Midpoint based balanced truncation for switched linear systems with known switching signal

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***Abstract*—We propose a novel model reduction approach for switched linear systems with known switching signal. The class of considered systems encompasses switched systems with mode-dependent state-dimension as well as impulsive systems. Our method is based on a suitable definition of (time-varying) reachability and observability Gramians and we show that these Gramians satisfy precise interpretations in terms of input and output energy. Based on balancing the midpoint Gramians, we propose a piecewise-constant projection based model reduction resulting in a switched linear system of smaller size.**

***Index Terms*—Balanced truncation, switched systems, Gramians.**

I. Introduction

During the last few years, considerable attention has been dedicated to the problem of model reduction for switched linear systems and several techniques have been proposed. In [1] a model reduction method has been proposed for switched systems with autonomous switching which depends on continuous outputs. In [2] a simultaneous balancing transformation is proposed which is based on reachability / observability Gramians for each mode and the assumption that all Gramians can simultaneously be transformed into a diagonal form.

An interesting model reduction method utilizing a so called envelope system for switched linear systems is proposed in [3], which is based on the idea of embedding the solution behaviors of the switched system into the solution behavior of a certain non switched system; then standard model reduction techniques can be applied to the envelope system.

Generalized Gramians based approaches are proposed e.g., in [4]–[9]; these approaches are considered only for quadratically stable systems, and the Gramians are then the solutions of certain linear matrix inequalities (LMIs). Furthermore, an improvement of the underlying approaches are reviewed in [10]–[15] by introducing new Gramians and its system theoretical properties; again, certain stability assumptions are made.

None of above model reduction approaches for switched systems consider the switched system as a piece-wise constant linear time-varying system, i.e. the question how to reduce a switched system for a given (and known) switching signal. We have recently studied the question of reduced realization of switched systems [16], [17] and observed that the specific mode sequence as well as the mode durations influence the size of the minimal realization. Hence it is reasonable to conclude that in case of a known switching signal the size of a reduced system which approximates the original input-output behavior sufficiently well will also depend on the particular switching signals. Consequently, all of the above approaches which do not consider a specific switching signal will usually not result in the "best" reduced model for this specific switching signal.

While there are some results on model reduction for general linear time-varying systems, e.g. [18]–[20], they usually assume at least continuity of the coefficient matrices (which of course is not satisfied for switched systems). Furthermore, even when relaxing the continuity assumptions (e.g. by approximating the switched system with a continuous piecewise-linear system as in [21]) the resulting reduced system is fully time-varying and not piecewise-constant as usually desired. Furthermore, our studies on reduced realization of switched systems also showed that the reduced switched system will in general have mode-dependent state-dimension and it is necessary to consider jump maps between the states of the different modes; likewise, in the case of model reduction it is reasonable to aim for a reduced switched system with mode-dependent state-dimension and to consider jumps of the states at the switches. In order to stay within the same system class we therefore consider in the following switched linear systems (SLSs) with a given switching signal of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)}x_k(t) + B_{\sigma(t)}u(t), \ t \in (s_k, s_{k+1}), \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)}x_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ y(t) = C_{\sigma(t)}x_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{1}$$

where $\sigma : \mathbb{R} \to \mathtt{M} = \{0, 1, 2, \dots, \mathtt{m}\} \subseteq \mathbb{N}$ is the given switching signal with finitely many switching times $s_1 < s_2 < \dots < s_\mathtt{m}$ in the bounded interval $[t_0, t_f)$ of interest and $x_k : (s_k, s_{k+1}) \to \mathbb{R}^{n_k}$ is the k-th piece of the state (whose dimension n_k may depend on the mode $p = \sigma(s_k^+)$ active on the interval (s_k, s_{k+1})). For notational convenience let $s_0 := t_0$, $s_{\mathtt{m}+1} := t_f$ and $\tau_k := s_{k+1} - s_k$, $k \in \mathtt{M}$. Assume a zero initial condition, i.e. set $x_{-1}(t_0^-) := 0$, and the input and output are given by $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^m$, respectively. Here, $x_k(t^-)$ and $x_k(t^+)$ denote, respectively, the left- and right-sided limit at t, assuming they exist.

For each mode $p \in \mathtt{M}$, the system matrices A_p, B_p, C_p of appropriate size describe the (continuous) dynamics corresponding to the linear system active for mode p. Furthermore, $J_{p,q} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is the jump map from mode q to mode p. Note that due to the different space dimensions the introduction of a jump map is necessary; on the other hand, in case all state dimensions are equal, the consideration of a jump map is "optional" and leads to so called impulsive systems (in

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particular, our approximation results will also preserve this system class).

The main goal is to find a reduced system which has a similar input-output behavior as (1) (with the same switching signal σ) of the form

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)}\widehat{x}_k(t) + \widehat{B}_{\sigma(t)}u(t),\ t \in (s_k, s_{k+1}), \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)}\widehat{x}_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ y(t) = \widehat{C}_{\sigma(t)}\widehat{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{2}$$

where $\widehat{x}_k \in \mathbb{R}^{r_k}$, $r_k \leq n_k$.

We will assume in the following that the switching signal is fixed, hence by suitable relabeling of the matrices, we can assume that $\sigma(t) = k$ on (s_k, s_{k+1}). Consequently, we can simply write $J_k := J_{\sigma(s_k^+),\sigma(s_k^-)} = J_{k,k-1}$ and $\widehat{J}_k := \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} = \widehat{J}_{k,k-1}$ in the following. Furthermore, in some slight abuse of notation, we will often speak in the following just of the solution $x(\cdot)$ instead of the different solution pieces $x_k(\cdot)$.

Our reduction approach is an adaptation of the well known balanced truncation approach [22], [23] to our system class. They key idea is to first define suitable (time-varying) Gramians which provide a quantitative measure of how difficult to reach and observe certain state directions are. With the help of the Gramian-based (mode-dependent) coordinate transformation a somewhat balanced system can be obtained, which means that simultaneously difficult to reach and difficult to observe states can be identified and easily removed. However, due to the time-varying and discontinuous nature as well as due to the consideration of finite intervals, classical properties of balanced truncation like stability preservation and error bounds cannot easily be generalized and are outside the scope of this technical note. Furthermore, we are not claiming that our method is in general better than existing ones, instead we see our main contribution in proposing a method utilizing the newly established input- and output-energy relations for a mode-wise reduction leading to mode-dependent reduced state space dimensions and corresponding jump maps.

This paper is organized as follows. In Section II, we first discuss why a naive model-reduction approach is not working, we then recall the definition and characterization of the time-varying reachable and unobservable spaces of the linear switched system (1). In Section III, the time-varying reachability and observability Gramians are proposed and their relationship to the corresponding reachable and unobservable spaces is highlighted; furthermore, a precise connection to input- and output energy is proven. We than propose in Section IV a mode-wise midpoint balanced truncation method to obtain a reduced model which disregards simultaneosly difficult to reach and difficult to observe states. We also provide a discussion about the numerical implementation of the algorithm and its feasibility for large scale systems. Finally, in Section V we provide a toy example to illustrate the individual steps of our method.

