A novel two stages funnel controller limiting the error derivative*

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Abstract

As a powerful adaptive control method for the output tracking problem, funnel control has attracted considerable attention in theoretical research and engineering practice. The funnel control strategy can guarantee both transient behavior and arbitrary good accuracy. A noticeable shortcoming is however that the derivative of the tracking error may become unnecessarily large resulting in a bouncing behavior of the tracking error between the funnel boundaries. To avoid this phenomenon, we present a novel two stages funnel control scheme to solve the output-tracking control problem for uncertain nonlinear systems with relative degree one and stable internal dynamics. This new scheme defines the control input in terms of a desired error derivative while still ensuring that the tracking error evolves within the prescribed funnel. In particular, we can quantify the range of the error derivative with a derivative funnel in terms of the known bounds of the system dynamics. Furthermore, we extend our approach to the situation where input saturations are present and extend the control law outside the funnel to ensure well-defined behavior in case the input saturations are too restrictive to keep the error within the funnel.

Keywords: Output tracking; Nonlinear systems; Relative degree one; Input saturations

1. Introduction

Funnel control is a well-established universal control method to achieve output tracking of unknown nonlinear relative degree one systems with arbitrary prespecified tracking accuracy. Compared to the other adaptive control methods like $\lambda-$tracking [1] or high-gain adaptive control, funnel controllers can ensure prespecified transient behavior and the boundedness of the adaptive gain. After the pioneering paper [2] there have been numerous extensions on funnel controller; we refer the reader to the survey [3] for the historical context and some early extensions and to [4] for a recent unifying approach encompassing most existing extensions. Applications of funnel control and other adaptive control strategies in a mechatronics context have been extensively discussed in [5].

The key feature of funnel control is to guarantee the transient behavior of the closed loop (see Figure 1 and Figure 3) which usually can not be found in other control methods, like sliding mode control, PI control, and fuzzy control. Transient behavior has been addressed in the context of prescribed performance control (PPC) [6, 7], which however requires a special (known) structure of the nonlinear dynamics and does not consider internal dynamics. Furthermore, in contrast to PPC, the funnel controller is an output-feedback controller and not a state-feedback controller (and also does not invoke any observers), which also distinguishes it from approaches utilizing barrier Lyapunov functions [8] and model reference adaptive control (MRAC) [9]; the more recent MRAC approach in [10] does however address input contraints and prescribed performance.

To motivate our novel approach and to highlight the shortcomings of the existing funnel controller methods, we consider the following simple scalar linear system

$$\dot{y} = ay + bu,\ a \in \mathbb{R},\ b > 0,\ y(0) = y^0, \tag{1}$$

with $a = 2$, $b = 1$, $y^0 = 0$. This system has a so-called high-gain property, i.e. the simple proportional feedback $u(t) = -ky(t)$ stabilizes the system for sufficiently large gain k (here any $k > a/b = 2$ is suitable). If the system parameters a and b are unknown, it may not be immediately clear which value for k is "sufficiently large"; furthermore, the "correct" (and not too large) choice of k to achieve a desired transient behavior and a desired final accuracy for tracking an output reference y_{ref} is not straightforward. Funnel control can resolve these problems by considering an "adaptive" gain in the (error) feedback

$$u(t) = -k(t)e(t), \tag{2}$$

where $e(t) := y(t) - y_{\mathrm{ref}}(t)$ and $k(t)$ is chosen adaptively based on a prespecified (time-varying) error bound ψ :

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$[0, \infty) \to (0, \infty)$:

$$k(t) := \frac{1}{\psi(t) - |e(t)|}. \tag{3}$$

The intuition behind this simple control law is that whenever the tracking error approaches the boundary of the funnel

$$\mathscr{F}_\psi := \{ \ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \ | \ |e(t)| < \psi(t) \ \},$$

i.e. $\psi(t) - |e(t)|$ becomes small, then the gain $k(t)$ gets very large and the high gain property of the system ensures that the magnitude of the error decreases.

The funnel controller has been shown to ensure that the error remains within the funnel and hence approximate tracking is achieved with prespecified transient behavior and any desired final accuracy. Interestingly, as a time-varying feedback rule, the internal-model-principle does *not apply*, i.e. it is possible to achieve asymptotic tracking of an arbitrary reference signal, see [11, 12, 13]; however, in practical applications, it may still be advantageous to add an internal model if the generator of the reference signal is known [5, Ch. 7].

For $y_{\text{ref}}(t) := \sin(50t)$ and $\psi(t) := 0.8e^{-5t} + 0.2$ the application of the funnel controller (3) to system (1) is illustrated in Figure 1.

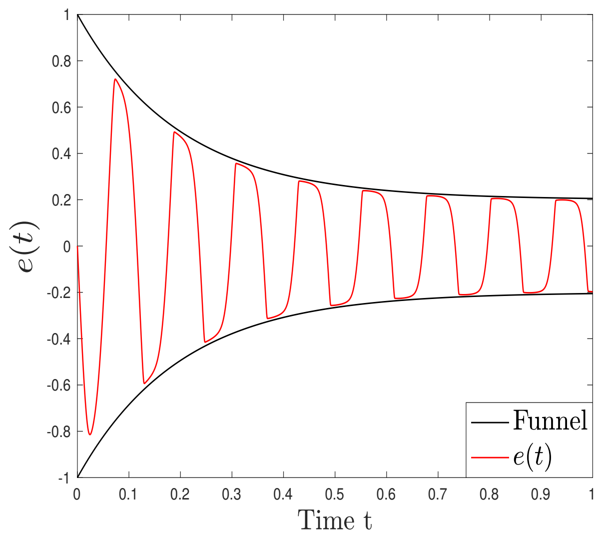


Figure 1: Traditional funnel controller (3).

This simulation shows clearly that the funnel controller (3) ensures that the error evolves within the prespecified funnel; however, the funnel controller pays no attention to the evolution of the error derivative, which results in a possibly undesired bouncing behavior of the error signal between the upper and lower funnel boundary.

The bouncing problem of funnel control may also occur for input-saturated systems. For the simple scalar system (1) with input saturation $|u(t)| \leq \hat{u}$ it can be guaranteed (straightforward worst-case analysis) that the error can be kept in the funnel if $\hat{u} > 0$ satisfies

$$b\hat{u} \geq |a|[||\psi||_\infty + ||y_{\text{ref}}||_\infty] + ||\dot{y}_{\text{ref}}||_\infty + \Lambda =: B_{\psi, y_{\text{ref}}} + \Lambda,$$

where Λ denotes the Lipschitz constant of ψ. However, with this choice of $\hat{u}$, the error derivative $\dot{e}$ evolves in general within a rather large range given by

$$|\dot{e}(t)| \leq b\hat{u} + |a|[||\psi||_\infty + ||y_{\text{ref}}||_\infty] + ||\dot{y}_{\text{ref}}||_\infty = b\hat{u} + B_{\psi, y_{\text{ref}}}.$$

This range is much larger than the necessary range $|\dot{e}(t)| \leq \Lambda$ which is actually needed to stay within the funnel. Consequently, an ad-hoc limiting of the input will not resolve the bouncing problem. In fact, simulations in [13] already showed that the bouncing behavior may lead to numerical issues for tight funnels. Apart from the latter observation, there seem to be no discussions in the literature to limit the magnitude of the error derivative. However, by taking the future funnel into account as proposed in [14] the magnitude of the error derivative may be reduced, but this has only a significant effect when the funnel is shrinking quickly and does not avoid bouncing in general.

In many applications, the bouncing behavior significantly degrades the control performance; in particular, limiting the bouncing, or the magnitude of the error derivative in general, is a desirable additional control objective. One of the major advantages of funnel control is its simplicity and universality (it does not depend at all on the to be controlled system model), but this feature is also a disadvantage, because the available knowledge about the system model cannot be utilized easily to further improve the performance of the controller. Our approach resolves this dilemma by directly incorporating known (not necessarily tight) bounds on the systems dynamics in the control design with the goal to additionally limit the range of the error derivative.

