Switch observability: A novel approach towards fault detection

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Joint work with my former PhD-student Ferdinand Küsters

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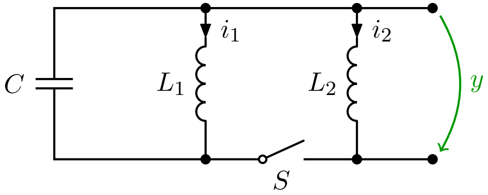
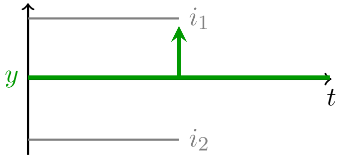
Motivational example





Switch		obsv.
open	$y \equiv 0$ for arbitrary internal state	✗
closed	equilibrium $i_1 = -i_2 =$ const $\rightarrow$ $y \equiv 0$	✗
closing	$y = 0$ jumps to $\neq 0$	✓
opening	non-equilibrium: $y \neq 0$ jumps to zero ($+$ Imp.)	✓
	equilibrium: $y(t) = 0\ \forall t$, but with impulse in y	✓

Transition "open→close" ($y \not\equiv 0$ on $(t_S, t_S + \varepsilon)$) distinguishable from transition "close→open" ($y \equiv 0$ on $(t_S, t_S + \varepsilon)$)

Discussion of example

Circuit is modelled by a switched differential-algebraic equation (DAE):

$$\boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x (+B_\sigma u) \\ y &= C_\sigma x \end{aligned}}$$

$\sigma : \mathbb{R} \to \{1, \ldots, P\}$ is the switching signal

Nonobservability on switch-free intervals

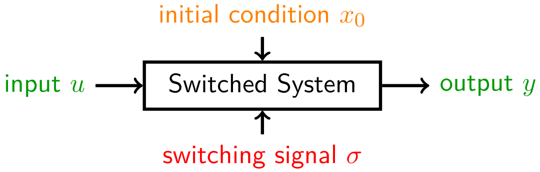
Using measurements only from switch-free intervals:

- Mode (i.e. switch position) cannot be recovered for some $x_0 \neq 0$
- Each individual mode is not state-observable

Observability around switch

- Modes before and after the switch can be recovered
- