Sampled-data funnel control with zero-order hold*

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***Abstract*— Output reference tracking for nonlinear systems with arbitrary relative degree via sampled-data feedback control with zero-order hold is studied. We propose a novel sample-and-hold feedback controller, which achieves output reference tracking with prescribed transient behaviour of the tracking error. Furthermore, we derive explicit bounds on the maximal required input signal and the global sampling time such that the proposed controller is feasible for all times. The results are illustrated with a numerical example.**

I. Introduction

In the context of output-reference tracking *funnel control* is an established adaptive high-gain control methodology, which guarantees satisfaction of a-priori fixed, possibly time-varying output constraints while only imposing structural assumptions (known relative degree, a high-gain property, and bounded-input-bounded-state properties of the internal dynamics), see [1] and the references therein. In particular, for every feasible initial value, one can show that the required input signal to satisfy the constraints is bounded. The control is continuously adapted based on the continuously measured output signal (and its higher-order derivatives for higher relative degree).

Although, funnel control has been successfully implemented in a sampled-data system with Zero-order Hold (ZoH) for a sufficiently small sampling time in [2], we are not aware of any results rigorously showing that the output signal stays within the funnel boundaries for ZoH funnel control. In addition to the structural assumptions in funnel control, further assumptions seem to be required to allow for a fixed uniform sampling rate. In this paper, we propose a novel sampled-data feedback controller with ZoH, for which we rigorously show that the output constraints induced by the funnel boundary are maintained on the infinite-time horizon. Based on *some* knowledge (bounds on the dynamics) about the system, we provide a bound both on the sampling rate and the maximal required control input, which are indeed sufficient for meeting the output constraints. Furthermore, the controller design is uniform for all initial values within a compact set. For relative degree one, this set coincides with the whole funnel, i.e., we in particular allow "to start on the funnel boundary" (for higher relative degree the last error variable is allowed to start on the boundary). This is a novelty for funnel control.

To the best of our knowledge, in funnel control uniform bounds on the input signal are only known if either the region of feasible initial values is further restricted *and* the dynamics are known [1] or alternatively, the given output constraints can be (dynamically) relaxed [3], [4], [5]. Ensuring a uniform bound on the input is strongly related to input saturations and there have been several attempts to deal with these for funnel control [6], [3], [7], [8], [5], [4]. While some approaches, which directly take the input saturation into account in the controller design, e.g., the bang-bang funnel controller [9], exhibit some similarity with our approach, the analysis of combining a ZoH with funnel control has not been done so far in the literature. Using a ZoH can also be interpreted as introducing a time-delay into the closed loop, hence there is also a relationship to the references [10], [11]. The controller proposed in the present article includes an "activation threshold" (see also [12] where a comparable approach is pursued in the form of an activation function) setting the input to zero for small tracking errors, which is similar to the λ-tracker [13] or, in general, the event-triggered controller designs. In the latter, however, the system is monitored continuously. Another activation strategy is used in self-triggered control, where the task is to compute the control as well as the next time instant at which the input can be updated; in the present article, the uniform sampling times are computed in advance, but adapting them to current requirements is straightforward. For a comprehensive review of existing results in event- and self-triggered controllers see e.g., [14] and the references therein.

Notation: $\mathbb{N}, \mathbb{R}$ denote the set of natural and real numbers, respectively. $\mathbb{R}_{\geq 0} := [0, \infty)$. The scalar product is denoted by $\langle \cdot, \cdot \rangle$, and $\|x\| := \sqrt{\langle x, x \rangle}$ is the Euclidean norm of $x \in \mathbb{R}^n$. $\mathcal{B}_\rho := \{ x \in \mathbb{R}^n \mid \|x\| < \rho \}$. $\mathcal{C}^p(V, \mathbb{R}^n)$ is the linear space of p-times continuously differentiable functions $f : V \to \mathbb{R}^n$, where $V \subset \mathbb{R}^m$ and $p \in \mathbb{N}_0 \cup \{\infty\}$. Further, we set $\mathcal{C}(V, \mathbb{R}^n) := \mathcal{C}^0(V, \mathbb{R}^n)$. For an interval $I \subset \mathbb{R}$, $L^\infty(I, \mathbb{R}^n)$ denotes the space of measurable essentially bounded functions $f : I \to \mathbb{R}^n$ with norm $\|f\|_\infty := \operatorname{ess\,sup}_{t \in I} \|f(t)\|$. $L^\infty_{\text{loc}}(I, \mathbb{R}^n)$ is the space of locally bounded measurable functions. Further, $W^{k,\infty}(I, \mathbb{R}^n)$ is the Sobolev space of all k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \dots, f^{(k)} \in L^\infty(I, \mathbb{R}^n)$.

* D. Dennstädt, L. Lanza, and K. Worthmann acknowledge funding by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-IDs 471539468 and 507037103.

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A. Problem formulation

We consider nonlinear control-affine systems

$$\begin{aligned} y^{(r)}(t) &= f\big(d(t), \mathbf{T}(y, \dots, y^{(r-1)})(t)\big) \\ &\quad + g\big(d(t), \mathbf{T}(y, \dots, y^{(r-1)})(t)\big)u(t), \\ y|_{[-\sigma,0]} &= y^0 \in \mathcal{C}^{r-1}([-\sigma, 0], \mathbb{R}^m), \end{aligned} \tag{1}$$

