Diracs

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u, \quad x(t_0^-) = x_0, \quad y = C_\sigma x + D_\sigma u \tag{swDAE}$$

Theorem

Let $\boldsymbol{U}^\nu := (u^\top, \dot{u}^\top, \ldots, u^{(\nu-1)^\top})^\top$, then there exist matrices such that

$$\begin{aligned}
\dot{z} &= A_k^{\text{diff}} z + B_k^{\text{diff}} u, \quad \text{on } (s_k, s_{k+1}), \quad z(t_0^-) = x_0 \\
z(s_k^+) &= J_k^x z(s_k^-) + J_k^v \boldsymbol{U}^{\nu_k - 1}(s_k^-), \quad k \geq 0, \\
w &= C_k z + D_k u + \boldsymbol{D}_k^{\text{imp}} \boldsymbol{U}^{\nu_k}, \quad \text{on } (s_k, s_{k+1}), \\
w[s_k] &= \sum_{i=0}^{\nu_k - 2} \left[C_k^i z(s_k^-) + \boldsymbol{D}_k^{\text{imp}-} \boldsymbol{U}^{\nu_k - 1}(s_k^-) - \boldsymbol{D}_k^{\text{imp}+} \boldsymbol{U}^{\nu_k}(s_k^+) \right] \delta_{s_k}^{(i)}
\end{aligned} \tag{swODE}$$

has the same input-output behavior as (swDAE)

Key ingredient: Quasi-Weierstrass form

Step 2: Reduced realization

$$
\begin{aligned}
\dot z &= A_k^{\mathsf{diff}} z + B_k^{\mathsf{diff}} u,\\
z(t_0^-) &= x_0 \in \mathcal{X}_0\\
z(s_k^+) &= J_k^x z(s_k^-) + J_k^v v_k,\\
w &= C_k z,\\
w[s_k] &= \sum_{i=0}^{\rho_k} C_k^i z(s_k^-)\delta_{s_k}^{(i)}
\end{aligned}
\quad\rightarrow\quad
\begin{aligned}
\dot{\widehat z} &= \widehat A_k \widehat z + \widehat B_k u,\\
\widehat z(t_0^-) &= \widehat x_0 = \Pi^{\mathcal{X}_0} x_0 \in \widehat{\mathcal{X}}_0\\
\widehat z(s_k^+) &= \widehat J_k^x \widehat z(s_k^-) + \widehat J_k^v v_k,\\
\widehat w &= \widehat C_k \widehat z,\\
\widehat w[s_k] &= \sum_{i=0}^{\rho_k} \widehat C_k^i \widehat z(s_k^-)\delta_{s_k}^{(i)}
\end{aligned}
$$

Key ingredients

- Extended reachability subspaces $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \ldots, \overline{\mathcal{R}}_{\mathtt{m}}$
- Restricted unobservability subspaces $\underline{\mathcal{U}}_{\mathtt{m}}, \underline{\mathcal{U}}_{\mathtt{m}-1}, \ldots, \underline{\mathcal{U}}_0$
- Weak Kalman decomposition based on the subspace pairs $(\overline{\mathcal{R}}_k, \underline{\mathcal{U}}_k)$

Step 2 - The subspaces $\overline{\mathcal{R}}_k$ and $\underline{\mathcal{U}}_k$

$\langle A \mid \mathcal{V} \rangle$ = smallest A-invariant subspace containing $\mathcal{V}$
$\langle \mathcal{V} \mid A \rangle$ = larges A-invariant subspace contained in $\mathcal{V}$

Ext. Reachability Subspaces

$$\overline{\mathcal{R}}_k := \mathcal{R}_k + \langle A_k^{\text{diff}} \mid J_k^x \overline{\mathcal{R}}_{k-1} + \operatorname{im} J_k^v \rangle, \quad k = 0, \ldots, \mathtt{m},$$

where $\mathcal{R}_k := \langle A_k^{\text{diff}} \mid \operatorname{im} B_k^{\text{diff}} \rangle$, $\quad \overline{\mathcal{R}}_0 := \mathcal{X}_0$

Restr. Unobservability Subspaces

$$\underline{\mathcal{U}}_k = \mathcal{U}_k \cap \langle (J_{k+1}^x)^{-1} \underline{\mathcal{U}}_{k+1} \cap \mathcal{U}_{k+1}^{\text{imp}} \mid A_k^{\text{diff}} \rangle, \quad k = \mathtt{m}-1, \ldots, 1, 0$$

where $\mathcal{U}_k := \langle \ker C_k \mid A_k^{\text{diff}} \rangle$, $\quad \mathcal{U}_k^{\text{imp}} := \ker[C_k^0 / C_k^1 / \ldots / C_k^{\rho_k}]$, $\quad \underline{\mathcal{U}}_{\mathtt{m}} := \mathcal{U}_{\mathtt{m}}$

Step 2: The weak Kalman decomposition

Lemma (Hossain & T. 2022, MATHMOD)

Given system (A, B, C) *and* A*-invariant subspaces* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$, $\underline{\mathcal{U}} \subseteq \ker C$.
Choose $T = [T_1, T_2, T_3, T_4]$ *invertible such that*

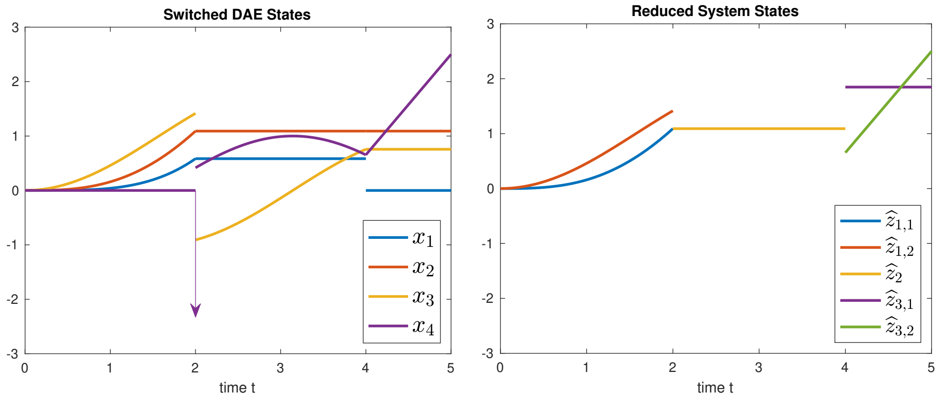
$$\operatorname{im} T_1 = \overline{\mathcal{R}} \cap \underline{\mathcal{U}}, \quad \operatorname{im}[T_1, T_2] = \overline{\mathcal{R}}, \quad \operatorname{im}[T_1, T_3] = \underline{\mathcal{U}}.$$

Then $(T^{-1}AT, T^{-1}B, CT)$ *has the form*

$$\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, [0\ C_2\ 0\ C_4] \right).$$

In particular, (A, B, C) *and* (A_{22}, B_2, C_2) *have identical input-output behavior.*

Example 1 - Reduced states after Step 2



Step 3 - Decoupling assumption

$$\dot{\widehat{z}} = \widehat{A}_k \widehat{z} + \widehat{B}_k u, \quad \widehat{z}(t_0^-) = \widehat{x}_0 \in \widehat{\mathcal{X}}_0,$$
$$\widehat{z}(s_k^+) = \widehat{J}_k^x \widehat{z}(s_k^-) + \widehat{J}_k^v v_k,$$
$$\widehat{w} = \widehat{C}_k \widehat{z}, \quad \widehat{w}[s_k] = \sum_{i=0}^{\rho_k} \widehat{C}_k^i \widehat{z}(s_k^-) \delta_{s_k}^{(i)}$$

Decoupling Assumption

$\exists$ coordinate transformations $T_k = [T_k^{\overline{\text{imp}}}, T_k^{\text{imp}}]$ such that

$$\widehat{C}_k^i T_{k-1}^{\overline{\text{imp}}} = 0, \quad T_k^{-1} \widehat{A}_k T_k = \begin{bmatrix} A_k^{\overline{\text{imp}}} & 0 \\ 0 & A_k^{\text{imp}} \end{bmatrix}, \quad T_k^{-1} \widehat{J}_k^x T_{k-1} = \begin{bmatrix} J_k^{x^{\overline{\text{imp}}}} & 0 \\ 0 & J_k^{x^{\text{imp}}} \end{bmatrix}$$

Step 3 - Decoupling assumption

$$\dot z^{\overline{\text{imp}}} = A_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad z^{\overline{\text{imp}}}(t_0^-) = x_0^{\overline{\text{imp}}} \in \mathcal{X}_0^{\overline{\text{imp}}},$$

$$\dot z^{\text{imp}} = A_k^{\text{imp}} z^{\text{imp}} + B_k^{\text{imp}} u, \quad z^{\text{imp}}(t_0^-) = x_0^{\text{imp}} \in \mathcal{X}_0^{\text{imp}},$$

$$z^{\overline{\text{imp}}}(s_k^+) = J_k^{x^{\overline{\text{imp}}}} z^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} v_k,$$

$$z^{\text{imp}}(s_k^+) = J_k^{x^{\text{imp}}} z^{\text{imp}}(s_k^-) + J_k^{v^{\text{imp}}} v_k,$$

$$\widehat{w} = C_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}} + C_k^{\text{imp}} z^{\text{imp}}, \quad \widehat{w}[s_k] = \sum_{i=0}^{\rho_k} C_k^{\text{imp},i} z^{\text{imp}}(s_k^-)\delta_{s_k}^{(i)}$$

