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Dynamical boundary conditions for the water hammer problem

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1 Introduction

We are using hyperbolic balanced laws to model the fluid flow in the water pipes, and specifically the partial differential equation (PDE) we use is a 1d Euler system with a semi-linear friction term. This coupled system in divergence form with semi-linearity is otherwise known as the isothermal Euler or Saint-Venant equations with friction. In [1] a switched differential algebraic equation (sDAE) model is used to approximate the water hammer model, however the approximation accuracy has not been investigated there. Therefore, the goal of this article is to show that a class of solutions of the sDAE model for the water hammer problem converges to a nonlinear PDE solution under some assumptions on the water density. Our main result is a theorem which gives an accurate quantitative description of the pressure on the valve at the time of the valve closure using dynamical boundary conditions. [2].

2 Statement of the Main Theorem

The 1-d Euler model with friction term, otherwise known as the Saint-Venant equations with friction term are given by

$$(1)\quad \begin{cases} \partial_t \rho + \partial_x q = 0 \quad \text{in} \quad [0,T]\times[0,2L] \\ \partial_t q + \partial_x \left(\frac{q^2}{\rho} + p\right) = -\frac{c_f q|q|}{2D\rho} \quad [0,T]\times[0,2L], \\ q(0,x) = f(x) \\ p(0,x) = g(x) \end{cases}$$

where $\rho(t,x)$ is the water density, $q(t,x)$ is the water flux and $p(t,x)$ is the water pressure inside the pipe, with $t > 0$. The coefficient c_f is the friction constant and D is the diameter of the pipe. The PDE (1) can be closed by the pressure law $p(\rho)$, which is given by

$$(2)\quad p(\rho) = p_a + K\frac{\rho - \rho_a}{\rho_a},$$

where $K > 0$ is the bulk modulus, $P_a > 0$ is atmospheric pressure, and ρ_a is the water density at the atmospheric pressure. Correspondingly the sDAE is given by

$$(3)\quad \begin{aligned} &\frac{\mathrm{d}Q}{\mathrm{d}t} + \frac{A}{L}(P_L(t) - P_0(t)) + \frac{c_f}{2D\rho_a A}Q|Q| = 0 \\ &Q(0) = Q_0 \quad Q_0 \in \mathbb{R}^+, \end{aligned}$$

The switching signal is from the valve open and closing. When the valve is switched on at $t = t_1$, $Q(t)$ in (3) becomes 0, and if we let $v(t,x) = q(t,x)A - Q(t)$ after the valve is switched on, we have a superposition model with boundary and initial

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conditions such that for $t > t_1, 0 < x < L$,

(4) $$\begin{cases} \partial_t\rho + \frac{1}{A}\partial_x v = 0 \\ \partial_t v + \partial_x\left(\frac{v^2}{A\rho} + Ap\right) = -\frac{c_f v|v|}{2DA\rho}, \end{cases} \quad \begin{cases} \rho(t_1, x) = \rho_{t_1}(x) \\ v(t_1, x) = (\text{computed right before t} = \text{t}_1) \\ \rho(t, 0) = \rho_{1,0} \\ v(t, L) = 0 \end{cases}.$$

We make the following assumptions for our model which we list below,

ASSUMPTION 1 *We assume:*
In equation (3)*, we have that* $\frac{A}{L}(P_L(t) - P_0(t)) = P_d$ *for* $0 \le t < t_1$*, and* $P_d < 0$.
In the PDE (1)*, there is a constant* $\rho_a > 0$ *such that* $\frac{\rho(t,x)}{\rho_a} = 1 + \epsilon(t, x)$ *for* $(t, x) \in [0, t_1) \times [0, L]$*, where* $\epsilon(t, x)$ *is such that* $||\epsilon||_{C^1([t_0,t_1)\times[0,L])} \le \epsilon_0 < 1/2$. *Then accordingly for the derivative on pressure we have* $K\frac{\partial_x \rho}{\rho_a} = \frac{(P_L(t)-P_0(t))}{L} + K\epsilon_1(t, x)$*, where* $||\epsilon_1||_{C^1([t_0,t_1)\times[0,L])} \le \epsilon_0$.
In addition, we assume for the boundary conditions that one end of the pipe satisfies Dirichlet boundary conditions ($x = 2L$) *and one end is connected to a reservoir at* ($x = 0$)*, which meets the boundary conditions of well-posedness of the PDE given by [2]. However the scenario will work on any pipeline system where the solution is well posed in* $[0, 2L]$.

With the problem set above we then achieve our main theorem which is the following,

THEOREM 2 *Let* Q_0 *and* ρ_a *be positive constants. Let* t_0 *be the initial time at which* $Aq = Q = Q_0$ *and* $\rho(t, x) = \rho_a$ *inside the pipeline. Let* t_1 *be the time of closure of the valve at* $x = L$*. We let* $P_L(t)$ *be given explicitly by the following formula:*

(5) $$P_L(t) = P_0 + \frac{L}{A}\frac{2P_dCe^{2\sqrt{-P_dC_f}t}}{C_f(Ce^{2\sqrt{-P_dC_f}t} + 1)^2} \quad \text{with} \quad C = \frac{\sqrt{-P_d/C_f} + Q_0}{\sqrt{-P_d/C_f} - Q_0}$$

where $C_f = \frac{c_f}{2D\rho_a A}$ *and* P_d *is a chosen constant pressure difference in* $[0, t_1)$*. Let* $P_L^c(t) = P_0 - P_L(t)\delta(t - t_1)$*. Let* $p(\rho(t, L))$ *be the pressure on the valve from the PDE model* (1)*, and* t_2 *is the when the water flow goes to rest. We assume that the integral of the pressure along the pipe* $[0, L]$ *exists. Let* t_δ *be some short time period before* t_1 *for integral convenience,* M *be some constant,* ϕ *be some bump function and* $\tilde{u} = q/\rho$*. We then have that*

(6) $$\left|\int_{t_1-t_\delta}^{t_2} (p(\rho(t, L)) - P_L^c(t))dt\right| \le \frac{1}{A}\|v(t_1 - t_\delta, \cdot)\|_{L^1(0,L)}$$

where $v(t, x) = Aq(t, x) - Q(t, x)$*, and* $v(t, x)$ *satisfies the inequality*

(7) $$\|v(t, \cdot)\|_{L^2(0,L)} \le \epsilon_0 e^{\left(\frac{Mc_f}{2DA\rho_a} + K||\phi^2||_{C^1} + ||\tilde{u}||_{C^1}\right)t} \int_0^t \left(A + \frac{2(1 + A)}{\rho_a}\right)\|q(s, \cdot)\|^2_{L^2(0,L)} + KA\, ds$$

The proof of our main theorem comes from Gronwall's inequality from [3].

In conclusion, we used a pipeline model for the water hammer problem and discussed it by dividing the scenario into 2 phases, which is before and after the valve closure. We proved the time averaged pressure for the sDAE and the PDE model converge to each other.

References

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