where $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ represents an unknown bounded disturbance, $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^m)$ is a drift term, the function $g \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^{m\times m})$ is the input distribution, and the operator $\mathbf{T}$ is causal, locally Lipschitz and satisfies a bounded-input bounded-output property; the operator is characterised in detail in Definition 1.1 and the class of systems under consideration is introduced in Definition 1.2. We emphasise that many physical phenomena such as *backlash* and *relay hysteresis*, and *nonlinear time delays* can be modelled by means of the operator $\mathbf{T}$ (σ corresponds to the initial delay). Moreover, systems with infinite-dimensional internal dynamics can be represented by (1). For a control function $u \in L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, system (1) has a solution in the sense of *Carathéodory*, meaning a function $\xi : [-\sigma, \omega) \to \mathbb{R}^{rm}$, $\omega > 0$, with $\xi|_{[-\sigma,0]} = (y^0, \dot{y}^0, \dots, (y^0)^{(r-1)})$ such that $\xi|_{[0,\omega)}$ is absolutely continuous and satisfies $\dot{\xi}_i(t) = \xi_{i+1}(t)$ for $i = 1, \dots, r-2$, and $\dot{\xi}_r(t) = f(d(t), \mathbf{T}(\xi(t))) + g(d(t), \mathbf{T}(\xi(t)))u(t)$ (which corresponds to (1)) for almost all $t \in [0, \omega)$. A solution ξ is said to be *maximal*, if it does not have a right extension which is also a solution. For later use, we introduce the compact notation

$$\chi(y)(t) := (y(t), \dot{y}(t), \dots, y^{(r-1)}(t)) \in \mathbb{R}^{rm}$$

for $y \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and $t \in \mathbb{R}_{\geq 0}$.
The control objective is to design a zero-order hold control strategy, i.e., for sampling time $\tau > 0$,

$$u(t) \equiv u \qquad \forall\, t \in [t_i, t_i + \tau),\ i \in \mathbb{N},$$

where the data are collected at uniform sample times $t_i = i \cdot \tau \in \mathbb{R}_{\geq 0}$, which achieves for a system (1) output tracking of a given reference $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ within pre-specified error bounds. To be more precise, the tracking error $t \mapsto e(t) := y(t) - y_{\text{ref}}(t)$ shall evolve within the prescribed performance funnel

$$\mathcal{F}_\varphi = \{\ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m\ \mid \varphi(t)\, \|e\| < 1\ \}.$$

This funnel is determined by the choice of the function φ belonging to

$$\mathcal{G} := \left\{\ \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R})\ \middle|\ \inf_{s \geq 0} \varphi(s) > 0\ \right\},$$

see Figure 1 for an illustration. The specific application usually dictates the constraints on the tracking error and thus indicates suitable choices for φ. To achieve the control objective, we propose the following controller structure for $i \in \mathbb{N}$

$$\forall t \in [t_i, t_i{+}\tau) :\ u(t) = \begin{cases} 0, & \|e_r(t_i)\| < \lambda, \\ -\beta \frac{e_r(t_i)}{\|e_r(t_i)\|^2}, & \|e_r(t_i)\| \geq \lambda, \end{cases} \tag{2}$$

where $\lambda \in (0, 1)$ is an activation threshold, $\beta > 0$ is the input gain, and $e_r \in \mathcal{B}_1$ is an error variable introduced in (3) in Section I-C. In particular, the control function u is uniformly bounded since we have

$$\forall t \geq 0 : \quad \|u(t)\| \leq \frac{\beta}{\lambda}.$$

Our designed controller can be considered a sample-and-hold version of the well known and widely established funnel controller, see [16], [15], [1].

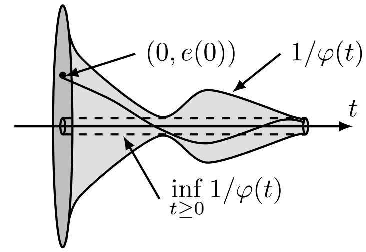

Fig. 1: Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $1/\varphi(t)$; the figure is based on [15, Fig. 1], edited for present purpose.

B. System class

In this section we formally introduce the system class under consideration. Prior to that, we state assumptions on the system parameters and characterise the operator $\mathbf{T}$.

Assumption 1: A bound $D > 0$ for the *unknown* disturbance $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ with $\|d\|_\infty \leq D$ is known.

Assumption 2: The matrix valued function $g \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^{m\times m})$ is strictly positive definite, that is

$$\forall \zeta \in \mathbb{R}^{p+q}\ \forall z \in \mathbb{R}^m \setminus \{0\} : \langle z, g(\zeta) z\rangle > 0.$$

Note that we could also allow the case of strictly negative g by changing the sign in (2). Next, we provide the defining properties of the class of operators to which $\mathbf{T}$ in (1) belongs.

Definition 1.1: For $n, q \in \mathbb{N}$ and $\sigma \geq 0$, the set $\mathcal{T}^{n,q}_\sigma$ denotes the class of operators $\mathbf{T} : \mathcal{C}([-\sigma, \infty), \mathbb{R}^n) \to L^\infty_{\text{loc}}(\mathbb{R}_{\geq 0}, \mathbb{R}^q)$ for which the following properties hold:

i) *Causality*: $\forall\, y_1, y_2 \in \mathcal{C}([-\sigma, \infty), \mathbb{R}^n)\ \ \forall\, t \geq 0$:
$$y_1|_{[-\sigma,t]} = y_2|_{[-\sigma,t]} \quad \Longrightarrow \quad \mathbf{T}(y_1)|_{[0,t]} = \mathbf{T}(y_2)|_{[0,t]}.$$

ii) *Local Lipschitz*: $\forall\, t \geq 0\ \ \forall\, y \in \mathcal{C}([-\sigma, t]; \mathbb{R}^n)$ $\exists\, \Delta, \delta, c > 0\ \ \forall\, y_1, y_2 \in \mathcal{C}([-\sigma, \infty); \mathbb{R}^n)$ with $y_1|_{[-\sigma,t]} = y = y_2|_{[-\sigma,t]}$ and $\|y_1(s) - y(t)\| < \delta$, $\|y_2(s) - y(t)\| < \delta$ for all $s \in [t, t + \Delta]$:
$$\operatorname*{ess\,sup}_{s\in[t,t+\Delta]} \|\mathbf{T}(y_1)(s) - \mathbf{T}(y_2)(s)\| \leq c \sup_{s\in[t,t+\Delta]} \|y_1(s) - y_2(s)\|.$$