Decoupling Assumption

$\exists$ coordinate transformations $T_k = [T_k^{\overline{\text{imp}}}, T_k^{\text{imp}}]$ such that

$$\widehat{C}_k^i T_{k-1}^{\overline{\text{imp}}} = 0, \quad T_k^{-1}\widehat{A}_k T_k = \begin{bmatrix} A_k^{\overline{\text{imp}}} & 0 \\ 0 & A_k^{\text{imp}} \end{bmatrix}, \quad T_k^{-1}\widehat{J}_k^x T_{k-1} = \begin{bmatrix} J_k^{x^{\overline{\text{imp}}}} & 0 \\ 0 & J_k^{x^{\text{imp}}} \end{bmatrix}$$

Step 4 - Midpoint balanced truncation

$$\begin{aligned}
\dot{z}^{\overline{\text{imp}}} &= A_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad z^{\overline{\text{imp}}}(t_0^-) = x_0^{\overline{\text{imp}}} \in \mathcal{X}_0^{\overline{\text{imp}}}, \\
z^{\overline{\text{imp}}}(s_k^+) &= J_k^{x^{\overline{\text{imp}}}} z^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} v_k, \\
\widehat{w} &= C_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}}
\end{aligned} \tag{swODE}$$

Lemma (Input-dependent jumps)

$z^{\overline{\text{imp}}}$ *solves (swODE)* $\iff$ $z^{\overline{\text{imp}}} = z_u + z_v$, *where*

z_u *is the solution of (swODE) with* $v_k = 0$ *and* $x_0^{\overline{\text{imp}}} = 0$ *and*

z_v *is the solution of (swODE) with* $u = 0$ $\hookrightarrow$ *discrete-time system*

Step 4 - Midpoint balanced truncation

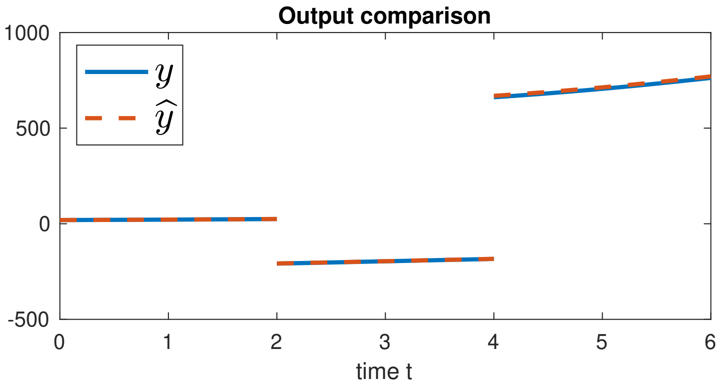
$$\begin{aligned} \dot{z}^{\overline{\text{imp}}} &= A_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}} + B_k^{\overline{\text{imp}}} u, \quad z^{\overline{\text{imp}}}(t_0^-) = x_0^{\overline{\text{imp}}} \in \mathcal{X}_0^{\overline{\text{imp}}}, \\ z^{\overline{\text{imp}}}(s_k^+) &= J_k^{x^{\overline{\text{imp}}}} z^{\overline{\text{imp}}}(s_k^-) + J_k^{v^{\overline{\text{imp}}}} v_k, \\ \widehat{w} &= C_k^{\overline{\text{imp}}} z^{\overline{\text{imp}}} \end{aligned} \qquad \text{(swODE)}$$

Midpoint balanced truncation method

1. Calculate midpoint reachability and observability Gramians for z_u, cf. Hossain & T. 2024, TAC
2. Calculate suitable discrete-time reachability Gramians for z_v
3. Define overall reachability Gramians as weighted sum of midpoint and discrete-time reachability Gramians
4. Apply balanced truncation with respect to Gramians, cf. Hossain & T. 2023, Automatica

Example 2 (Illustration of Step 4)

(swODE) size: $n_0 = 50$, $n_1 = 60$, $n_2 = 40$, $m = p = 1$, $\dim \mathcal{X}_0 = 5$
Truncation balance for Hankel singular values: $10^{-3} \rightsquigarrow \widehat{n}_0 = 8$, $\widehat{n}_1 = 10$, $\widehat{n}_2 = 6$
Simulations for input $u(t) = \cos(t)$ and with random initial value:



Summary

$$
\begin{aligned}
E_\sigma \dot{x} &= A_\sigma x + B_\sigma u, \quad x(t_0^-) = x_0 \in \mathcal{X}_0, \\
y &= C_\sigma x + D_\sigma u
\end{aligned}
$$

- Model reduction method for general (regular) switched DAEs - arbitrary index
- Consideration of finite-time interval - no stability assumptions
- Properly handles jumps, Dirac impulses and non-zero initial values
- Matlab implementation available on Zenodo `doi:10.5281/zenodo.8133789`

Remaining issues

1. No guaranteed error bounds
2. Steps 1 and 2 needs exact rank decisions
3. Decoupling assumption in Step 3 not constructive
4. Step 4 needs large matrix exponentials
5. Switching signal needs to been known a priori