II. Preliminaries

We first recall [21, Example 1] which showed that a naive mode-wise reduction approach does not work in general. There are two underlying main reason why mode-wise balanced truncation will in general not work for switched systems: 1) Balancing each mode individually results in a mode-dependent coordinate transformation, so even without reducing the state-dimension, the resulting switched system will not preserve the input-output behavior *unless an additional state-jump is introduced to take into account the mode-dependent coordinate transformations*; 2) A mode-wise reduction removes difficult to observe and difficult to reach states in each mode, however, *a difficult to observe state in one mode may be easily observable in another mode*, hence in order to approximately preserve the input-output behavior one should not remove such a state.

In order to resolve the second issue, we recall in the following the (time-varying) reachability and unobservable spaces of a switched system.

We first highlight that the solution of (1) can be expressed recursively by, for $t \in [s_k, s_{k+1})$ and $k = 1, \dots, \mathtt{m}$,

$$x(t) := e^{A_k(t-s_k)}J_k x(s_k^-) + \int_{s_k}^t e^{A_k(t-\tau)}B_k u(\tau)\,\mathrm{d}\tau, \tag{3}$$

with corresponding output

$$y(t) := C_k x(t),\ t \in [s_k, s_{k+1}),\ k = 0, 1, \dots, \mathtt{m}. \tag{4}$$

Let us now introduce the following formal definition of the reachable space and unobservable space of (1) on the intervals $[t_0, t)$ and $[t, t_f)$, respectively.

Definition 1: The reachable space of the switched system (1) on time interval $[t_0, t)$ is

$$\mathcal{R}_{[t_0,t)} := \left\{ x(t^-) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \text{ of } (1) \\ \text{with } x(t_0^-) = 0 \end{array} \right\}.$$

We call the switched system (1) *reachable on* (t_0, t) for $t \in (s_k, s_{k+1}]$ if, and only if, $\mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$. △

Definition 2: The unobservable space of the switched system (1) on time interval $[t, t_f)$ is

$$\mathcal{U}_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u=0) \text{ of } (1) \\ \text{such that } y = 0 \text{ on } [t, t_f) \end{array} \right\}.$$

We call the switched system (1) *observable on* $[t, t_f)$ if, and only if, $\mathcal{U}_{[t,t_f)} = \{0\}$. △

In [17], [24], it is shown that the exact (time-varying) reachable and unobservable spaces can be calculated by the following recursive formulas. The reachable spaces are given by, for $k = 1, 2, \dots, \mathtt{m}$,

$$\begin{aligned} &\mathcal{R}_{[t_0,t)} := \mathcal{R}_0, && t \in (t_0, s_1], \\ &\mathcal{R}_{[t_0,t)} := e^{A_k\tau_k}J_k\mathcal{R}_{[t_0,s_k)} + \mathcal{R}_k, && t \in (s_k, s_{k+1}], \end{aligned} \tag{5}$$

and the unobservable spaces are defined by, for $k = \mathtt{m}-1, \mathtt{m}-2, \dots, 0$,

$$\begin{aligned} &\mathcal{U}_{[t,t_f)} := \mathcal{U}_\mathtt{m}, && t \in [s_\mathtt{m}, t_f), \\ &\mathcal{U}_{[t,t_f)} := \left(e^{-A_k(s_{k+1}-t)}J_{k+1}^{-1}\mathcal{U}_{[s_{k+1},t_f)}\right) \cap \mathcal{U}_k, && t \in [s_k, s_{k+1}). \end{aligned} \tag{6}$$

Here $\mathcal{R}_k := \operatorname{im}[B_k, A_kB_k, \dots, A_k^{n_k-1}B_k]$ and $\mathcal{U}_k := \ker[C_k^\top, (C_kA_k)^\top, \dots, (C_kA_k^{n_k-1})^\top]^\top$ are the classical local reachable and unobservable spaces of mode k and J_{k+1}^{-1} stands for the pre-image (the jump maps are not assumed to be invertible and are rectangular in general anyway).

It should be noted that these spaces do not contain quantitative information about how easy/difficult it is to reach a reachable state or observe an observable state. Consequently, while these spaces are quite helpful to derive a reduced realization as in [17] where unobservable and unreachable states are removed, they cannot be used directly to obtain a reduced model which discards difficult to reach and difficult to observe states. To quantify reachability and observability it is necessary to introduce suitable Gramians.

III. Exact (time-varying) Gramians for switched linear systems

Our proposed reduction is a generalization of the well-established balanced truncation methods and therefore relies on a suitable definition of Gramians which then need to be balanced.

A. Reachability Gramian

We propose the following recursive definition for the reachability Gramian of (1):

$$\begin{aligned}\mathcal{P}_0^\sigma(t) &:= P_0(t),\ t \in (t_0, s_1],\\ \mathcal{P}_k^\sigma(t) &:= e^{A_k(t-s_k)} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top(t-s_k)} + P_k(t), \quad (7)\\ & \qquad t \in (s_k, s_{k+1}],\end{aligned}$$

where $P_k(t) = \int_{s_k}^t e^{A_k(\tau - s_k)} B_k B_k^\top e^{A_k^\top(\tau-s_k)}\,\mathrm{d}\tau$ is the classical reachability Gramian of mode k on the interval (s_k, t). The intuition behind the sequence is as follows: Since the first system starts with zero initial values, the reachability Gramian in the first mode is the classical reachability Gramian of the first mode. Continuing recursively, the Gramian between the switching time s_k and s_{k+1} is obtained by propagating forward the Gramian just before switch k in time, i.e., first jump via J_k and then propagating according to the matrix exponential and finally, take into account the classical reachability Gramian for mode k.

We first show that the reachability Gramian $\mathcal{P}_k^\sigma(t)$ indeed spans the reachable space $\mathcal{R}_{[t_0,t)}$.

Lemma 3: For all $k \in \mathtt{M}$ and $t \in (s_k, s_{k+1}]$

$$\operatorname{im} \mathcal{P}_k^\sigma(t^-) = \mathcal{R}_{[t_0,t)}.$$

In particular, (1) is reachable on $[t_0, t)$ i.e., $\mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ for $t \in (s_k, s_{k+1}]$ if, and only if, $\mathcal{P}_k^\sigma(t^-)$ is positive definite (and hence nonsingular).