iii) *Bounded-input bounded-output (BIBO)*: $\forall\, c_0 > 0$ $\exists\, c_1 > 0\ \ \forall\, y \in \mathcal{C}([-\sigma, \infty), \mathbb{R}^n)$:
$$\sup_{t\in[-\sigma,\infty)} \|y(t)\| \leq c_0 \implies \sup_{t\in[0,\infty)} \|\mathbf{T}(y)(t)\| \leq c_1.$$

While the first property (causality) introduced in Definition 1.1 is quite intuitive, the second (locally Lipschitz) is of a more technical nature, required to guarantee existence and uniqueness of solutions. The third property (BIBO) can be motivated from a practical point of view as an infinite-dimensional extension of minimum-phase. Various examples for the operator $\mathbf{T}$ can be found in [16], [1].

With Assumptions 1 and 2 and Definition 1.1 we formally introduce the system class under consideration.

Definition 1.2: For $m, r \in \mathbb{N}$ we say that a system (1) belongs to the system class $\mathcal{N}^{m,r}$, written $(d, f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$, if, for some $p, q \in \mathbb{N}$ and $\sigma \geq 0$, the following holds: $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ satisfies Assumption 1, $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q, \mathbb{R}^m)$, g satisfies Assumption 2, and $\mathbf{T} \in \mathcal{T}_\sigma^{rm,q}$.

Note that all linear minimum-phase systems with relative degree $r \in \mathbb{N}$ are contained in the system class $\mathcal{N}^{m,r}$, cf. [1].

C. Error variables and auxiliary results

In order to formulate the main result, we introduce some notation and establish two auxiliary results in this section. First, we introduce the error variables used in the control law (2). We define for $\varphi \in \mathcal{G}$, $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, a bijection $\alpha \in \mathcal{C}^1([0,1), [1,\infty))$, $t \geq 0$, and $\zeta := (\zeta_1, \ldots, \zeta_r) \in \mathbb{R}^{rm}$ the error variables

$$
\begin{aligned}
&e_1(t,\zeta) := \varphi(t)(\zeta_1 - y_{\text{ref}}(t)), \qquad (3)\\
&e_{k+1}(t,\zeta) := \varphi(t)(\zeta_{k+1} - y_{\text{ref}}^{(k)}(t)) + \alpha(\|e_k(t,\zeta)\|^2) e_k(t,\zeta),
\end{aligned}
$$

for $k = 1, \ldots, r-1$. A suitable choice for the bijection is $\alpha(s) := 1/(1-s)$. To guarantee that the tracking error $e = y - y_{\text{ref}}$ evolves within the boundary of $\mathcal{F}_\varphi$, we want to address the problem of ensuring that $\chi(y)(t)$ is at every time $t \geq 0$ an element of the set

$$
\mathcal{D}_t^r := \left\{ \zeta \in \mathbb{R}^{rm} \;\middle|\; \begin{array}{l} \|e_k(t,\zeta)\| < 1, \;\; k = 1, \ldots, r-1, \\ \|e_r(t,\zeta)\| \leq 1 \end{array} \right\}.
$$

We define the set of all functions $\zeta \in \mathcal{C}^r([-\sigma,\infty), \mathbb{R}^m)$ which coincide with y^0 on the interval $[-\sigma, 0]$ and for which $\chi(y)(t) \in \mathcal{D}_t^r$ on the interval $[t_0, \delta)$ for $\delta \in (0,\infty]$:

$$
\mathcal{Y}_\delta^r := \left\{ \zeta \in \mathcal{C}^r([-\sigma,\infty), \mathbb{R}^m) \;\middle|\; \begin{array}{l} \zeta|_{[-\sigma,0]} = y^0, \\ \forall\, t \in [0,\delta) : \chi(\zeta)(t) \in \mathcal{D}_t^r \end{array} \right\}.
$$

Now we may infer the existence of bounds for the error variables e_k defined in (3) for all functions in $\mathcal{Y}_\delta^r$ independent of the system dynamics (1).

Lemma 1.1: Let $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\varphi \in \mathcal{G}$, and $y^0 \in \mathcal{C}^{r-1}([-\sigma, 0], \mathbb{R}^m)$ with $\chi(y^0) \in \mathcal{D}_0^r$ be given. Then there exist constants $\varepsilon_k, \mu_k > 0$ such that for all $\delta \in (0,\infty]$ and all $\zeta \in \mathcal{Y}_\delta^r$ the functions e_k defined in (3) satisfy

i) $\|e_k(t, \chi(\zeta)(t))\| < \varepsilon_k < 1$,
ii) $\|\frac{\mathrm{d}}{\mathrm{d}t} e_k(t, \chi(\zeta)(t))\| < \mu_k$,

for all $t \in [0,\delta)$ and for all $k = 1, \ldots, r-1$.

Proof: The proof consists of two steps. In the first step we define the constants $\varepsilon_k, \mu_k > 0$ recursively. In the second step we invoke a Lyapunov-like argument to prove the assertions. To improve legibility, we use the notation $e_k(t) := e_k(t, \chi(\zeta)(t))$ for $\zeta \in \mathcal{Y}_\delta^r$.