Towards this goal we introduce a novel two stages error-derivative-limiting (EDL) funnel controller for relative-degree one systems (cf. Remark 1). The first step is the design of an "optimal" convergence rate $e_{\text{op}}(e(t), t)$ depending on the current tracking error and the current funnel. The idea is to ensure that in the closed loop $\dot{e}(t)$ is not too far away from $e_{\text{op}}(t)$ because $\dot{e}(t) = e_{\text{op}}(e(t), t)$ would be sufficient to stay within the funnel (see Definition 1). The second step, which we call *orientated* funnel controller, is then to calculate the input $u(t)$ in terms of $e_{\text{op}}(e(t), t)$, the reference signal, and known bounds on the system's dynamics. The overall controller structure is illustrated in Figure 2.

To illustrate this two stage design let us discuss a simplified version of the actual EDL funnel controller (presented in Section 3) for the simple linear system (1). One possible choice for e_{op} is

$$e_{\text{op}}(t) = \frac{e(t)}{\psi(t)} \dot{\psi}(t). \tag{4}$$

The intuition behind this choice is that if $e(t)$ is close to the funnel boundary $\psi(t)$, then $e_{\text{op}}(t)$ is approaching $\dot{\psi}$ (i.e. in

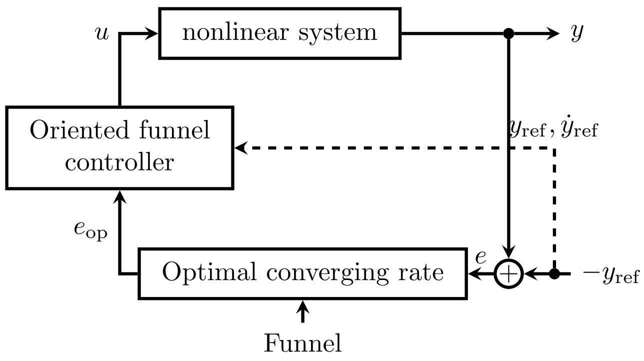


Figure 2: Overall system structure.

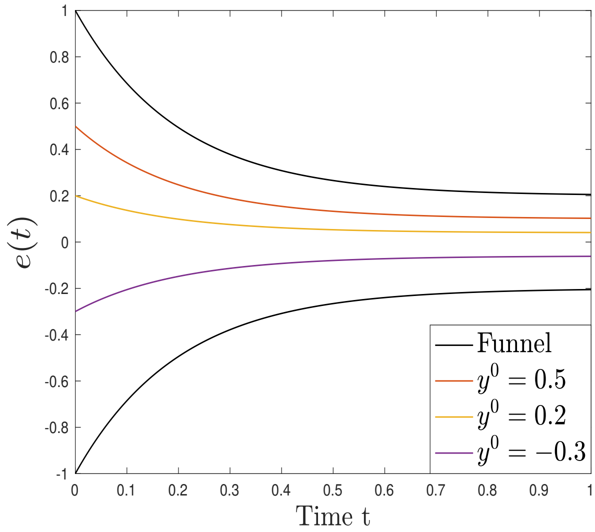


Figure 3: Funnel controller (5) with e_{op} (4).

case $e(t) > 0$, the error decreases as quickly as the funnel), on the other hand, if $e(t)$ is far away from the funnel boundary or $\dot{\psi}$ is small then also $e_{\text{op}}(t)$ is small. In fact, it is easily seen that if $\dot{e}(t) = e_{\text{op}}(e(t), t)$ then the funnel $\mathcal{F}_\psi$ is positively invariant, i.e. if the initial error $e(0)$ is in the funnel, then the error $e(t)$ will stay within the funnel for all times. Under the assumption that we have full knowledge of all parameters in (1) (which is only assumed here for illustrating purposes and will *not* be assumed for the actual controller proposed in Section 4) as well as access to the derivative of the reference signal, we can define the following oriented funnel controller

$$u(t) = \frac{e_{\text{op}}(t) + \dot{y}_{\text{ref}}(t) - a \cdot y(t)}{b}. \tag{5}$$

Plugging controller (5) into (1), we obtain

$$\begin{aligned}
\dot{e} &= \dot{y} - \dot{y}_{\text{ref}} \\
&= a \cdot y + b\frac{e_{\text{op}} + \dot{y}_{\text{ref}} - a \cdot y}{b} - \dot{y}_{\text{ref}} \\
&= e_{\text{op}}.
\end{aligned}$$

Simulation results for this case are shown in Figure 3, which clearly show that the error remains within the funnel without exhibiting any bouncing behavior.

The remainder of this paper is structured as follows. In Section 2 we present the actual system class with corresponding assumptions and highlight some boundedness properties of this system class. Afterward, in Section 3 we present our proposed optimal convergence rate for the two stages EDL funnel controller. In Section 4 we motivate and propose our oriented funnel controller in terms of the optimal convergence rate and prove that this choice indeed ensures that the error evolves within the funnel. Furthermore, we also analyze the behavior of the error derivative and derive some bounds for the error derivative in Section 5. The theoretical results are illustrated with some simulations. Finally, in Section 6 we also study the case of input saturations and how to adjust the EDL funnel controller accordingly. In particular, we provide conditions under which it can be guaranteed that the error remains within the funnel. We also study the case, when the input saturations are too restrictive to keep the error within the funnel; in this case, we provide an outer funnel in which the error is guaranteed to stay within.

2. Problem formulation

We consider nonlinear system of the following input affine form

$$\begin{aligned}
\dot{y} &= f\big(p_f, y, z\big) + g\big(p_g, y, z\big)u, \qquad & y(0) &= y^0, \qquad (6a)\\
\dot{z} &= h\big(p_h, y, z\big), & z(0) &= z^0, \qquad (6b)
\end{aligned}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$ represents the output of the controlled system, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ denotes the control input and $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1}$ is the internal state of order $n-1 \in \mathbb{N}$. The functions f, g: $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and h: $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are assumed to be locally Lipschitz continuous; $p_f, p_g, p_h : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ are locally integrable perturbations (and/or unknown d-dimensional, time-varying parameters).

Furthermore, we make the following additional assumptions for (6).

(**A**1) Relative degree one with positive "high frequency gain": $g(p_g, y, z) > 0$ for all p_g, y and z.

(**A**2) Bounded perturbations: p_f, p_g and p_h are assumed to be globally bounded on $\mathbb{R}_{\geq 0}$ and we assume knowledge of these (not necessarily tight) bounds, say $p_f^{\max}$, $p_g^{\max}$, $p_h^{\max}$, respectively.

(**A**3) BIBO-stability of internal dynamics: There exists a continuous function $b_z : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all continuous p_h, y the solutions of (6b) satisfy

$$\|z(t)\| \leq b_z\big(\|p_{h[0,t)}\|_\infty, \|y_{[0,t)}\|_\infty, \|z^0\|\big).$$

Furthermore, assume $z_0 \in Z_0$ for some bounded $Z_0 \subset \mathbb{R}^{n-1}$.

Remark 1. Our controller design actually works without change for systems of the general form

$$\dot{x} = G(p_G, x, u), \quad y = H(p_H, x), \tag{7}$$

with $G : \mathbb{R}^d \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ and $H : \mathbb{R}^d \times \mathbb{R}^n \to \mathbb{R}$ as long as there is a (nonlinear) coordinate transformation $x \mapsto (y, z)$ which transforms (7) into (6). The existence of such a coordinate transformation is strongly related to the property that (7) has relative degree one [15]. However, we need knowledge about some system bounds, see Lemma 2, which are formulated in terms of the system description (6), hence we only consider the latter instead of the original form (7).

The overall control objective is to ensure that the output y of (6) follows a given reference output $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ in such a way that the error $e := y - y_{\text{ref}}$ satisfies the time-varying error bound $\psi_-(t) \leq e(t) \leq \psi_+(t)$ for some given functions $\psi_\pm : \mathbb{R}_{\geq 0} \to \mathbb{R}$. In other words, we want to achieve that the error evolves in the (possibly unsymmetric) funnel

$$\mathscr{F}_{\psi_\pm} := \{ \ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \ | \ \psi_-(t) \leq e \leq \psi_+(t) \ \}. \tag{8}$$

We will make the following assumptions for the funnel boundaries and the reference signal.

(**PR**1) The funnel boundaries $\psi_+ : \mathbb{R}_{\geq 0} \to (0, \infty)$, $\psi_- : \mathbb{R}_{\geq 0} \to (-\infty, 0)$ are continuously differentiable and bounded with bounded derivative. Furthermore, we assume that $\psi_{+(-)}$ is convex (concave).

(**PR**2) The reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable, bounded and with bounded derivative.

(**PR**3) The initial tracking error $e^0 := y^0 - y_{\text{ref}}(0)$ satisfies $\psi_-(0) \leq e^0 \leq \psi_+(0)$.

Note that the convexity/concavity assumption in (**PR**1) (together with boundedness of $\psi_\pm$) implies that

$$\forall t \geq 0 : \quad \dot{\psi}_+(t) \leq 0 \quad \text{and} \quad \dot{\psi}_-(t) \geq 0 \tag{9}$$

and consequently

$$\forall t \geq 0 : \quad 0 < \psi_+(t) \leq \psi_+(0) \quad \text{and} \quad 0 > \psi_-(t) \geq \psi_0(0).$$

A consequence of the above assumptions is the existence of certain bounds which we will later use in the controller design and analysis of the closed loop.

Lemma 2. *Consider a nonlinear system (6) satisfying assumptions (**A**1)-(**A**3) together with funnel boundaries and a reference signal satisfying (**PR**1)-(**PR**3). Then there exist constants $Y_{\max}$, $Y_{\min}$, $Z_{\max}$, $G_{\min}$, $F_{\max}$, $F_{\min} \in \mathbb{R}$ such that*

$$\begin{aligned}
&Y_{\max} > \sup_{t \geq 0} y_{\text{ref}}(t) + \psi_+(0), \\
&Y_{\min} < \inf_{t \geq 0} y_{\text{ref}}(t) + \psi_-(0), \\
&Z_{\max} > \max_{\|p_h\| \leq p_h^{\max}, y \in [Y_{\min}, Y_{\max}], z_0 \in Z_0} b_z(\|p_h\|, |y|, \|z_0\|), \\
&0 < G_{\min} \leq \min_{\|p_g\| \leq p_g^{\max}, y \in [Y_{\min}, Y_{\max}], |z| \leq Z_{\max}} g(p_g, y, z), \\
&F_{\max} \geq \max_{\|p_f\| \leq p_f^{\max}, y \in [Y_{\min}, Y_{\max}], |z| \leq Z_{\max}} f(p_f, y, z), \\
&F_{\min} \leq \min_{\|p_f\| \leq p_f^{\max}, y \in [Y_{\min}, Y_{\max}], |z| \leq Z_{\max}} f(p_f, y, z).
\end{aligned}$$

Proof. This is a direct consequence of the boundedness of y_{ref} and the properties of continuous functions considered on compact domains. □

The utility of the above Lemma is that as long as $e(t)$ remains in the funnel, it can be concluded (utilizing monotonicity of the funnel boundaries) that $y(t) \in (Y_{\min}, Y_{\max})$, which then can be used to conclude that $|z(t)| < Z_{\max}$, $g(p_g(t), y(t), z(t)) \geq G_{\min}$ and $f(p_f(t), y(t), z(t)) \in [F_{\min}, F_{\max}]$. Furthermore, the bounds can be sharpened by choosing $Y_{\max}$ and $Y_{\min}$ as follows:

$$\begin{aligned}
Y_{\max} &> \sup_{t \geq 0} (y_{\text{ref}}(t) + \psi_+(t)), \\
Y_{\min} &< \inf_{t \geq 0} (y_{\text{ref}}(t) + \psi_-(t)).
\end{aligned}$$

However, the more conservative bounds in Lemma 2 are required when considering input saturations in Section 6.

Remark 3. Existence of a solution of (6) considered on the open domain $\mathcal{D} := (Y_{\min}, Y_{\max}) \times \{ \ z \ | \ \|z\| < Z_{\max} \ \}$ is guaranteed by standard ODE theory for any (continuously defined) feedback rule; this solution is in general only defined on a finite time interval $[0, \omega)$. Furthermore, it is also well known that if $\omega < \infty$ then the maximal solution leaves any compact subset of the domain $\mathcal{D}$; in particular, by considering the compact set $[Y^*_{\min}, Y_{\max}*] \times \{ \ z \ | \ \|z\| \leq Z^*_{\max} \ \} \subseteq \mathcal{D}$, where $Y^*_{\max} := \sup_{t \geq 0} y_{\text{ref}}(t) + \psi_+(0)$, $Y^*_{\min} := \sup_{t \geq 0} y_{\text{ref}}(t) + \psi_-(0)$, $Z_{\max} := \max_{\|p_h\| \leq p_h^{\max}, y \in [Y_{\min}, Y_{\max}], z_0 \in Z_0} b_z(\|p_h\|, |y|, \|z_0\|)$, we can then conclude that there exists $t \in [0, \omega)$ such that $e(t) = y(t) - y_{\text{ref}}(t) \notin [\psi_-(t), \psi_+(t)]$. Consequently, a maximal solution which remains within the funnel for all $t \in [0, \omega)$ implies that $\omega = \infty$. Note that this is in contrast to classical funnel control theory, where the domain of the ODE is usually restricted to the interior of the funnel (because the classical funnel feedback rule is undefined on the boundary), and hence necessarily any maximal solution evolves within the funnel and showing $\omega = \infty$ requires some extra effort.

3. Optimal converging rate

As motivated in the introduction our approach is based on designing a desired rate of the change of the error signal such that the error remains in the funnel but at the same time the error derivative is not unnecessarily large. In particular, when the error is already in a (possibly time-varying) neighborhood of zero then the error derivative should ideally be zero. On the other hand, closer to the funnel boundary, the desired error derivative should in magnitude be at least so large to prevent crossing the funnel boundary, but not much larger. These intuitive requirements are formalized by the following definition of the "optimal" convergence rate.

Definition 1. Consider a funnel with boundaries $\psi_\pm$ satisfying (**PR**1) and a desired "zero error derivative region"

$$\mathscr{F}_0 := \{\ (e,t) \in \mathbb{R} \times \mathbb{R}_{\geq 0}\ \mid \lambda_-(t) \leq e \leq \lambda_+(t)\ \} \subseteq \mathscr{F}_{\psi_\pm}$$

for some $\lambda_-, \lambda_+ : \mathbb{R}_{\geq 0} \to \mathbb{R}$ with $\psi_-(t) < \lambda_-(t) < 0 < \lambda_+(t) < \psi_+(t)$, for all $t \geq 0$. Any function $e_{\mathrm{op}} : \mathbb{R} \times \mathbb{R}_{\geq 0}, (e,t) \mapsto e_{\mathrm{op}}(e,t)$ is a suitable desired optimal convergence rate if it satisfies the following properties:

(**EO**1) e_{op} is locally Lipschitz continuous.

(**EO**2) $e_{\mathrm{op}}(e,t) = 0$ for all $(e,t) \in \mathscr{F}_0$, i.e. $e_{\mathrm{op}}(e,t) = 0$ if $\lambda_-(t) \leq e \leq \lambda_+(t)$.

(**EO**3) For any continuous non-negative $e : [a,b] \to \mathbb{R}_{\geq 0}$ on some interval $[a,b] \subseteq \mathbb{R}_{\geq 0}$ with $e(a) \leq \psi_+(a)$ the solution $\eta : [a,b] \to \mathbb{R}$ of

$$\dot\eta(t) = e_{\mathrm{op}}(e(t),t), \quad \eta(a) = e(a)$$

satisfies $\eta(t) \leq \psi_+(t)$ for all $t \in [a,b]$. Analogously, for any continuous non-positive $e : [a,b] \to \mathbb{R}_{\leq 0}$ with $e(a) \geq \psi_-(a)$ the solution $\eta : [a,b] \to \mathbb{R}$ of

$$\dot\eta(t) = e_{\mathrm{op}}(e(t),t), \quad \eta(a) = e(a)$$

satisfies $\eta(t) \geq \psi_-(t)$ for all $t \in [a,b]$.

A possible choice for e_{op} satisfying the above properties is the following:

$$e_{\mathrm{op}}(e,t) = \begin{cases} \dfrac{e-\lambda_+(t)}{\psi_+(t)-\lambda_+(t)}\dot\psi_+(t), & e \geq \lambda_+(t), \\ 0, & \lambda_-(t) < e < \lambda_+(t), \\ \dfrac{e-\lambda_-(t)}{\psi_-(t)-\lambda_-(t)}\dot\psi_-(t), & e \leq \lambda_-(t). \end{cases} \tag{10}$$

The key idea is now to construct a controller which ensures that in the closed loop the following implications are true:

$$\begin{aligned} e(t) \geq \lambda_+(t) \quad &\Longrightarrow \quad \dot e(t) \leq e_{\mathrm{op}}(e(t),t), \\ e(t) \leq \lambda_-(t) \quad &\Longrightarrow \quad \dot e(t) \geq e_{\mathrm{op}}(e(t),t), \end{aligned} \tag{11}$$

because then property (**EO3**) ensures that the error remains inside the funnel. This intuition is formalized in the following theorem.

Theorem 4. *Consider the nonlinear system (6), an output reference signal $y_{\mathrm{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$ and a funnel $\mathscr{F}_{\psi_\pm}$ as in (8) satisfying (**PR**1)-(**PR**3). For a given optimal convergence rate e_{op} as in Definition 1 assume that there exists a controller which ensures that in the corresponding closed loop the implications (11) are satisfied on the domain $[0,\omega)$ of a solution, then*

$$\psi_-(t) \leq e(t) \leq \psi_+(t), \quad \forall t \in [0,\omega),$$

i.e. the objective of funnel control is achieved on $[0,\omega)$.

Proof. Let $e : [0,\omega) \to \mathbb{R}$ be a maximal solution of the closed loop. By assumption, $e^0 \in [\psi_-(0), \psi_+(0)]$. Seeking a contradiction, assume there exists $t_1 > 0$ such that $e(t_1) \notin [\psi_-(t_1), \psi_+(t_1)]$. We consider only the case $e(t_1) > \psi_+(t_1)$, the case $e(t_1) < \psi_-(t_1)$ is completely analogous and omitted. Then there exists $t_0 \in [0,t_1)$ such that $e(t_0) = \psi_+(t_0)$ and $e(t) > \psi_+(t)$ for all $t \in (t_0,t_1]$. Consequently, $e(t) > \lambda_+(t) > 0$ for all $t \in [t_0,t_1]$ and by (**EO**3) the solution $\eta : [t_0,t_1] \to \mathbb{R}$ of $\dot\eta = e_{\mathrm{op}}(e,t)$, $\eta(t_0) = e(t_0) = \psi_+(t_0)$ satisfies $\eta(t) \leq \psi_+(t)$ for all $t \in [t_0,t_1]$. Furthermore, implication (11) yields that $\dot e(t) \leq e_{\mathrm{op}}(e(t),t)$ for all $t \in [t_0,t_1]$. Hence (since $e(t_0) = \eta(t_0)$) it follows that $e(t) \leq \eta(t)$ for all $t \in [t_0,t_1]$, which leads to the contradiction $\psi_+(t_1) < e(t_1) \leq \eta(t_1) \leq \psi_+(t_1)$. ☐

Remark 5. Under the assumptions of Theorem 4, we concluded that $\psi_-(t) \leq e(t) \leq \psi_+(t)$ for all $t \in [0,\omega)$ if implications (11) are satisfied on the domain $[0,\omega)$. Based on Remark 3, the funnel $\mathscr{F}_{\psi_\pm}$ is positively invariant, and finite escape time can not occur; consequently, $\omega = \infty$, i.e. the existence of the solution of the closed loop is guaranteed on the on the whole positive time axis $[0,\infty)$.

Remarks 6. (i) Validity of Theorem 4 does not explicitly rely on the assumptions (**A**1)-(**A**3) for the nonlinear system (6). However, the existence of a controller which ensures the implications (11) can only be guaranteed when these assumptions are satisfied, see Section 4.

(ii) The conclusion of Theorem 4 is independent of the control action carried out when $e(t) \in [\lambda_-(t), \lambda_+(t)]$. This freedom allows to "switch" to a different controller, whenever the error is sufficiently close to the origin, e.g. using some form of PI-controller to reduce the steady state error. However, this "switching" must be carefully designed to avoid discontinuities in the resulting overall control law, otherwise, some sliding solutions may occur along the switching boundary, which could lead to chattering when implemented.

(iii) If $e^0 > \lambda_+(0)$ or $e^0 < \lambda_-(0)$, then the closed loop error satisfies

$$e(t) \leq (\geq) \eta(t,e^0), \quad \text{for all } t \geq 0, \tag{12}$$

where η is the solution of

$$\dot{\eta}(t, e^0) = e_{\text{op}}(\eta(t, e^0), t), \quad \eta(0, e^0) = e^0, \tag{13}$$

see Figure 4. The proof closely resembles that of Theorem 4 and is omitted.

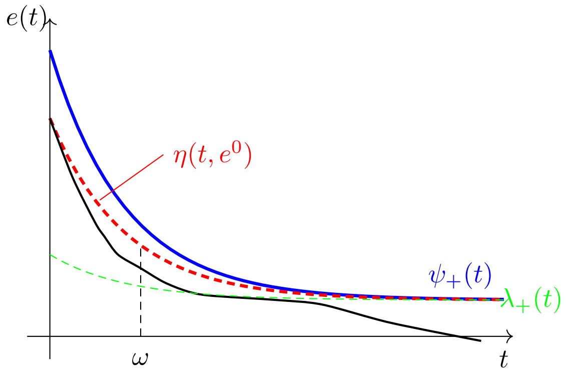


Figure 4: Illustration of $\eta(t, e^0)$, $e^0 > \lambda_+$.

4. Oriented funnel controller

Consider the nonlinear system (6) with an output reference signal y_{ref}. Then the tracking error $e := y - y_{\text{ref}}$ satisfies

$$\dot{e} = \dot{y} - \dot{y}_{\text{ref}} = f(p_f, y, z) - \dot{y}_{\text{ref}} + g(p_g, y, z) \cdot u. \tag{14}$$

If we had full knowledge of the system dynamics (including the internal states and perturbations), we could simply choose $u = u^{\text{ideal}}$ with

$$u^{\text{ideal}}(t) = \frac{e_{\text{op}}(e(t), t) - f(p_f(t), y(t), z(t)) + \dot{y}_{\text{ref}}(t)}{g(p_g(t), y(t), z(t))},$$

because then the implication (11) would be satisfied with equality $\dot{e}(t) = e_{\text{op}}(e(t), t)$ (and, in fact, independently of $e(t)$). Of course, such detailed knowledge of the system (including the perturbations and internal state) is unrealistic, instead, we will assume knowledge of the (not necessarily tight) bounds from Lemma 2 guaranteed by the structural assumptions (**A**1)-(**A**3). Since we will use different constants depending on the sign of the error, we call this approach *orientated* funnel controller. In fact, we define

$$u_+(e, t, \dot{y}_{\text{ref}}) := \min\left\{0, \frac{e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\max}}{G_{\min}}\right\} \le 0,$$
$$u_-(e, t, \dot{y}_{\text{ref}}) := \max\left\{0, \frac{e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\min}}{G_{\min}}\right\} \ge 0,$$

and

$$u(t) := \begin{cases} u_+\big(e(t), t, \dot{y}_{\text{ref}}(t)\big), & \text{if } e(t) \ge \lambda_+(t), \\ u_-\big(e(t), t, \dot{y}_{\text{ref}}(t)\big), & \text{if } e(t) \le \lambda_-(t), \\ \text{arbitrary}, & \text{otherwise.} \end{cases} \tag{15}$$

Remark 7. Our proposed oriented funnel controller assumes knowledge of $\dot{y}_{\text{ref}}(t)$ at any current time t (in addition to the value $y_{\text{ref}}(t)$ needed to calculate the tracking error). Differentiation of a given signal may not be feasible in all situations; however, the reference signal is often produced via a filter $\dot{y}_{\text{ref}}(t) = a y_{\text{ref}}(t) + v_{\text{ref}}(t)$ for some known a and know $v_{\text{ref}}(t)$ (to ensure the required boundedness of $\dot{y}_{\text{ref}}$), in which case $\dot{y}_{\text{ref}}$ can be assumed to be available for the controller design. In case $\dot{y}_{\text{ref}}(t)$ is *not* available to the controller, our approach can easily be adopted by replacing $\dot{y}_{\text{ref}}(t)$ by constants $\dot{Y}_{\text{ref}}^{\min}$ and $\dot{Y}_{\text{ref}}^{\max}$ in $u_+(e, t, \dot{y}_{\text{ref}})$ and $u_-(e, t, \dot{y}_{\text{ref}})$, respectively, where

$$\dot{Y}_{\text{ref}}^{\min} \le \inf_{\tau \ge 0} \dot{y}_{\text{ref}}(\tau), \quad \dot{Y}_{\text{ref}}^{\max} \ge \sup_{\tau \ge 0} \dot{y}_{\text{ref}}(\tau).$$

The key property of the proposed controller (15) is that indeed implication (11) is guaranteed:

Lemma 8. *Consider the closed loop of (6) with a continuous error feedback (15) based on an optimal convergence rate e_{op} as in Definition 1 and under the assumptions (**A**1)-(**A**3), (**PR**1)-(**PR**3). Furthermore, we consider solutions only on the domain $\mathcal{D}$ as in Remark 3. Then every maximal solution $e : [0, \omega) \to \mathbb{R}$ satisfies implication (11).*

Proof. By considering the closed loop on the domain $\mathcal{D}$ with continuous error feedback, we know from classical ODE theory that there exists a maximal solution on some interval $[0, \omega)$, for which $y(t) \in (Y_{\min}, Y_{\max})$ and $\|z(t)\| < Z_{\max}$ for all $t \in [0, \omega)$. Consequently, for all $t \in [0, \omega)$,

$$F_{\min} \le f\big(p_f(t), y(t), z(t)\big) \le F_{\max},$$

and

$$G_{\min} \le g\big(p_g(t), y(t), z(t)\big).$$

The latter implies that (omitting most time dependencies)

$$\begin{aligned} & g(p_g, y, z) \cdot u_+(e, t, \dot{y}_{\text{ref}}) \\ & = \tfrac{g(p_g, y, z)}{G_{\min}} \min\{0, e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\max}\} \\ & \le \min\{0, e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\max}\} \\ & \le e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\max}. \end{aligned}$$

Note that above we used $\frac{g(p_g, y, z)}{G_{\min}} \ge 1$, which (because it is multiplied with a non-positive number) results in an upper bound. If $u_+(e, t, \dot{y}_{\text{ref}})$ would not be sign-semidefinite we could not make this conclusion, which is the reason we have to truncate all possible positive values occurring in the definition of $u_+(e, t, \dot{y}_{\text{ref}})$ to zero. Hence if $e(t) \ge \lambda_+(t)$ we have

$$\begin{aligned} \dot{e}(t) & = f(p_f, y, z) + g(p_g, y, z) u(t) \\ & = f(p_f, y, z) + g(p_g, y, z) u_+(e, t, \dot{y}_{\text{ref}}) \\ & \le F_{\max} - \dot{y}_{\text{ref}} + e_{\text{op}}(e, t) + \dot{y}_{\text{ref}} - F_{\max} \\ & = e_{\text{op}}(e, t). \end{aligned}$$

Analogously, if $e(t) \le \lambda_-(t)$, we have

$$\begin{aligned} \dot e(t) &= f(p_f, y, z) + g(p_g, y, z)u(t) \\ &= f(p_f, y, z) + g(p_g, y, z)u_-(e, t, \dot y_{\mathrm{ref}}) \\ &\ge F_{\min} - \dot y_{\mathrm{ref}} + e_{\mathrm{op}}(e,t) + \dot y_{\mathrm{ref}} - F_{\min} \\ &= e_{\mathrm{op}}(e,t). \end{aligned}$$

This concludes the proof. □

In order to ensure the existence and uniqueness of solutions it is crucial to define the oriented funnel controller as a continuous function of $e(t)$; this can easily be achieved by e.g. just linearly interpolating the values in (15) for $e(t) \in [\lambda_-(t), \lambda_+(t)]$ as follows:

$$u(t) = u_-^\lambda(t) + \tfrac{e(t)-\lambda_-(t)}{\lambda_+(t)-\lambda_-(t)}\left(u_+^\lambda(t) - u_-^\lambda(t)\right), \qquad (16)$$

where (using $e_{\mathrm{op}}(\lambda_\pm(t), t) = 0$)

$$\begin{aligned} u_-^\lambda(t) &:= u_-(\lambda_-(t), t, \dot y_{\mathrm{ref}}(t)) = \max\left\{0, \tfrac{\dot y_{\mathrm{ref}}(t) - F_{\min}}{G_{\min}}\right\}, \\ u_+^\lambda(t) &:= u_+(\lambda_+(t), t, \dot y_{\mathrm{ref}}(t)) = \min\left\{0, \tfrac{\dot y_{\mathrm{ref}}(t) - F_{\max}}{G_{\min}}\right\}. \end{aligned}$$

Combining Lemma 8, Theorem 4 and Remark 3 we arrive at our desired main result of the section:

Corollary 9. *Under the assumptions of Lemma 8, we have that the oriented funnel controller (15) (with continuization (16)) based on a given desired convergence rate e_{op} as in Definition 1 ensures that all solutions of the closed loop are defined on $[0, \infty)$ and that the tracking error evolves within the funnel for all times. Furthermore, if e_{op} is chosen to be bounded (e.g. as in (10)), then the input is uniformly bounded.*

Remarks 10. (i) Our framework covers *asymptotic* tracking because the assumptions do not exclude that $\psi_\pm(t) \to 0$ as $t \to \infty$. In that case, a feasible choice for $\mathcal{F}_0$ is e.g. $\lambda_\pm(t) = \frac{1}{2}\psi_\pm(t)$; note that e_{op} given by (10) remains bounded for $t \to \infty$ (because $|\frac{e-\lambda_\pm(t)}{\psi_\pm(t)-\lambda_\pm(t)}| \le 1$ as long as e is in the funnel) and hence also the control input u remains bounded.

(ii) In case only practical tracking is desired, i.e. $\lim_{t\to\infty} \psi_\pm(t) \neq 0$, it is possible to choose a constant zero error derivative region $\mathscr{F}_0$ via sufficiently small constants $\lambda_+(t) := \lambda_+^c > 0$ and $\lambda_-(t) := \lambda_-^c < 0$. In that case we see that implication (11) results in

$$\begin{aligned} e(t) = \lambda_+^c \quad &\Longrightarrow \quad \dot e(t) \le e_{\mathrm{op}}(\lambda_+^c, t) = 0, \\ e(t) = \lambda_-^c \quad &\Longrightarrow \quad \dot e(t) \ge e_{\mathrm{op}}(\lambda_-^c, t) = 0, \end{aligned}$$

i.e. the region $\mathscr{F}_0$ is *positively* invariant. In particular, the bouncing behavior in the transient phase seen in Figure 1 is avoided.

We illustrate the theoretical result of this section by revisiting the example from the introduction.

Example 1. *Consider again the scalar example (1) with an additional bounded disturbance term and specific constants a and b:*

$$\dot y(t) = 2 + 0.9\sin(t) + u(t), \;\; y^0 = 1. \qquad (17)$$

As a reference signal, we consider $y_{\mathrm{ref}}(t) = \sin(20t)$, which we want to track with an accuracy given by the funnel boundaries $\psi_+(t) = e^{-4t} + 0.02$, $\psi_-(t) = -\psi_+(t)$. With $\lambda_+(t) = 0.02$, $\lambda_-(t) = -0.02$, it is easily seen that all assumptions of the EDL funnel controller approach are satisfied and we can choose $F_{\min} = 1.1$, $F_{\max} = 2.9$, $G_{\min} = G_{\max} = 1$. Simulation results for the classical funnel controller (3) and the EDL funnel controller (15) with e_{op} given by (10) and with the input interpolation (16) are shown in Figure 5. As expected both controllers ensure that the error evolves within the funnel, however, clearly the EDL funnel controller avoids a strong bouncing behavior. The simulations also clearly show that the magnitude of the error derivative is significantly reduced and the input signal is much smoother.

Example 2. *As a second example we revisit an example from [16], given by*

$$\begin{aligned} \dot y(t) =& p_f(t) + |y(t)|y(t) + z(t) + \mathrm{sat}_{[\underline{u},\overline{u}]}\, u(t), \\ \dot z(t) =& - z(t) - z^3(t) + [1 + z^2(t)]y(t), \\ y(0) =& y^0, \;\; z(0) = z^0, \end{aligned}$$

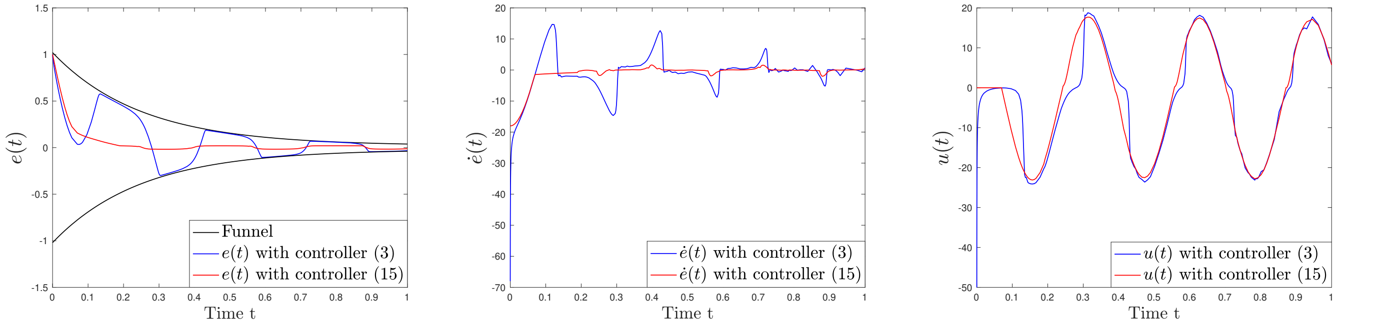
with reference signal $y_{\mathrm{ref}}(t) = \xi_1(t)$ and perturbation $p_f(t) = -\xi_2(t)$ given by the solution of the chaotic Lorenz system

$$\begin{aligned} \dot\xi_1(t) =& \xi_2(t) - \xi_1(t), \\ \dot\xi_2(t) =& \left(\frac{28\xi_1(t)}{10}\right) - \left(\frac{\xi_2(t)}{10}\right) - \xi_1(t)\xi_3(t), \\ \dot\xi_3(t) =& \xi_1(t)\xi_2(t) - \left(\frac{8\xi_3(t)}{30}\right), \\ \xi_1(0) =& 1, \;\; \xi_2(0) = 0, \;\; \xi_3(0) = 3. \end{aligned}$$

The reference signal and perturbation satisfy $\|y_{\mathrm{ref}}\|_\infty \le 2$, $\|\dot y_{\mathrm{ref}}\|_\infty \le 1$ and $\|p_f\|_\infty \le 2.4$. The prescribed funnel boundaries are chosen as $\psi_+(t) = 2e^{-0.1t} + 0.1$, and $\psi_-(t) = -\psi_+(t)$. $y^0 = 2$, $z^0 = 1$. Suitable bounds for Lemma 2 are given by $F_{\max} = -F_{\min} = 27$, $G_{\min} = G_{\max} = 1$. The simulation results are shown in Figure 6.

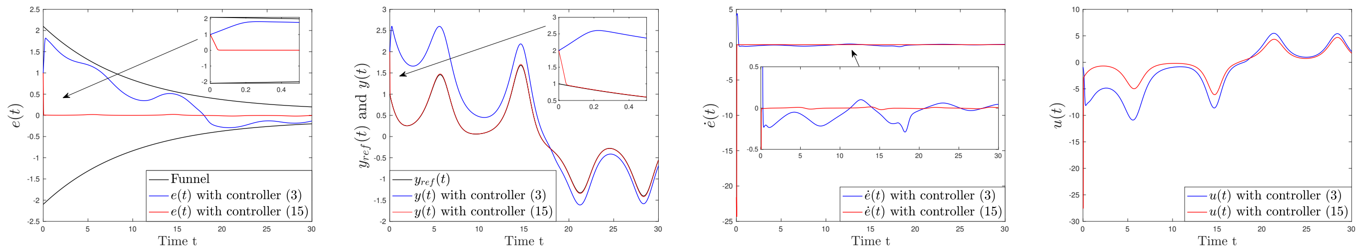
5. Bounds for error derivatives

As explained above the idea of the EDL funnel controller is to design a feedback controller in such a way that (in addition to keeping the error within the funnel) the error derivative is close to a prespecified "optimal" error convergence rate $e_{\mathrm{op}}(e,t)$. Lemma 8 shows that the oriented funnel controller given by (15) indeed ensures that the error derivate near the funnel boundary is upper/lower bounded by $e_{\mathrm{op}}(e,t)$. This section will provide further bounds on



(a) Error comparison within funnel. (b) Error derivative comparison. (c) Input comparison.

Figure 5: Simulation results for Example 1.



(a) Error comparison within funnel (b) Output comparison. (c) Error derivative comparison. (d) Input comparison.

Figure 6: Simulations for Example 2

the error derivatives, first (implicitly) in terms of e_{op} and then explicitly by defining a suitable error derivative funnel

$$\mathscr{F}^{e_0}_{\psi^d_\pm} := \left\{ \, (t, \dot{e}) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \psi^d_-(t) \leq \dot{e} \leq \psi^d_+(t) \, \right\},$$

whose boundaries will depend on the initial error $e(0) = e_0$.

The first result provides a bound for $\dot{e}$ in terms of e_{op} in the form of a "band" around the desired value e_{op}; this band for $\dot{e}$ is illustrated in Figure 7.

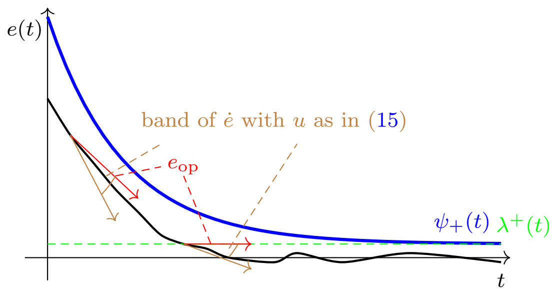


Figure 7: Band of $\dot{e}$ with certain u ($e \geq \lambda_+$).

Corollary 11. *Under the assumption of Lemma 8 and with $G_{\max} > 0$ being an upper bound of $g(p_g, y, z)$ analogously defined as $G_{\min}$, the error derivative $\dot{e}(t)$ in the closed loop satisfies (omitting the time dependencies)*

$$F_{\min} - \dot{y}_{\mathrm{ref}} + \tfrac{G_{\max}}{G_{\min}} \min\{0, e_{\mathrm{op}} + \dot{y}_{\mathrm{ref}} - F_{\max}\} \leq \dot{e} \leq e_{\mathrm{op}}$$

if $e(t) \geq \lambda_+(t)$, and

$$e_{\mathrm{op}} \leq \dot{e} \leq F_{\max} - \dot{y}_{\mathrm{ref}} + \tfrac{G_{\max}}{G_{\min}} \max\{0, e_{\mathrm{op}} + \dot{y}_{\mathrm{ref}} - F_{\min}\}$$

if $e(t) \leq \lambda_+(t)$. In particular, if g is constant and known (i.e. $G_{\max} = G_{\min}$) and $u(t) \neq 0$, then either

$$e_{\mathrm{op}} - (F_{\max} - F_{\min}) \leq \dot{e} \leq e_{\mathrm{op}}$$

or

$$e_{\mathrm{op}} \leq \dot{e} \leq e_{\mathrm{op}} + F_{\max} - F_{\min}.$$

In the following, we provide explicit constant bounds on $\dot{e}$ for the situation that $e^0 \in [\lambda_-(0), \lambda_+(0)]$ and that the zero error derivative region $\mathscr{F}_0$ has constant boundaries. In that case, by Remark 10(ii), the region $[\lambda^c_-, \lambda^c_+]$ is positively invariant for the error signal and the input defined by (16) satisfies $u(t) \in [u^\lambda_-(t), u^\lambda_+(t)]$, from which the following constant bounds for the error derivative can easily be derived.