Proof. It is well known (see e.g. [23]) that $\mathcal{R}_{[t_0,t)} = \mathcal{R}_0 = P_0(t) = \operatorname{im} \mathcal{P}_0^\sigma(t^-)$ for all $t \in (t_0, s_1]$. We proceed inductively and assume that for some $k \leq \mathtt{m}$,

$$\operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-) = \mathcal{R}_{[t_0,s_k)}.$$

From (5), the reachable space at $t \in (s_k, s_{k+1}]$ is given by

$$\mathcal{R}_{[t_0,t)} = e^{A_k(\tau_k(t-s_k)} J_k \mathcal{R}_{[t_0,s_k)} + \mathcal{R}_k. \quad (8)$$

Furthermore, $\mathcal{P}_{k-1}^\sigma(s_k^-)$ is by construction symmetric and positive semi-definite, hence $\mathcal{P}_{k-1}^\sigma(s_k^-)^{1/2}$ is well defined and $\operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-) = \operatorname{im} \mathcal{P}_{k-1}^\sigma(s_k^-)^{1/2}$ and, therefore,

$$\operatorname{im} e^{A_k(\tau_k(t-s_k)} J_k (\mathcal{P}_{k-1}^\sigma(s_k^-))^{\frac{1}{2}} = e^{A_k(\tau_k(t-s_k)} J_k \mathcal{R}_{[t_0,s_k)}.$$

Note that for any matrix M, $\operatorname{im} M = \operatorname{im}(MM^\top)$ and consequently

$$\operatorname{im} e^{A_k\tau_k} J_k (\mathcal{P}_{k-1}^\sigma(s_k^-))^{\frac{1}{2}} = \operatorname{im} e^{A_k\tau_k} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top \tau_k}.$$

Together with $\mathcal{R}_k = \operatorname{im} P_k(t)$ and the general fact that $\operatorname{im}(M_1 + M_2) = \operatorname{im} M_1 + \operatorname{im} M_2$ for any two symmetric positive semi-definite matrices M_1 and M_2 of the same size, we can now conclude from (8) the desired subspace equation

$$\begin{aligned}\operatorname{im} \mathcal{P}_k^\sigma(t^-) &= \operatorname{im} e^{A_k\tau_k} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top\tau_k} + \operatorname{im} P_k(t)\\ &= \mathcal{R}_{[t_0,t)}.\end{aligned} \qquad \square$$

Remark 4 (Singularity of reachability Gramian): If the switched system (1) is reachable for all $t \in (t_0, t_f)$, i.e. $\mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ for $t \in (s_k, s_{k+1}]$, Lemma 3 implies that the reachability Gramians $\mathcal{P}_k^\sigma(t^-)$ are all non-singular. However, the right limits at the switching times $\mathcal{P}_k^\sigma(s_k^+) = J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top$ are in general singular, because we did not make a full row rank assumption on the jump map J_k (which in fact cannot hold if $n_k > n_{k-1}$). In particular, any time-varying coordinate transformation defined in terms of the nonsingular $\mathcal{P}_k^\sigma(t^-)$ will result in an unbounded behavior to the right of each switching time (unless the corresponding jump map J_k has full row rank). $\triangle$

We now present one of our main theoretical results which is the connection of the reachability Gramian with minimal input energy required to reach a given final state in a given time.

Theorem 5 (Reachability Gramian and input energy): Consider the switched system (1) with zero initial value and given switching signal. For some $t \in (s_k, s_{k+1}]$ assume that the corresponding reachability Gramian $\mathcal{P}_k^\sigma(t^-)$ as well as the classical Gramian $P_k(t)$ and all previous Gramians $\mathcal{P}_j^\sigma(s_{j+1}^-)$, $j = 0, 1, \ldots, k-1$ are positive definite. Then, for all $x_t \in \mathbb{R}^{n_k}$,

$$\min_u \int_{t_0}^t u(\tau)^\top u(\tau)\mathrm{d}\tau = x_t^\top \mathcal{P}_k^\sigma(t^-)^{-1} x_t,$$

where the minimum is taken over all $u : [t_0, t) \to \mathbb{R}^m$ which result in a solution of (1) with $x(t_0) = 0$ and $x(t^-) = x_t$. In other words, the directions of eigenvectors of $\mathcal{P}_k^\sigma(t^-)$ corresponding to the smallest eigenvalues require the most energy to be reached from zero.

Proof. It is well known (see e.g. [23]) that for a classical linear system $\dot{x} = Ax + Bu$ with reachability Gramian $P(t) := \int_{t_0}^t e^{A(\tau-t_0)} BB^\top e^{A^\top(\tau-t_0)}\mathrm{d}\tau$ the input

$$u(t) := B^\top e^{A^\top(t_1-t)} P(t_1)^{-1}(x_1 - e^{A(t_1-t_0)}x_0)$$

steers the state from x_0 towards x_1 on the interval $[t_0, t_1]$ and that this is the input with minimal energy $\int_{t_0}^{t_1} u^\top u = (x_1 - e^{A(t_1-t_0)}x_0)^\top P(t_1)^{-1}(x_1 - e^{A(t_1-t_0)}x_0)$ achieving this.

For $t \in (t_0, s_1]$ this already shows the claim of the theorem (with $x_0 = 0$). Proceeding inductively, assume that the claim is shown for $t = s_k$, and we want to show it for $t \in (s_k, s_{k+1}]$. In particular, we know that the minimal input energy to reach any z from zero on the interval $[t_0, s_k)$ is given by $z^\top P_{k-1}^\sigma(s_k^-) z$ and from the above statement we know that the minimal input energy to reach any x_t from $J_k z$ is given by $(x_t - e^{A_k(t-s_k)} J_k z)^\top P_k(t)^{-1}(x_t - e^{A_k(t-s_k)} J_k z)$, where

$P_k(t) := \int_{s_k}^t e^{A_k(\tau-s_k)}B_kB_k^\top e^{A_k^\top(\tau-s_k)}\mathrm{d}\tau$. Clearly, to reach x_t from zero on the time interval $[t_0,t)$ we have to find z^* which minimize the sum of the minimal energy to reach z on $[t_0,s_k)$ and the minimal energy to reach x_t from J_kz on $[s_k,t)$, i.e. we have to show that

$$\begin{aligned}&\min_z \big(z^\top P_{k-1}^\sigma(s_k^-)^{-1}z+\\ &\quad (x_t-e^{A_k(t-s_k)}J_kz)^\top P_k(t)^{-1}(x_t-e^{A_k(t-s_k)}J_kz)\big)\\ &= x_t^\top P_k^\sigma(t)^{-1}x_t.\end{aligned}$$

The above minimization has the form

$$\min_z(z^\top Mz-2c^\top z+\alpha)$$

with $M = P_{k-1}^\sigma(s_k^-)^{-1}+J_k^\top e^{A_k^\top(t-s_k)}P_k(t)^{-1}e^{A_k(t-s_k)}J_k$, $c^\top = x_t^\top P_k(t)^{-1}e^{A_k(t-s_k)}J_k$, $\alpha = x_t^\top P_k(t)^{-1}x_t$. Note that M is the sum of a symmetric positive definite matrix $P_{k-1}^\sigma(s_k^-)^{-1}$ and a symmetric positive semi-definite matrix and hence it is itself a symmetric positive definite matrix. Consequently, the unique minimizer is given by $z^* := M^{-1}c$ and the minimal value is then $\alpha - c^\top M^{-1}c$. Hence it remains to be shown that

$$\begin{aligned}&P_k(t)^{-1}-P_k(t)^{-1}e^{A_k(t-s_k)}J_kM^{-1}J_k^\top e^{A_k^\top(t-s_k)}P_k(t)^{-1}\\ &= P_k^\sigma(t)^{-1}\\ &= \Big(e^{A_k(t-s_k)}J_k\mathcal{P}_{k-1}^\sigma(s_k^-)J_k^\top e^{A_k^\top(t-s_k)}+P_k(t)\Big)^{-1}.\end{aligned}\tag{9}$$