Step 1. We define the constants ε_k, μ_k. Let $\hat{\varepsilon}_1 \in (0,1)$ be the unique point such that $\alpha(\hat{\varepsilon}_1^2)\hat{\varepsilon}_1 = \left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty + 1$, and set $\varepsilon_1 := \max\{\|e_1(0)\|, \hat{\varepsilon}_1\}$, $\mu_1 := \|\frac{\dot{\varphi}}{\varphi}\|_\infty + 1 + \alpha(\varepsilon_1^2)\varepsilon_1$, $\bar{\gamma}_0 := 0$ and $\bar{\gamma}_1 := 2\alpha'(\varepsilon_1^2)\varepsilon_1^2\mu_1 + \alpha(\varepsilon_1^2)\mu_1$. Successively for $k = 2, \ldots, r-1$ define

$$
\begin{aligned}
&\hat{\varepsilon}_k \in (0,1) \text{ s.t. } \alpha(\hat{\varepsilon}_k^2)\hat{\varepsilon}_k = \left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \alpha(\varepsilon_{k-1}^2)\varepsilon_{k-1}) + 1 + \bar{\gamma}_{k-1},\\
&\varepsilon_k := \max\{\|e_k(0)\|, \hat{\varepsilon}_k\} < 1, \qquad (4)\\
&\mu_k := \left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \alpha(\varepsilon_{k-1}^2)\varepsilon_{k-1}) + 1 + \alpha(\varepsilon_k^2)\varepsilon_k + \bar{\gamma}_{k-1},\\
&\bar{\gamma}_k := 2\alpha'(\varepsilon_k^2)\varepsilon_k^2\mu_k + \alpha(\varepsilon_k^2)\mu_k.
\end{aligned}
$$

Step 2. Let $\delta \in (0,\infty]$ and $\zeta \in \mathcal{Y}_\delta^r$ be arbitrary but fixed. We define the auxiliary function $\gamma_k(t) := \alpha(\|e_k(t)\|^2)e_k(t)$, and set $\gamma_0(\cdot) = \dot{\gamma}_0(\cdot) = 0$. Note that for $k = 1, \ldots, r-1$ each of the error signals defined in (3) satisfies for $t \in [0,\delta)$ the differential equation

$$
\begin{aligned}
\dot{e}_k &= \dot{\varphi} e^{(k-1)} + \varphi e^{(k)} + \dot{\gamma}_{k-1}\\
&= \frac{\dot{\varphi}}{\varphi}(e_k - \gamma_{k-1}) + e_{k+1} + \dot{\gamma}_{k-1} - \alpha(\|e_k\|^2)e_k,
\end{aligned}
$$

where the dependency on t has been omitted and $e^{(k)}$ denotes the k-th derivative of $e(t) = \zeta(t) - y_{\text{ref}}(t)$. We observe

$$
\dot{\gamma}_k = 2\alpha'(\|e_k\|^2)\langle e_k, \dot{e}_k\rangle e_k + \alpha(\|e_k\|^2)\dot{e}_k.
$$

Seeking a contradiction, we assume that for at least one $\ell \in \{1, \ldots, r-1\}$ there exists $t^* \in (0,\delta)$ such that $\|e_\ell(t^*)\|^2 > \varepsilon_\ell$. W.l.o.g. we assume that this is the smallest possible ℓ. Invoking $\chi(y^0) \in \mathcal{D}_0^r$ and continuity of the involved functions we may define $t_* := \max\left\{ t \in [0,t^*) \;\middle|\; \|e_\ell(t)\|^2 = \varepsilon_\ell \right\}$. Then, for $t \in [t_*, t^*]$ we calculate, omitting again the dependency on t,

$$
\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|e_\ell\|^2 = \langle e_\ell, \dot{e}_\ell\rangle\\
&= \left\langle e_\ell, \tfrac{\dot{\varphi}}{\varphi}(e_\ell - \gamma_{\ell-1}) + e_{\ell+1} + \dot{\gamma}_{\ell-1} - \alpha(\|e_\ell\|^2)e_\ell \right\rangle\\
&\leq \|e_\ell\| \left(\left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \alpha(\varepsilon_{\ell-1}^2)\varepsilon_{\ell-1}) + 1 + \bar{\gamma}_{\ell-1} - \alpha(\varepsilon_\ell^2)\varepsilon_\ell \right) \leq 0,
\end{aligned}
$$

in the last line we used the monotonicity of $\alpha(\cdot)$, the definition of ε_ℓ, and that $\dot{\gamma}_{\ell-1}$ is bounded by minimality of ℓ. Therefore, the contradiction $\varepsilon_\ell = \|e_\ell(t^*)\|^2 < \|e(t_*)\|^2 < \varepsilon_\ell$ arises. This yields boundedness of e_ℓ, γ_ℓ, and using the derived bounds we estimate

$$
\|\dot{e}_\ell\| \leq \left\|\frac{\dot{\varphi}}{\varphi}\right\|_\infty (1 + \alpha(\varepsilon_{\ell-1}^2)\varepsilon_{\ell-1}) + 1 + \alpha(\varepsilon_\ell^2)\varepsilon_\ell + \bar{\gamma}_{\ell-1} = \mu_\ell.
$$

Hence, we conclude $\|e_k(t)\| \leq \varepsilon_k < 1$ and $\|\dot{e}_k(t)\| \leq \mu_k$ for all $k = 1, \ldots, r-2$ and all $t \in [0,\delta)$. For $k = r-1$ the same arguments are valid invoking $e_r : [0,\delta) \to \overline{\mathcal{B}_1}$. ■

In the next result we derive bounds on the right-hand side of system (1).