Corollary 12. *Consider a constant region $\mathscr{F}_0$ for e_{op}, i.e. $\lambda_\pm(t) = \lambda^c_\pm \in \mathbb{R}$ in Definition 1, and assume that $e^0 \in [\lambda^c_-, \lambda^c_+]$. Then under the assumptions as in Corollary 11 together with the continuation (16) of the input we have*

$$\psi^{d,c}_- \leq \dot{e}(t) \leq \psi^{d,c}_+, \quad \forall t \geq 0, \tag{18}$$

where $\psi_+^{d,c} := \Psi_-(0)$ *and* $\psi_-^{d,c} := \Psi_+(0)$ *with*

$$\begin{aligned}\Psi_+^d(e_{\rm op}) :=& F_{\max} - \inf_{\tau\geq 0} \dot y_{\rm ref}(\tau) \\ &+ G_{\max}\max\left\{0, \frac{e_{\rm op} + \sup_{\tau\geq 0}\dot y_{\rm ref}(\tau) - F_{\min}}{G_{\min}}\right\}, \\ \Psi_-^d(e_{\rm op}) :=& F_{\min} - \sup_{\tau\geq 0} \dot y_{\rm ref}(\tau) \\ &+ G_{\max}\min\left\{0, \frac{e_{\rm op} + \inf_{\tau\geq 0}\dot y_{\rm ref}(\tau) - F_{\max}}{G_{\min}}\right\}.\end{aligned}$$

Utilizing the error bound from Remark 6(iii) in terms of $\eta(t, e_0)$ given by (13) and using the specific choice (10) for $e_{\rm op}$, we can conclude that

$$e_{\rm op}\big(e(t), t\big) \geq e_{\rm op}\big(\eta(t, e_0), t\big) \quad \forall t \geq 0$$

if $e_0 > \lambda_+^c$ and

$$e_{\rm op}\big(e(t), t\big) \leq e_{\rm op}\big(\eta(t, e_0), t\big) \quad \forall t \geq 0$$

if $e_0 < \lambda_-^c$. Hence, we can derive an explicit bound (not depending on the error signal) for the error derivative from the previous two corollaries as follows.

Corollary 13. *Consider the setup and notation of Corollary 12 and for* $e_0 > \lambda_+^c$ *or* $e_0 < \lambda_-^c$ *let* $\eta(t, e_0)$ *be the solution of* (13)*. Then the error derivative satisfies*

$$\psi_-^d(t) \leq \dot e(t) \leq \psi_+^d(t) \quad \forall t \geq 0,$$

where

$$\psi_+^d(t, e^0) = \begin{cases} \Psi_+^d(0) = \psi_+^{d,c}, & e^0 > \lambda_+^c, \\ \Psi_+^d\Big(e_{\rm op}\big(\eta(t, e^0), t\big)\Big), & e^0 < \lambda_+^c, \end{cases}$$

$$\psi_-^d(t, e^0) = \begin{cases} \Psi_-^d\Big(e_{\rm op}\big(\eta(t, e^0), t\big)\Big), & e^0 > \lambda_-^c, \\ \Psi_-^d(0) = \psi_-^{d,c}, & e^0 < \lambda_-^c. \end{cases}$$

The final result concerning the bounding of the error derivative is illustrated in Figure 8.

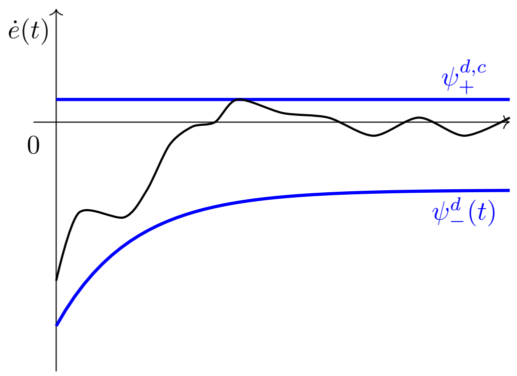


Figure 8: Illustration of $\mathscr{F}_{\psi_\pm^d}$, $e^0 > \lambda_+^c$.

6. Input saturations

Consider now nonlinear system (6) with input saturation

$$u \mapsto {\rm sat}_{[\underline u, \overline u]}(u) = \begin{cases} \underline u, & u < \underline u, \\ u, & u \in [\underline u, \overline u], \\ \overline u, & u > \overline u, \end{cases} \tag{19}$$

where the threshold values satisfy $\underline u < 0 < \overline u$.

For the feasibility of the tracking problem, the following implications concerning the maximal control inputs should hold for the error dynamics given by (14):

$$\begin{aligned} u(t) = \underline u &\implies \dot e(t) < 0, \\ u(t) = \overline u &\implies \dot e(t) > 0. \end{aligned}$$

In view of input saturation, sufficient conditions to ensure the validity of the above implications are the following two assumptions

(**T**1) $0 > \underline d(\underline u) := F_{\max} - \inf_{\tau\geq 0}\dot y_{\rm ref}(\tau) + G_{\min}\underline u$,

(**T**2) $0 < \overline d(\overline u) := F_{\min} - \sup_{\tau\geq 0}\dot y_{\rm ref}(\tau) + G_{\min}\overline u$.

Clearly, these two assumptions are satisfied if the input bounds are sufficiently large in magnitude.

Even with assumptions (**T**1) and (**T**2) satisfied it cannot be expected that the error can be kept in an arbitrary funnel, because if the funnel boundaries are shrinking too rapidly the input saturation may limit the ability of the error to shrink sufficiently fast to stay in the funnel, see Figure 9.

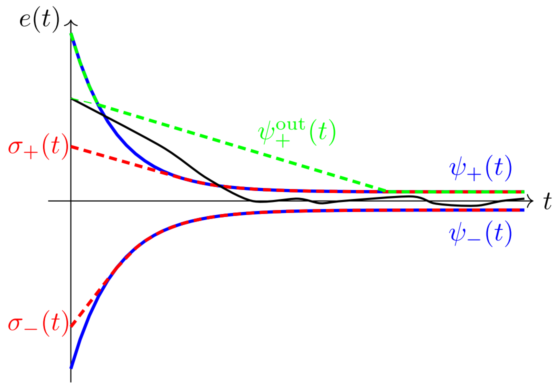


Figure 9: Error signal (in black) leaves the funnel (blue) due to input saturations. Proposed safety functions σ_+ and σ_- (dashed red) and outer funnel given by $\psi_+^{\rm out}$ (dashed green).

There are three ways to deal with this problem:

1) *Funnel is strict.* Ensure that the input is powerful enough to keep the error within the desired funnel shape, which is the case if $\underline d(\underline u) \leq \inf_{t\geq 0}\dot\psi_+(t)$ and $\overline d(\overline u) \geq \sup_{t\geq 0}\dot\psi_-(t)$.

2) *Input constraints are strict.* Choose a funnel shape that is sufficiently slowly shrinking, which is the case if $\inf_{t\geq 0}\dot\psi_+(t) \geq \underline d(\underline u)$ and $\sup_{t\geq 0}\dot\psi_-(t) \leq \overline d(\overline u)$.

3) *Given funnel shape is desired and input saturations are strict.* If possible, the error should be kept within the funnel, however, temporarily leaving the funnel is allowed.