Recall the well known Woodbury matrix identity which states that for any $P_0\in\mathbb{R}^{n_0\times n_0}$, $P_1\in\mathbb{R}^{n_1\times n_1}$ invertible and $F_1\in\mathbb{R}^{n_1\times n_0}$ we have that,

$$\begin{aligned}&(F_1P_0F_1^\top+P_1)^{-1}\\ &\qquad = P_1^{-1}-P_1^{-1}F_1(P_0^{-1}+F_1^\top P_1^{-1}F_1)^{-1}F_1^\top P_1^{-1}.\end{aligned}$$

With $P_0 = P_{k-1}^\sigma(s_k^-)$, $P_1 = P_k(t)$, $F_1 = e^{A_k(t-s_k)}J_k$ this identity equals exactly the desired relationship (9). $\square$

B. Observability Gramian

We propose the following recursive definition for the (time-varying) observability Gramian for the switched system (1):

$$\begin{aligned}&\mathcal{Q}_{\mathtt{m}}^\sigma(t) := Q_{\mathtt{m}}(t),\quad t\in[s_{\mathtt{m}},t_f),\\ &\mathcal{Q}_k^\sigma(t) := e^{A_k^\top(s_{k+1}-t)}J_{k+1}^\top\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)J_{k+1}e^{A_k(s_{k+1}-t)}\\ &\quad +Q_k(t),\quad t\in[s_k,s_{k+1}),\end{aligned}\tag{10}$$

where $Q_k(t) := \int_t^{s_{k+1}} e^{A_k^\top(s_{k+1}-\tau)}C_k^\top C_ke^{A_k(s_{k+1}-\tau)}\,\mathrm{d}\tau$ is the classical observability Gramian of mode k on the interval (t,s_{k+1}). The intuition for this definition is that starting from the time-limited observability Gramian of the last mode, the observability Gramian for the interval (t,t_f) is composed of the Gramian for $(k+1)$-st mode on (s_{k+1},t_f) which is propagated backwards in time under the jump J_{k+1} together with the matrix exponential of k-th mode and the classical observability Gramian of mode k on the interval (t,s_{k+1}). We first show that the kernel of the observability Gramian is indeed the unobservable space of the switched system.

Lemma 6: For all $k\in\mathtt{M}$ and $t\in[s_k,s_{k+1})$,

$$\mathcal{U}_{[t,t_f)} = \ker\mathcal{Q}_k^\sigma(t^+).$$

In particular, (1) is observable on $[t,t_f)$ i.e., $\mathcal{U}_{[t,t_f)}=\{0\}$ if, and only if, $\mathcal{Q}_0^\sigma(t^+)$ is positive definite.

Proof. It is well known (see e.g. [23]) that $\mathcal{U}_{[t,t_f)} = \mathcal{U}_{\mathtt{m}} = Q_{\mathtt{m}}(t) = \ker\mathcal{Q}_{\mathtt{m}}^\sigma(t)$ for all $t\in[s_{\mathtt{m}},t_f)$. Proceeding inductively, assume now that for some $k<\mathtt{m}$,

$$\mathcal{U}_{[s_{k+1},t_f)} = \ker\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)$$

and we will then show that for $t\in[s_k,s_{k+1})$

$$\mathcal{U}_{[t,t_f)} = \ker\mathcal{Q}_k^\sigma(t^+).$$

We first observe that $\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)^{1/2}$ is well defined because $\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)$ is symmetric and positive semi-definite, furthermore, $\ker\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+) = \mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)^{1/2}$. Together with the general property $M^{-1}\ker N = \ker NM$ for arbitrary suitable matrices M and N we have

$$\begin{aligned}&e^{-A_k(s_{k+1}-t)}J_{k+1}^{-1}\ker\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)\\ &\qquad\qquad = \ker\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)^{1/2}J_{k+1}e^{A_k(s_{k+1}-t)}.\end{aligned}$$

Utilizing further that for any matrix M it holds that $\ker M = \ker M^\top M$, we have that $\mathcal{U}_{[t,t_f)}$ is equal to

$$\ker e^{A_k^\top(s_{k+1}-t)}J_{k+1}^\top\mathcal{Q}_{k+1}^\sigma(s_{k+1})J_{k+1}e^{A_k(s_{k+1}-t)}\cap\mathcal{U}_k.$$

Since $\mathcal{U}_k = \ker Q_k(t)$ for all $t\in[s_k,s_{k+1})$, the claim follows from the fact that $\ker(M_1+M_2) = \ker(M_1)\cap\ker(M_2)$ for any two positive semi-definite matrices M_1 and M_2. $\square$

Before stating the relationship between the observability Gramian and the output energy, we would like to make a remark about the singularity of the observability Gramian similar to Remark 4.

Remark 7 (Singularity of observability Gramian): By Lemma 6 the observability Gramian $\mathcal{Q}_k^\sigma(t^+)$ is non-singular for all $t\in[s_k,s_{k+1})\subseteq[t_0,t_f)$ if the switched system is observable for all $t\in[t_0,t_f)$, i.e. $\mathcal{U}_{[t,t_f)}=\{0\}$. However, the left limit at the switching times $\mathcal{Q}_k^\sigma(s_{k+1}^-) = J_{k+1}^\top\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)J_{k+1}$ is in general singular, because we did not make any full column rank assumptions on the jump matrix J_{k+1} (which in fact cannot be satisfied if $n_k>n_{k+1}$). $\triangle$

Theorem 8 (Observability Gramian and output energy): Consider a solution of the switched system (1) with zero input on the interval $[t,t_f)$ for $t\in[s_k,s_{k+1})$ with corresponding observability Gramian $\mathcal{Q}_k^\sigma(t)$. Then the corresponding output satisfies

$$\int_t^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(t^+)^\top\mathcal{Q}_k^\sigma(t^+)x(t^+).\tag{11}$$

In other words, states values at time t which are in the direction of an eigenvector corresponding to the smallest eigenvalue of $\mathcal{Q}_k^\sigma(t^+)$ produce only a small amount of output energy and are therefore hard to observe.