Lemma 1.2: Consider (1) with $(d, f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$. Let $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, $\varphi \in \mathcal{G}$, $y^0 \in \mathcal{C}^{r-1}([-\sigma, 0], \mathbb{R}^m)$ with $\chi(y^0)(0) \in \mathcal{D}_0^r$, and $D > 0$ from Assumption 1. Then, there exist constants $f_{\max}, g_{\max}, g_{\min} > 0$ such that for every $\delta \in (0,\infty]$, $\zeta \in \mathcal{Y}_\delta^r$, $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ with $\|d\|_\infty \leq D$,

$z \in \mathbb{R}^n \setminus \{0\}$, and $t \in [0, \delta)$

$$\begin{aligned} f_{\max} &\geq \left\| f((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}) \right\|_\infty, \\ g_{\max} &\geq \left\| g((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}) \right\|_\infty, \\ g_{\min} &\leq \frac{\langle z, g((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}(t)) z \rangle}{\|z\|^2}. \end{aligned} \tag{5}$$

Proof: To prove the assertion, we invoke continuity of the system functions f, g and the resulting boundedness on compact sets. According to Lemma 1.1, there exist $\varepsilon_k \in (0,1)$ for $k = 1, \ldots, r-1$, and $\mu_{r-1} > 0$ such that

$$\forall \zeta \in \mathcal{Y}_\infty^r \ \forall t \in \mathbb{R}_{\geq 0} \ \forall k = 1, \ldots, r-1 : \|e_k(t, \zeta(t))\| < \varepsilon_k$$

and $\|\frac{\mathrm{d}}{\mathrm{d}t} e_{r-1}(t, \chi(\zeta)(t))\| < \mu_{r-1}$. Thus, due to the definition of e_k in (3), there exists a compact set $K_\zeta \subset \mathbb{R}^{rm}$ with

$$\forall \zeta \in \mathcal{Y}_\infty^r \ \forall t \in \mathbb{R}_{\geq 0} : \quad \chi(\zeta)(t) \in K_\zeta.$$

Due to the BIBO property of the operator $\mathbf{T}$, there exists a compact set $K_q \subset \mathbb{R}^q$ with $\mathbf{T}(\xi)(\mathbb{R}_{\geq 0}) \subset K_q$ for all $\xi \in \mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^{rm})$ with $\xi(\mathbb{R}_{\geq 0}) \subset K_\zeta$. For arbitrary $\delta \in (0, \infty)$ and $\zeta \in \mathcal{Y}_\delta^r$, we have

$$\forall t \in [0, \delta) \ \forall k = 1, \ldots, r-1 : \quad \|e_k(t, \zeta(t))\| < \varepsilon_k,$$

and $\|\frac{\mathrm{d}}{\mathrm{d}t} e_{r-1}(t, \chi(\zeta)(t))\| < \mu_{r-1}$ according to Lemma 1.1. Thus, $\chi(\zeta)(t) \in K_\zeta$ for all $t \in [0, \delta)$. For every element $\zeta \in \mathcal{Y}_\delta^r$ the function $\chi(\zeta)|_{[0,\delta)}$ can smoothly be extended to a function $\tilde{\zeta} \in (\mathcal{C}(\mathbb{R}_{\geq 0}, \mathbb{R}^m))^r$ with $\tilde{\zeta}(t) \in K_\zeta$ for all $t \in \mathbb{R}_{\geq 0}$. Due to the BIBO property of the operator $\mathbf{T}$, we have $\mathbf{T}(\tilde{\zeta})(t) \in K_q$ for all $t \in \mathbb{R}_{\geq 0}$. Since $\mathbf{T}$ is causal, this implies $\mathbf{T}(\chi(\zeta))|_{[0,\delta)}(t) \in K_q$ for all $t \in [0, \delta)$ and $\zeta \in \mathcal{Y}_\delta^r$. Define the compact set $K := \overline{\mathcal{B}_D} \times K_q \subset \mathbb{R}^{p+q}$. Since $f(\cdot)$ and $g(\cdot)$ are continuous, the constants $f_{\max} := \max_{x \in K} f(x)$ and $g_{\max} := \max_{x \in K} g(x)$ exist. For every $\delta \in (0, \infty]$, $\zeta \in \mathcal{Y}_\delta^r$, and $d \in L^\infty(\mathbb{R}_{\geq 0}, \mathbb{R}^p)$ with $\|d\|_\infty \leq D$ we have

$$\forall t \in [0, \delta) : \quad (d(t), \mathbf{T}(\chi(\zeta))(t)) \in K.$$

Therefore, we obtain $f_{\max} \geq \left\| f((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}) \right\|_\infty$ and $g_{\max} \geq \left\| g((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}) \right\|_\infty$. Since $g(x)$ is positive definite, for every $x \in K$ there exists $g_{\min} > 0$ such that $g_{\min} \leq \frac{\langle z, g((d, \mathbf{T}(\chi(\zeta)))|_{[0,\delta)}(t)) z \rangle}{\|z\|^2}$ for all $z \in \mathbb{R}^m \setminus \{0\}$. ■

II. Main Result

With the introductory results presented in the previous section, we are now in a position to formulate the main result. To phrase it, Theorem 2.1 yields that the ZoH controller (2) achieves the control objective discussed in Section I-A for a system (1) with $(d, f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$, if the sampling time τ satisfies condition (6).