In the remainder of this section, we will focus on the third situation as this is in many practical applications the most realistic and least conservative approach. There are actually two aspects: a) stay in the funnel if possible, and b) return to funnel as quickly as possible. The latter can easily be achieved by applying the maximal input whenever outside the funnel, whereas the first property is not so straightforward. For example, just applying the maximum available input whenever the error is very close to the funnel boundary may not be enough, because at that moment the funnel boundary may shrink too quickly for the error to stay within the funnel. However, by looking ahead one could have applied a more aggressive control action earlier to prevent the error from getting too close to the funnel boundary in the first place. This idea is formalized by the notion of a *safety function* defined as follows.

Definition 2. Consider a nonlinear system (6) satisfying assumptions (**A**1)-(**A**3), prescribed funnel boundary and reference signal satisfying (**PR**1)-(**PR**3). Furthermore, consider input saturations that satisfy (**T**1) and (**T**2). Let

$$\begin{aligned} t_d^+ &:= \min\left\{ t \geq 0 \;\middle|\; \dot{\psi}_+(t) \geq \underline{d} \right\}, \\ t_d^- &:= \min\left\{ t \geq 0 \;\middle|\; \dot{\psi}_-(t) \leq \overline{d} \right\}, \end{aligned}$$

which are well defined since by assumption (**PR**1) $\lim_{t\to\infty} \dot{\psi}_\pm(t) = 0$. Then the safety function is defined as

$$\begin{aligned} \sigma^+(t) &:= \begin{cases} \underline{d}(\underline{u}) \cdot (t - t_d^+) + \psi_+(t_d^+), & t \in [0, t_d^+], \\ \psi_+(t), & t \in [t_d^+, \infty), \end{cases} \\ \sigma^-(t) &:= \begin{cases} \overline{d}(\overline{u}) \cdot (t - t_d^-) + \psi_-(t_d^-), & t \in [0, t_d^-], \\ \psi_-(t), & t \in [t_d^-, \infty). \end{cases} \end{aligned} \tag{20}$$

The safety functions are illustrated in Figure 9. Based on the safety region we can now define a saturation-aware optimal convergence rate as follows.

Definition 3. *For prescribed funnel satisfying (**PR1**), input saturation values satisfy (**T1**) and (**T2**), let the saturation aware optimal converging rate be given as*

$$e_{op}^{\text{sat}}(e,t) = \begin{cases} \underline{d}(\underline{u}), & e > \sigma_+(t), \\ \dfrac{e(t) - \lambda_+(t)}{\sigma_+(t) - \lambda_+(t)} \dot{\sigma}_+(t), & \sigma_+(t) \geq e \geq \lambda_+(t), \\ 0, & \lambda_+(t) > e > \lambda_-(t), \\ \dfrac{e(t) - \lambda_-(t)}{\sigma_-(t) - \lambda_-(t)} \dot{\sigma}_-(t), & \lambda_-(t) \geq e \geq \sigma_-(t), \\ \overline{d}(\overline{u}), & \sigma_-(t) > e. \end{cases} \tag{21}$$

The intuition behind (21) is that within the safety region, we simply replace the original funnel boundaries in (10) by the safety boundaries and outside the safety region the optimal convergence rate is set to the guaranteed decrease/increase rate in view of the input saturations. With this choice we immediately have the following result:

Corollary 14. *Consider a nonlinear system (6) satisfying assumptions (**A**1)-(**A**3), prescribed funnel boundary and reference signal satisfying (**PR**1)-(**PR**3). Furthermore, consider input saturations that satisfy (**T**1) and (**T**2) with corresponding safety functions* (20)*. Then the EDL funnel controller* (15) *(with continuation* (16)*) based on the saturation aware optimal convergence rate $e_{\text{op}}^{\text{sat}}$ given by* (21) *achieves the following properties in closed loop:*

1. *The solution of the closed loop exists on* $[0,\infty)$*.*
2. *The safety region is positively invariant for the error signal; in particular, if* $e(0) \in [\sigma_-(0), \sigma_+(0)]$ *then the error will remain within the funnel for all times.*
3. *The input (before saturation) satisfies* $u(t) \in [\underline{u}, \overline{u}]$ *for all* $t \geq 0$*.*

If the initial error is not in the safety region, we cannot guarantee that the error evolves within the originally given funnel, however, we are able to define the following "outer" funnel (depending on e^0) for which it can be guaranteed that the error remains within:

$$\mathscr{F}_{\text{out}}^{e^0} := \left\{ (t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \psi_-^{\text{out}}(t) \leq e(t) \leq \psi_+^{\text{out}}(t) \right\}, \tag{22}$$

where

$$\begin{aligned} \psi_+^{\text{out}}(t) &= \max\{e^0 + \underline{d} \cdot t, \psi_+(t)\}, \\ \psi_-^{\text{out}}(t) &= \min\{e^0 + \overline{d} \cdot t, \psi_-(t)\}. \end{aligned}$$

Note that $\psi_+^{\text{out}}(t) = \psi_+(t)$ for all $t \geq 0$ if $e^0 \leq \sigma_+(0)$ and $\psi_-^{\text{out}}(t) = \psi_-(t)$ if $e^0 \geq \sigma_-(0)$. The case $e^0 > \sigma_+(0)$ is illustrated in Figure 9.

Note that from $e^0 \in [\psi_-(0), \psi_+(0)]$ and monotonicity of $\psi_\pm$ it follows that

$$\begin{aligned} &\psi_+(t) \leq \psi_+^{\text{out}}(t) \leq \psi_+(0) \text{ and} \\ &\psi_-(t) \geq \psi_-^{\text{out}}(t) \leq \psi_-(0). \end{aligned}$$

In particular, as long as $e(t) \in [\psi_-^{\text{out}}(t), \psi_+^{\text{out}}(t)]$ we can still conclude that $y(t) \in [Y_{\min}, Y_{\max}]$ and hence the same bounds for z, g and f hold as before. Furthermore, the choice of $e_{\text{op}}^{\text{sat}}$ satisfies the properties (**EO**1)-(**EO**3) w.r.t. the outer funnel $\mathscr{F}_{\text{out}}^{e^0}$, hence we arrive at the following result:

Corollary 15. *Under the same assumptions as in Corollary 14, we have that the closed loop satisfies*

$$\psi_-^{\text{out}}(t) \leq e(t) \leq \psi_+^{\text{out}}(t), \quad \forall t \geq 0.$$

Remark 16. It should be noted that $e_{\text{op}}(e(t),t) = \underline{d}(\underline{u})$ does *not* in general imply that $u(t) = \underline{u}$, this is because in general, if $e(t) > \lambda_+$,

$$\begin{aligned} u(t) = u_+(e, t, \dot{y}_{\text{ref}}(t)) &\geq \frac{\underline{d}(\underline{u}) + \dot{y}_{\text{ref}}(t) - F_{\max}}{G_{\min}} \\ &= \underline{u} + \dot{y}_{\text{ref}}(t) - \inf_{\tau \geq 0} \dot{y}_{\text{ref}}(\tau) > \underline{u}. \end{aligned}$$

In particular, if $e(t)$ is outside the funnel, our approach is not using the full force to push the error back into the funnel. This can easily fixed by redefining the control input in such a way that $(t) = \underline{u}$ whenever $e(t) > \psi_+(t)$ (or even if $e(t) > \sigma_+(t)$). In order to keep a continuous input rule (to guarantee existence of solutions in the closed loop) it is then however necessary to add an additional "buffer" region, which in principle is easily done but adds another level of technicality and therefore is not presented here.

7. Conclusion

To avoid the error bouncing between prescribed funnel boundaries, a novel funnel control scheme has been proposed in this paper. In addition, to keep to error within the funnel, we have proposed the error derivative limiting (EDL) funnel controller as a two stages controller, where we first introduced the concept of the desired optimal convergence rate and then the oriented funnel controller which is defined in such a way that it tries to match the error derivative as close as possible to the desired convergence rate. We prove that this approach indeed achieves the desired control objectives: The error evolves within the funnel and the error derivative is limited to a band whose width is expressed in terms of uncertainty about the system. Furthermore, we exploit the structure of this EDL funnel controller to handle input saturation. In case the input saturations are too restrictive, our controller allows the error to temporarily leave the funnel and the return to the inside of the funnel is guaranteed.

Future work is concerned with extending these ideas to the multi-input multi-output setting as well as to higher relative degree systems.

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