Proof. For $t\in[s_{\mathtt{m}},t_f)$ we have $y(\tau)=e^{A_{\mathtt{m}}(\tau-t)}x(t^+)$ and hence $\int_t^{t_f} y(\tau)^\top y(\tau)\mathrm{d}\tau = x(t^+)^\top Q_{\mathtt{m}}(t)x(t^+) = \mathcal{Q}_{\mathtt{m}}^\sigma(t^+)$. Proceeding inductively assume now that for some $k=\mathtt{m}-1,\mathtt{m}-2,\dots,0$, we have

$$\int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\,\mathrm{d}\tau = x(s_{k+1}^+)^\top\mathcal{Q}_{k+1}^\sigma(s_{k+1}^+)x(s_{k+1}^+)\tag{12}$$

and we will show that then for all $t \in [s_k, s_{k+1})$ the equation (11) holds for any solution $x(\cdot)$ of (1) with zero input on $[t, t_f)$. Then

$$\int_t^{t_f} y(\tau)^\top y(\tau)\mathrm{d}\tau = \int_t^{s_{k+1}} y(\tau)^\top y(\tau)\mathrm{d}\tau + \int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\mathrm{d}\tau.$$

For $\tau \in [t, s_{k+1})$ we have $y(\tau) = e^{A_k(\tau - t)}x(t^+)$ and hence

$$\int_t^{s_{k+1}} y(\tau)^\top y(\tau)\mathrm{d}\tau = x(t^+)^\top Q_k(t)x(t^+).$$

From (12) together with $x(s_{k+1}^+) = J_{k+1}e^{A_k(s_{k+1}-t)}x(t^+)$ we can conclude that $\int_{s_{k+1}}^{t_f} y(\tau)^\top y(\tau)\mathrm{d}\tau$ is equal to

$$x(t^+)^\top e^{A_k^\top(s_{k+1}-t)} J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(s_{k+1}^+) J_{k+1} e^{A_k(s_{k+1}-t)} x(t^+).$$

Altogether we arrive at (11). □

IV. Midpoint balanced truncation

A. Motivation and algorithm

For any specific time $t \in (s_k, s_{k+1}) \subseteq (t_0, t_f)$ the corresponding reachability Gramian $\mathcal{P}_k^\sigma(t^-)$ and observability Gramian $\mathcal{Q}_k^\sigma(t^+)$ give us precise quantitative information about which state direction is difficult to reach from zero on the interval $[t_0, t)$ and which state direction is difficult to observe from the output on $[t, t_f)$. While in the reduced realization context, the property of unreachability or unobservability is to some degree independent from the actual mode-duration, this is not the case for the quantitative measure of reachability or observability. In fact, the values of the integrals in the definitions of the Gramians explicitly depend on the mode-duration and due to the positive-semidefinite nature of the involved matrices the magnitude of the Gramians will increase with an increased mode-duration. This implies that it is in principle not possible to have a good model reduction method for switched systems which is independent of the mode-duration.

In addition to the dependence of the Gramians on the mode-duration we also have that the Gramians are also depending on the time $t \in (s_k, s_{k+1})$ within a given mode. Since we aim to obtain a switched linear system of the form (2) as a reduced model, each mode needs to be reduced in a time-invariant fashion and it needs to be decided which information of the time-varying Gramians is utilized for the reduction method. We propose in the following to take the Gramians evaluated at the midpoints

$$g_k = \frac{s_k + s_{k+1}}{2},\ k = 0, 1, \ldots, \mathtt{m}.$$

of each mode as the basis of the model reduction, i.e. we consider

$$\overline{\mathcal{P}}_k^\sigma := \mathcal{P}_k^\sigma(g_k) \quad \text{and} \quad \overline{\mathcal{Q}}_k^\sigma := \mathcal{Q}_k^\sigma(g_k).$$

In particular, we make the implicit assumption that a state-direction which is difficult to reach/observe at the midpoint of a mode is also difficult to reach/observe in the whole time interval (s_k, s_{k+1}) in which mode k is active.

The key intuition behind the midpoint Gramians is that mode k needs to be active for a while so that one can really see which states are easy to reach in this mode (i.e. the corresponding reachability Gramian is changed sufficiently). The same applies to the observability Gramian, but there it is needed do stay long enough in mode k. So, in this sense, the middle point is 'optimal' as any other choice would make relative reachability and observability properties smaller compared to the already calculated reachability / observability properties from the past / future. Another aspect to consider is the potential singularity of the Gramians around the switching times as pointed out in Remarks 4 and 7, which indicates that balancing each mode based on Gramians close to the switching times would lead to coordinate transformations which are close to being singular; again this motivates to choose the midpoint Gramians because they are "farthest away" from potential singularities.

As a first step of the model reduction we first need to identify states which are simultaneously difficult to reach and difficult to observe (quantified via the midterm Gramians). This can be achieved by constructing a mode-dependent coordinate transformation $\widetilde{x}_k = T_k x_k$ in such a way that the corresponding midpoint Gramians (w.r.t. to the new coordinates) become equal and diagonal (i.e. balanced). Before continuing the discussion, let us highlight how a mode-wise coordination transformation effects the form of the switched system and the corresponding Gramians.

Lemma 9: Consider the switched systems (1) and a mode-wise coordinate transformation $\widetilde{x}_k = T_k x_k$ for a family of invertible matrices $T_k \in \mathbb{R}^{n_k \times n_k}$, $k \in \mathtt{M}$. Then the input-output behavior of (1) with zero initial value is equal to the input-output behavior of

$$\widetilde{\Sigma}_\sigma : \begin{cases} \dot{\widetilde{x}}_k(t) = \widetilde{A}_k \widetilde{x}_k(t) + \widetilde{B}_k u(t),\ t \in (s_k, s_{k+1}), \\ \widetilde{x}_k(s_k^+) = \widetilde{J}_k \widetilde{x}_{k-1}(s_k^-), \quad k \in \mathtt{M}, \\ y(t) = \widetilde{C}_k \widetilde{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{13}$$

where $\widetilde{A}_k = T_k A_k T_k^{-1}$, $\widetilde{B}_k = T_k B_k$, $\widetilde{C}_k = C_k T_k^{-1}$ and $\widetilde{J}_k = T_k J_k T_{k-1}^{-1}$. Furthermore, the corresponding Gramians satisfy

$$\widetilde{\mathcal{P}}_k^\sigma(t) = T_k \mathcal{P}_k^\sigma(t) T_k^\top, \quad \widetilde{\mathcal{Q}}_k^\sigma(t) = T_k^{-\top} \mathcal{Q}_k^\sigma(t) T_k^{-1},$$

in particular, the eigenvalues of $\mathcal{P}_k^\sigma(t)\mathcal{Q}_k^\sigma(t)$ are invariant under mode-wise coordinate transformations.

Proof. This can easily be verified inductively. □

Remark 10 (Necessity of jumps): It should be noted that a mode-wise coordinate transformation applied to a linear switched system without jumps necessarily introduced jumps of the form $\widetilde{J}_k = T_k T_{k-1}^{-1}$. △

The following lemma is a well known result and shows how a balancing coordinate transformation can be found.

Lemma 11 ([22]): For $P, Q \in \mathbb{R}^{n \times n}$ symmetric and positive definite, there exists invertible $T \in \mathbb{R}^{n \times n}$ and a diagonal positive definite matrix $\Xi \in \mathbb{R}^{n \times n}$ such that

$$TPT^\top = \Xi = T^{-\top} Q T^{-1}.$$

In fact, $T = \Xi^{-1/2} V^\top L^\top$ and $T^{-1} = RU\Xi^{-1/2}$ where $P = RR^\top$ and $Q = LL^\top$ is a Cholesky decomposition and $R^\top L = U \Xi V^\top$ is a singular value decomposition.