Theorem 2.1: Given a reference $y_{\text{ref}} \in W^{r,\infty}(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ and a funnel function $\varphi \in \mathcal{G}$ consider a system (1) with $(d, f, g, \mathbf{T}) \in \mathcal{N}^{m,r}$. With the constants given in (4) (if $r = 1$, then we define $\varepsilon_0 := 0$) we set

$$\kappa_0 := \left\| \frac{\dot{\varphi}}{\varphi} \right\|_\infty (1 + \alpha(\varepsilon_{r-1}^2) \varepsilon_{r-1}) + \|\varphi\|_\infty (f_{\max} + \|y_{\text{ref}}^{(r)}\|_\infty) + \bar{\gamma}_{r-1},$$

define the input gain

$$\beta > \frac{2\kappa_0}{g_{\min} \inf_{s \geq 0} \varphi(s)},$$

and the constant

$$\kappa_1 := \left\| \frac{\dot{\varphi}}{\varphi} \right\|_\infty (1 + \alpha(\varepsilon_{r-1}^2) \varepsilon_{r-1}) + \|\varphi\|_\infty (f_{\max} + g_{\max} \beta + \|y_{\text{ref}}^{(r)}\|_\infty) + \bar{\gamma}_{r-1}.$$

Assume that the initial condition satisfies $\chi(y^0)(0) \in \mathcal{D}_0^r$, i.e., the error variables from (3) (here we omit the dependence on $\chi(y) = (y, \ldots, y^{(r-1)})$) satisfy

$$\forall k = 1, \ldots, r-1 : \|e_k(0)\| < 1, \quad \|e_r(0)\| \leq 1,$$

and, for an activation threshold $\lambda \in (0,1)$, let the sampling time satisfy

$$\tau \leq \min \left\{ \frac{\kappa_0}{\kappa_1^2}, \frac{1 - \lambda}{\kappa_0} \right\}. \tag{6}$$

Then the ZoH controller (2) applied to a system (1) yields that $\|e_k(t)\| < 1$ for all $k = 1, \ldots, r-1$ and $\|e_r(t)\| \leq 1$ for all $t \geq 0$. This is initial and recursive feasibility of the ZoH control law (2). In particular, the tracking error satisfies $\|e(t)\| < 1/\varphi(t)$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof: The proof consists of two main steps. In the first step we establish the existence of a solution of the initial value problem (1), (2). In the second step we show feasibility of the proposed control law, i.e., all error variables are bounded by ε_k and the tracking error evolves within the funnel boundaries.

Step 1. The application of the control signal (2) to system (1) leads to an initial value problem. If this problem is considered on the interval $[0, \tau]$, then there exists a unique maximal solution on $[0, \omega)$ with $\omega \in (0, \tau]$. If all error variables e_k evolve within the set $\mathcal{B}_1$ for all $t \in [0, \omega)$, then $\|\chi(y)(\cdot)\|$ is bounded on the interval $[0, \omega)$ and, as a consequence of the BIBS condition of the operator, $\mathbf{T}(\cdot)$ is bounded as well. Then $\omega = \tau$, cf. [17, § 10, Thm. XX] and there is nothing else to show. Seeking a contradiction, we assume the existence of $t \in [0, \omega)$ such that $\|e_k(t)\| \geq 1$ for at least one $k = 1, \ldots, r$. Invoking Lemma 1.1 it remains only to show that the last error variable e_r satisfies $\|e_r(t)\| \leq 1$ for all $t \in [0, \omega)$. Before we do so, we record the following observation. For $\gamma_{r-1}(t) := \alpha(\|e_{r-1}(t)\|^2) e_{r-1}(t)$ we calculate for $\zeta(\cdot) := (d(\cdot), \mathbf{T}(\chi(y))(\cdot))$

$$\begin{aligned} &\dot{e}_r(t) - \varphi(t) g(\zeta(t)) u = \dot{\varphi}(t) e^{(r-1)}(t) + \varphi(t) e^{(r)}(t) \\ &\quad + \dot{\gamma}_{r-1}(t) - \varphi(t) g(\zeta(t)) u \\ &= \frac{\dot{\varphi}(t)}{\varphi(t)} (e_r(t) - \gamma_{r-1}(t)) + \dot{\gamma}_{r-1}(t) + \varphi(t) (f(\zeta(t)) \\ &\quad + g(\zeta(t)) u - y_{\text{ref}}^{(r)}(t) - g(\zeta(t)) u) \\ &= \frac{\dot{\varphi}(t)}{\varphi(t)} (e_r(t) - \gamma_{r-1}(t)) + \dot{\gamma}_{r-1}(t) \\ &\quad + \varphi(t) (f(\zeta(t)) - y_{\text{ref}}^{(r)}(t)) =: J(t). \end{aligned} \tag{7}$$

Step 2. We show $\|e_r(t)\| \leq 1$ for all $t \in [0, \omega)$. We separately investigate the two cases $\|e_r(0)\| < \lambda$ and $\|e_r(0)\| \geq \lambda$.

Step 2.a We consider $\|e_r(0)\| < \lambda$. In this case we have $u = 0$. Seeking a contradiction, we suppose that there exists $t^* := \inf \{ \ t \in (0, \omega) \ \mid \ \|e_r(t)\| > 1 \ \}$. For the function $J(\cdot)$

introduced in (7) we observe $\|J|_{[0,t^*)}\|_\infty \le \kappa_0$ according to Lemmata 1.1 and 1.2. Then we calculate for $t \in [0, t^*]$

$$\begin{aligned} 1 &= \|e_r(t^*)\| \le \|e_r(0)\| + \int_0^{t^*} \|\dot{e}_r(s)\| \,\mathrm{d}s \\ &= \|e_r(0)\| + \int_0^{t^*} \|J(s)\| \,\mathrm{d}s \le \|e_r(0)\| + \int_0^{t^*} \kappa_0 \,\mathrm{d}s \\ &< \lambda + \kappa_0\omega < 1, \end{aligned}$$

where we used $t^* < \omega \le \tau < (1-\lambda)/\kappa_0$. This contradicts the definition of t^*.

Step 2.b We consider $\|e_r(0)\| \ge \lambda$. In this case we have $u = -\beta e_r(0)/\|e_r(0)\|^2$. We show again $\|e_r(t)\| \le 1$ for all $t \in [0, \omega)$. To this end, seeking a contradiction, we suppose the existence of $t^* = \inf\{\ (0,\omega)\ |\ \|e_r(t)\| > 1\ \}$. Invoking the initial conditions and continuity of the involved functions, and utilising Lemma 1.2 and (7), we calculate for $t \in [0, t^*]$

$$\begin{aligned} &\tfrac{\mathrm{d}}{\mathrm{d}t}\tfrac{1}{2}\|e_r(t)\|^2 = \langle e_r(t), \dot{e}_r(t)\rangle = \left\langle e_r(0) + \int_0^t \dot{e}_r(s)\,\mathrm{d}s, \dot{e}_r(t)\right\rangle \\ &\le \|e_r(0)\|\|J(t)\| + \omega\|\dot{e}_r|_{[0,t^*]}\|_\infty^2 + \varphi(t)\langle e_r(0), g(\zeta(t))u\rangle \\ &= \|e_r(0)\|\|J(t)\| + \omega\|\dot{e}_r|_{[0,t^*]}\|_\infty^2 - \varphi(t)\beta\tfrac{\langle e_r(0), g(\zeta(t))e_r(0)\rangle}{\|e_r(0)\|^2} \\ &\le \|e_r(0)\|\kappa_0 + \omega\|\dot{e}_r|_{[0,t^*]}\|_\infty^2 - \inf_{s\ge0}\varphi(s)g_{\min}\beta \\ &\le \kappa_0 + \omega\kappa_1^2 - \inf_{s\ge0}\varphi(s)g_{\min}\beta \le 2\kappa_0 - \inf_{s\ge0}\varphi(s)g_{\min}\beta < 0, \end{aligned}$$

the third line due to $t^* < \omega \le \tau$, the penultimate line via the definition of τ and the last line by definition of β; moreover, we used $\|\dot{e}_r|_{[0,t^*]}\| \le \kappa_1$ and $\|J|_{[0,t^*]}\|_\infty \le \kappa_0$. In particular this yields $\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_r(t)|_{t=0}\|^2 < 0$, by which $t^* > 0$. Therefore, we find the contradiction $1 = \|e_r(t^*)\|^2 < \|e_r(0)\|^2 \le 1$. Repeated application of the arguments in *Steps 1* and *2* on the interval $[t_i, t_i+\tau]$, $i \in \mathbb{N}$, yields recursive feasibility. ∎

Remark 2.1: We comment on some aspects of the main result Theorem 2.1.

i) The parameter $\lambda \in (0,1)$ is an activation threshold (cf. event-triggered control [14]), which is chosen by the designer. A large value of λ implies that the controller will be inactive for small values of the last error variable, which, in case of relative degree one, corresponds to a small tracking error.

ii) The sampling time τ in (6) strongly depends on the funnel function and its derivative, and on the higher-order derivatives of the reference y_{ref}. This means, that for stabilisation, i.e., $y_{\mathrm{ref}} \equiv 0$, the sampling time can be chosen larger compared to tracking of non-constant references. Moreover, adapting the sampling time to the evolution of the known reference and the known funnel function will yield larger sampling times for most of the time. For instance, consider the case of setpoint transition, where the reference is constant y^0_{ref} in the first period and constant $y^1_{\mathrm{ref}} \neq y^0_{\mathrm{ref}}$ in the last period, and in between it rapidly changes. Since the reference and the funnel functions are known in advance, the sampling time can be computed by taking the reference's and the funnel function's evolution into account. This idea results in solving an implicit inequality of the form $\tau_{i+1} \le R(\chi(y_{\mathrm{ref}}|_{[\tau_i,\tau_{i+1}]}), \varphi|_{[\tau_i,\tau_{i+1}]}, f_{\max}, g_{\max}, g_{\min})$ for some continuous function R.

iii) Note that $\|u\|_\infty \le \beta/\lambda$, i.e., an explicit bound on the control input can be computed in advance. This bound depends on the system parameters, namely the constants derived in Lemma 1.2. However, precise knowledge about the functions f, g and the operator $\mathbf{T}$ is not necessary to apply the controller. Mere (conservative) estimates on the bounds $f_{\max}$, $g_{\max}$, and $g_{\min}$ as in (5) are sufficient.

Remark 2.2: Note that the results in Theorem 2.1 are also valid for $\|e_r(0)\| = 1$. This is in contrast to continuous time funnel control, where all r error variables (3) initially have to be bounded away from 1 to guarantee boundedness of the input. To illustrate this, consider $\dot{y}(t) = u(t)$. Given a reference $y_{\mathrm{ref}} \in W^{1,\infty}(\mathbb{R}_{\ge0}, \mathbb{R})$, let $e(t) := y(t) - y_{\mathrm{ref}}(t)$, $\varphi \in \mathcal{G}$ and choose the bijection $\alpha(s) = 1/(1-s)$. According to [1] the control is given by $u(t) = -\frac{e(t)}{1-\varphi(t)^2e(t)^2}$. Now, for a sequence of initial values $y_j(0)$, $j \in \mathbb{N}$, such that $\varphi(0)^2(y_j(0) - y_{\mathrm{ref}}(0))^2 \to 1$ for $j \to \infty$, the sequence of corresponding initial controls $u_j(0)$ becomes unbounded. On the other hand, for the same sequence of initial values the controller (2) yields a bounded signal (bounded by β/λ).