The idea is now to carry out a mode-wise balancing of the original switched system (1) based on the midpoint Gramians to obtained the transformed switched system (13) whose corresponding midpoint Gramians are equal and diagonal. We can now remove all state components (in the balanced coordinate system) corresponding to sufficiently small entries in the diagonal balanced midpoint Gramians to obtained a reduced systems which will have a similar input-output behavior because only those state components have been removed which are simultaneously difficult to reach and difficult to observe.

The overall midpoint balanced truncation method is summarized in Algorithm 1. Note that the algorithm will only be able to run successfully if each midterm Gramian is nonsingular, which in view of Lemmas 3 and 6 implies that the switched system is reachable and observable at each midpoint of each mode. If this condition is not satisfied it is possible to first eliminate unreachable and unobservable states via the method proposed by us recently [17].

Algorithm 1: Midpoint balanced truncation

Data: Modes (A_k, B_k, C_k, J_k), $k = 0, 1, \dots, \mathtt{m}$, switching times s_k, $k = 0, \dots, \mathtt{m}+1$, reduction threshold ε_k or desired reduction size $r_k \leq n_k$, $k = 0, 1, \dots, \mathtt{m}$.
Result: Reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k, \widehat{J}_k)$, $k = 0, \dots, \mathtt{m}$.

1 Compute the sequence of midpoint reachability Gramians $\overline{\mathcal{P}}_0^\sigma, \overline{\mathcal{P}}_1^\sigma, \cdots, \overline{\mathcal{P}}_{\mathtt{m}}^\sigma$.
2 Compute the sequence of midterm observability Gramians $\overline{\mathcal{Q}}_{\mathtt{m}}^\sigma, \overline{\mathcal{Q}}_{\mathtt{m}-1}^\sigma, \cdots, \overline{\mathcal{Q}}_0^\sigma$.
3 **for** $k = 0, 1 \dots, m$ **do**
4   **if** $\overline{\mathcal{P}}_k^\sigma$ *and* $\overline{\mathcal{Q}}_k^\sigma$ *nonsingular* **then**
5   Compute Cholesky decompositions $\overline{\mathcal{P}}_k^\sigma =: R_k R_k^\top$ and $\overline{\mathcal{Q}}_k^\sigma =: L_k L_k^\top$;
6   Compute singular value decomposition $R_k^\top L_k =: U_k \Xi_k V_k^\top$ with decreasing diagonal entries in Ξ_k;
7   In case threshold ε_k is given: choose maximal $r_k \leq n_k$ such that r_k-th entry of Ξ_k is bigger than ε_k;
8   Calculate transformation matrices $T_k := \Xi^{-1/2} V_k^\top L_k^\top$, $T_k^{-1} := R_k U_k \Xi^{-1/2}$;
9   Define left-projector Π_k^l as the first r_k rows of T_k and the right-projector Π_k^r as the first r_k columns of T_k^{-1};
10   Compute: $\widehat{A}_k := \Pi_k^l A_k \Pi_k^r$, $\widehat{B}_k := \Pi_k^l B_k$, $\widehat{C}_k := C_k \Pi_k^r$;
11   **if** $k > 0$ **then**
12   Compute: $\widehat{J}_k := \Pi_k^l J_k \Pi_{k-1}^r$;
13   **end**
14   **end**
15   **else**
16   Abort: Midterm Gramians not positive definite, no balanced truncation possible, apply reduced realization algorithm [17] first;
17   **end**
18 **end**

B. Numerical aspects

The main motivation for model reduction is usually that the state-dimensions of the original system is very large so that running (many) simulations or designing feedback controllers is not feasible. Hence it is necessary to critically reflect whether the proposed reduction method is in fact feasible for large scale systems. Clearly, the calculations of the midpoint Gramians (lines 1 and 2 in Algorithm 1 are by far the most expansive part of the whole method, followed by the Cholesky decomposition (line 5) and the singular value decomposition (line 6). Since the latter are also used in classical balanced truncation methods, there are already many efficient implementations available and we will not further discuss those.

In the following we will only discuss the calculation of the reachability Gramians, because the calculation of the observability Gramians can be carried out analogously by considering the transpose of it.

In order to obtain the midpoint reachability Gramians, we need efficient methods for 1) the calculation of the classical reachability Gramians $P_k(g_k)$ and $P_k(s_{k+1})$ for each mode k on the intervals $[s_k, g_k)$ and $[s_k, s_{k+1})$; and 2) the left- and right multiplication action of the matrix exponential $e^{A_k(g_k - s_k)} = e^{A_k \tau_k / 2}$ on the already calculated matrix $J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top$. Note that $e^{A_k(s_{k+1} - s_k)} = e^{A_k \tau_k} = (e^{A_k \tau_k / 2})^2$ which can be utilized in the calculation of the matrix $e^{A_k \tau_k} J_k \mathcal{P}_{k-1}^\sigma(s_k^-) J_k^\top e^{A_k^\top \tau_k}$.

The calculation of the classical reachability Gramian $\int_0^t e^{A\tau} B B^\top e^{A^\top \tau} \mathtt{d}\tau$ for given matrices A, B and a time-duration t has already been addressed in the context of time-limited balanced truncation [25], with further investigations in [26]–[28]. In particular, it can be shown that the reachability Gramian $P(t_1) = \int_{t_0}^{t_1} e^{A(\tau - t_0)} B B^\top e^{A^\top(\tau - t_0)} \mathtt{d}\tau$ for $\dot{x} = Ax + Bu$ considered on the interval $(t_0, t_1]$ is the solution of the Lyapunov equation

$$AP + PA^\top + BB^\top - e^{A(t_1 - t_0)} B B^\top e^{A^\top(t_1 - t_0)} = 0.$$

Hence, the calculation of $P_k(g_k)$ and $P_k(s_{k+1})$ reduces to the ability to efficiently calculate the matrix exponential and the ability to efficiently solve a Lyapunov equation. These are standard numerical tasks and efficient implementations exists for example in Matlab. The overall calculation of the midpoint reachability Gramian is summarized in Algorithm 2.

Algorithm 2: Midpoint reachability Gramians

Data: Modes (A_k, B_k, J_k), $k = 0, 1, \dots, \mathtt{m}$ and mode durations τ_k, $k = 0, \dots, \mathtt{m}$.
Result: Midpoint Gramians $\overline{\mathcal{P}}_k^\sigma$, $k = 0, \dots, \mathtt{m}$.

Initialization: $P_{-1}^\sigma = 0$;
for $k = 0, 1 \dots, m$ **do**
  Calculate: $F_{k,1/2} := e^{A_k \tau_k / 2}$;
  Obtain $P_{k,1/2}$ as solution of the Laypunov equation $A_k P_{k,1/2} + P_{k,1/2} A_k^\top + B_k B_k^\top - F_{k,1/2} B_k B_k^\top F_{k,1/2}^\top = 0$;
  Calculate: $\overline{\mathcal{P}}_k^\sigma := F_{k,1/2} J_k P_{k-1}^\sigma J_k^\top F_{k,1/2}^\top + P_{k,1/2}$;
  Calculate $F_k := (F_{k,1/2})^2$;
  Obtain P_k as solution of the Laypunov equation $A_k P_k + P_k A_k^\top + B_k B_k^\top - F_k B_k B_k^\top F_k^\top = 0$;
  if $k < m$ **then**
    Calculate $P_{k+1}^\sigma := F_k J_k P_{k-1}^\sigma J_k^\top F_k^\top + P_k$;
  end
end

For switched systems (1) with state dimensions up to one thousand, the matrix exponentials and the solution of the

Laypunov equations can be obtained on a current laptop within seconds with the standard Matlab functions `expm` and `lyap`, so the proposed method is already feasible for many practical problems without any further code optimization and sophisticated approximation techniques. How to adapt our approach to very large scale systems (state dimensions in the order of millions) is a numerical challenge but outside the scope of this contribution.

V. Numerical results

In the following we present an academic toy example to illustrate our proposed method. The main purpose of this example is to illustrate the individual calculation steps and show intermediate results; this is only possible for tiny state-space dimension for which one usually would not need to apply model reduction, but for more realistically sized examples, we would not be able to provide specific intermediate results.

Example 12: Consider a switched linear system with modes:

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 0.2 & 0.1 & 0.01 & 0.02 \\ 0.02 & 0.1 & 0.2 & 0.01 \\ 0.3 & 0.02 & 0.5 & 0.01 \\ 0.04 & 0.1 & 0.01 & 0.6 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0.7 \\ 1 \\ 0.01 \end{bmatrix}^\top \right),$$

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} -0.2 & 0.01 & 0 \\ 0.1 & 0.1 & 0.2 \\ 0 & 0.1 & -0.3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0.2 \\ -0.02 \end{bmatrix}, \begin{bmatrix} 0.1 \\ 0.01 \\ 0.004 \end{bmatrix}^\top \right),$$

$$(A_2, B_2, C_2) = \left(\begin{bmatrix} 0.8 & 0.1 & 0 & -0.1 & 0.01 \\ 0.07 & 0.5 & 0 & 0.1 & 0 \\ 0.1 & 0.2 & 0.3 & 0.01 & 0 \\ 0.1 & 0 & 0 & 0.1 & 0.01 \\ 0 & 0 & 0.1 & 0 & 0.4 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -0.2 \\ 0.1 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0.2 \\ 0.1 \\ 0.2 \end{bmatrix}^\top \right),$$

$$J_{1,0} = \begin{bmatrix} 0.3 & 1 & 0 & 0 \\ 0.1 & 0.2 & 0.1 & -1 \\ 0 & 0.1 & 0 & 1 \end{bmatrix}, \quad J_{2,1} = \begin{bmatrix} 1 & 0.1 & 0 \\ 0.02 & -0.2 & 0.1 \\ 0 & 0.01 & 0.1 \\ 0.1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

and the switching signal is given by $\sigma(t) = 0$ on $[0, 2)$, $\sigma(t) = 1$ on $[2, 3)$ and $\sigma(t) = 2$ on $[3, 5)$.

Step 1. Following Algorithm 2, we calculate the midpoint Gramians of the time intervals $[0, 2)$, $[2, 3)$ and $[3, 5)$:

$$\overline{\mathcal{P}}_0^\sigma = \begin{bmatrix} 5.7006 & 7.0498 & -5.1347 & 3.8804 \\ 7.0498 & 8.8090 & -6.3863 & 4.7298 \\ -5.1347 & -6.3863 & 4.6396 & -3.4675 \\ 3.8804 & 4.7298 & -3.4675 & 2.6942 \end{bmatrix},$$

$$\overline{\mathcal{P}}_1^\sigma = \begin{bmatrix} 23.6495 & -11.9148 & 2.6197 \\ -11.9148 & 8.4388 & -1.4232 \\ 2.6197 & -1.4232 & 0.2991 \end{bmatrix},$$

$$\overline{\mathcal{P}}_2^\sigma = \begin{bmatrix} 91.0695 & 19.9927 & 0.9495 & 14.3635 & 5.0961 \\ 19.9927 & 10.1823 & -1.8749 & 2.2150 & 1.0600 \\ 0.9495 & -1.8749 & 0.8944 & 0.5362 & 0.0800 \\ 14.3635 & 2.2150 & 0.5362 & 2.4461 & 0.8301 \\ 5.0961 & 1.0600 & 0.0800 & 0.8301 & 0.3060 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_0^\sigma = \begin{bmatrix} 14.2249 & 7.2736 & 6.3552 & -2.8392 \\ 7.2736 & 10.4901 & 4.4036 & -7.4309 \\ 6.3552 & 4.4036 & 3.0521 & -2.2824 \\ -2.8392 & -7.4309 & -2.2824 & 5.8998 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_1^\sigma = \begin{bmatrix} 7.9665 & 3.6209 & -1.0210 \\ 3.6209 & 1.6646 & -0.4353 \\ -1.0210 & -0.4353 & 0.1814 \end{bmatrix},$$

$$\overline{\mathcal{Q}}_2^\sigma = \begin{bmatrix} 2.1060 & -3.6360 & 0.3586 & -0.1436 & 0.3697 \\ -3.6360 & 6.3082 & -0.6224 & 0.2310 & -0.6410 \\ 0.3586 & -0.6224 & 0.0614 & -0.0226 & 0.0632 \\ -0.1436 & 0.2310 & -0.0226 & 0.0193 & -0.0237 \\ 0.3697 & -0.6410 & 0.0632 & -0.0237 & 0.0651 \end{bmatrix}.$$

Step 2. The corresponding balanced midpoint Gramians are

$$\Xi_0 = \begin{bmatrix} 10.3447 & 0 & 0 & 0 \\ 0 & 1.0422 & 0 & 0 \\ 0 & 0 & 0.0020 & 0 \\ 0 & 0 & 0 & 10^{-6} \end{bmatrix},$$

$$\Xi_1 = \begin{bmatrix} 10.5854 & 0 & 0 \\ 0 & 0.2399 & 0 \\ 0 & 0 & 0.0063 \end{bmatrix},$$

$$\Xi_2 = \begin{bmatrix} 10.6192 & 0 & 0 & 0 & 0 \\ 0 & 0.5082 & 0 & 0 & 0 \\ 0 & 0 & 0.0014 & 0 & \\ 0 & 0 & 0 & 10^{-5} & 0 \\ 0 & 0 & 0 & 0 & 10^{-10} \end{bmatrix}.$$

With a truncation threshold of 0.1, the reduced dimensions is two for all three modes.