III. Numerical Example

To illustrate the proposed controller (2) we consider the ODE with stable internal dynamics

$$\begin{aligned} \dot{y}(t) &= \sin(y(t)) + d(t) + \eta(t) + u(t), \quad & y(0) &= -0.195, \\ \dot{\eta}(t) &= -\eta(t) + y(t), & \eta(0) &= 0, \end{aligned}$$

and $d(t) = 0.1\sin(13t)$ as a bounded disturbance. Obviously, $g_{\max} = g_{\min} = 1$. The task is to track the reference within the error margins ± 0.2

$$y_{\mathrm{ref}}(t) = \begin{cases} 1, & 0 \le t \le 0.25, \\ 0.5(1-\cos(\pi(t-0.25))), & 0.25 \le t \le 1.25, \\ 0, & t \ge 1.25, \end{cases}$$

which mimics a setpoint transition from 0 to 1. With this, we have $f_{\max} = 2.3$. We emphasise that, although we can simply infer the bounds from the system parameters, the system's equations are not used within the controller. We choose the activation threshold $\lambda = 0.8$ and $\alpha(s) = 1/(1-s)$. With this we compute the sampling time $\tau = 4.1 \cdot 10^{-3}$, and the input gain $\beta \approx 10.88$, which already gives $\|u_{\mathrm{ZoH}}\|_\infty \le \beta/\lambda < 14$. We compare the proposed controller (2) with the continuous funnel controller [1]; corresponding signals have the subscript FC, e.g., y_{FC}. Moreover, a simulation of the ZoH controller was even successful for $\tau = 2.5 \cdot 10^{-2}$ and $\beta = 2$; corresponding signals are labelled with a circumflex, e.g., $\hat{y}_{\mathrm{ZoH}}$. Figure 2 shows the system's output and the reference plus/minus funnel boundary. All controllers achieve the tracking task. The red and black lines show some abrupt changes; these come from the discontinuity of the ZoH control law. In Figure 3 the controls are depicted. The ZoH input consists of separated pulses - for two reasons. First, since we do not assume knowledge of the system's equation, the control law (2) uses (undirected) worst-case estimations $g_{\min}, g_{\max}$ and $f_{\max}$ to compute the input signal. Therefore, the control signal is, at many time instances,

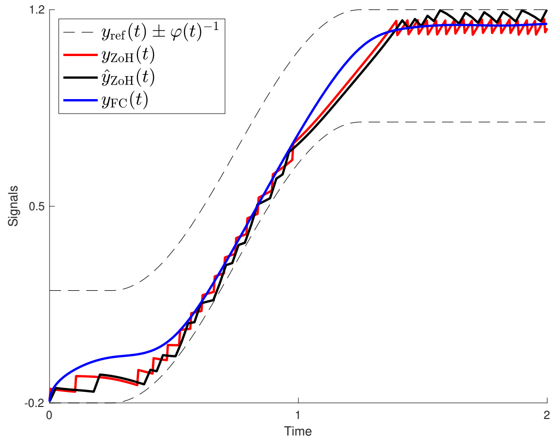

Fig. 2: Outputs, reference and funnel boundaries.

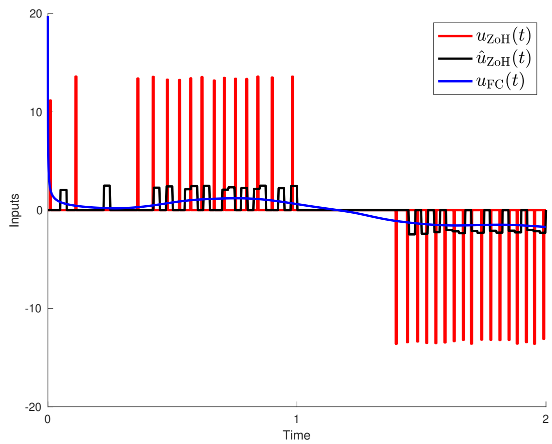

Fig. 3: Controls.

unnecessary large; however, it is ensured that the control signal is sufficiently large for all times. Second, the control law involves an activation threshold λ, i.e., the controller is inactive, if the tracking error is small. If at sampling the tracking error is above this threshold, the applied input is sufficiently large (due to the worst case estimations) to push the error back below the activation threshold. Therefore, at the next sampling instance the input is determined to be zero. The worst-case estimations and the ZoH setting make it inevitable that the control signal looks peaky. The control signal $\hat{u}_{\mathrm{ZoH}}$ (black) is also peaky, but not so large in magnitude (smaller β) and with a larger width (larger τ). Overall, $\hat{u}_{\mathrm{ZoH}}$ is comparable with u_{FC}. The success of the simulation with these parameters gives rise to the hope of finding better estimates for sufficient control parameters β, τ in future work. Note that the control signal u_{FC} also has a large peak at the beginning, hence $\|u_{\mathrm{FC}}\|_\infty \approx 20$. For simulation we used MATLAB, for integration of the dynamics the routine `ode15s` with $\mathrm{AbsTol} = \mathrm{RelTol} = 10^{-6}$, using adaptive time steps Δ. Integrating the funnel controller [1] we observe that $\Delta_{\max} \approx 3.04 \cdot 10^{-2}$ and $\Delta_{\min} \approx 4.05 \cdot 10^{-6}$. This means, the largest step $\Delta_{\max}$ is about seven times larger than τ. However, the smallest time step $\Delta_{\min}$ is about one thousand times smaller than τ.

IV. Conclusion and Future Work

In the present article we proposed a novel sampled-data zero-order hold feedback controller, which achieves output reference tracking with prescribed error performance for nonlinear systems with arbitrary relative degree. Bounds for the sampling time τ and the input signal are derived and given explicitly. Based on the presented results, future work will aim to reduce the conservatism of the proposed controller and to investigate the interplay with observers and/or the funnel pre-compensator [18], [19] to alleviate the strict assumption of not only the output but also its derivatives. Moreover, we plan to improve the proposed controller design and perform a comprehensive comparison (simulation study) with other sampled-data controllers.

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