Step 3. We calculate the left- and right projectors according to Algorithm 1:

$$(\overline{\Pi}_0^l, \overline{\Pi}_0^r) = \left(\begin{bmatrix} -1.1180 & 1.0923 \\ -0.8135 & -1.8677 \\ -0.5399 & 0.0753 \\ 0.4345 & 1.9427 \end{bmatrix}^\top, \begin{bmatrix} -0.7395 & 0.2032 \\ -0.9227 & -0.0423 \\ 0.6694 & -0.0653 \\ -0.4972 & 0.3624 \end{bmatrix} \right),$$

$$(\overline{\Pi}_1^l, \overline{\Pi}_1^r) = \left(\begin{bmatrix} -0.8674 & -0.0995 \\ -0.3935 & -0.3247 \\ 0.1122 & -0.4054 \end{bmatrix}^\top, \begin{bmatrix} -1.4672 & 1.8930 \\ 0.6475 & -4.0771 \\ -0.1586 & 0.3347 \end{bmatrix} \right),$$

$$(\overline{\Pi}_2^l, \overline{\Pi}_2^r) = \left(\begin{bmatrix} -0.4453 & -0.0207 \\ 0.7694 & -0.2092 \\ -0.0759 & 0.0226 \\ 0.0301 & 0.1375 \\ -0.0782 & 0.0182 \end{bmatrix}^\top, \begin{bmatrix} -2.3741 & -7.8328 \\ -0.0888 & -4.4533 \\ -0.1811 & 0.9209 \\ -0.4449 & -0.7821 \\ -0.1374 & -0.4051 \end{bmatrix} \right).$$

Step 4. Applying the left- and right projectors according to Algorithm 1 we obtain the reduced switched system (2) given by

$$\begin{aligned} (\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) &= \left(\overline{\Pi}_0^l A_0 \overline{\Pi}_0^r, \overline{\Pi}_0^l B_0, C_0 \overline{\Pi}_0^r \right) \\ &= \left(\begin{bmatrix} 0.0264 & 0.0389 \\ -1.1032 & 0.4960 \end{bmatrix}, \begin{bmatrix} -3.1623 \\ -1.6263 \end{bmatrix}, \begin{bmatrix} -2.2001 \\ 0.5182 \end{bmatrix}^\top \right), \\ (\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) &= \left(\overline{\Pi}_1^l A_1 \overline{\Pi}_1^r, \overline{\Pi}_1^l B_1, C_1 \overline{\Pi}_1^r \right) \\ &= \left(\begin{bmatrix} -0.2028 & 0.3664 \\ 0.0385 & 0.2969 \end{bmatrix}, \begin{bmatrix} -0.9483 \\ -0.1563 \end{bmatrix}, \begin{bmatrix} -0.1466 \\ 0.1619 \end{bmatrix}^\top \right), \\ (\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) &= \left(\overline{\Pi}_2^l A_2 \overline{\Pi}_2^r, \overline{\Pi}_2^l B_2, C_2 \overline{\Pi}_2^r \right) \\ &= \left(\begin{bmatrix} 0.6373 & 0.7872 \\ 0.0500 & 0.6001 \end{bmatrix}, \begin{bmatrix} 1.1555 \\ -0.4875 \end{bmatrix}, \begin{bmatrix} -2.3047 \\ 1.0988 \end{bmatrix}^\top \right), \end{aligned}$$

$$\widehat{J}_1 = \begin{bmatrix} 0.8566 & 0.1279 \\ 0.0722 & 0.1011 \end{bmatrix}, \quad \widehat{J}_2 = \begin{bmatrix} 0.4940 & 0.0110 \\ 0.0205 & -0.1383 \end{bmatrix}.$$

Figure 1(a) depicts the output of the original switched system and its approximation for the input $u(t) = 0.5 \sin(0.5t)$. Clearly, both outputs match nicely. The related relative errors between the two outputs are depicted in Figure 1(b), which shows that the relative error is less then 0.5%.

Next, we compute another reduced system by considering the larger threshold 0.25, then the dimensions of the reduced modes will be 2, 1 and 2, respectively. Finally, we consider another larger threshold 1.5, then the dimension of each reduced mode will be one. The input-output behavior is depicted in Figure 1(c) which results in an acceptable approximation for the threshold 0.25, but a bad approximation for threshold 1.5, especially in the last mode. $\triangle$

The example shows that there is clear relationship between the size of the removed eigenvalues of the balanced Gramians and the error between the output of the original and reduced system. However, it is not clear whether an explicit error bound similar to the classical balanced truncation method can be obtained.

VI. Conclusions

In this paper, we have proposed a reduction method for switched linear systems. We have defined suitable reachability and observability Gramians such that they provide precise quantitative information about how difficult to reach/observe a state is at a specified time. Based on this information we propose a mode-wise midpoint balanced truncation method which results in a reduced switched system whose input-output

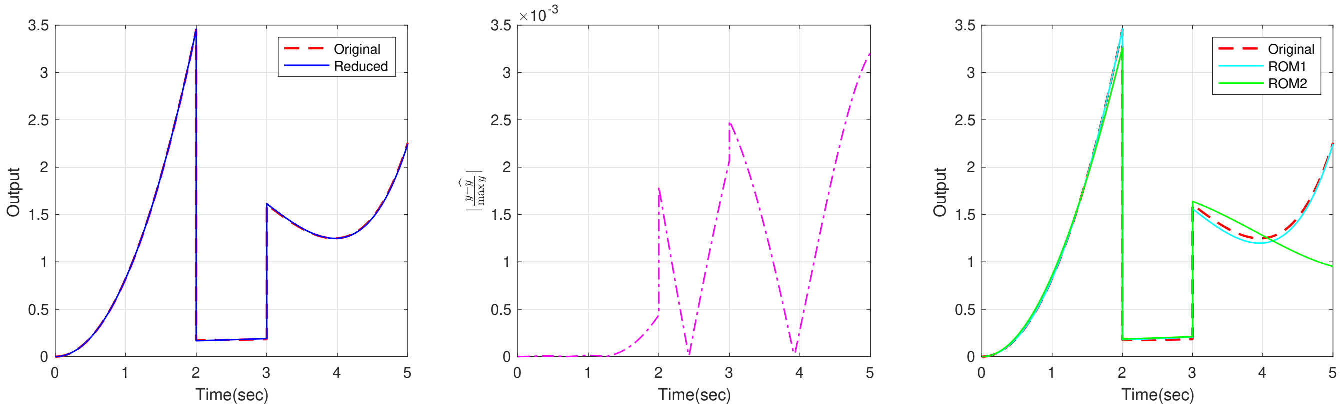

Fig. 1. (a) (left): Outputs of the original system and the proposed reduced system with truncation threshold 0.1 and input $u = 0.5\sin(0.5t)$. (b) (middle): Relative errors of original system and the proposed reduced system with truncation threshold 0.1. (c) (right): Outputs of the original system and the proposed reduced system with truncation threshold 0.25 (ROM1) and threshold 1.5 (ROM2).

behavior is similar to the original one. We have discussed numerical issues and for moderately large sized original systems our method is applicable, while for very large scale systems (e.g. millions of state variables) our method is not directly applicable and further adjustments are necessary. Furthermore, we cannot provide an error bound at the moment, although we believe that due to the proven energy interpretation our methods results in a very good approximation of the input-output behavior.

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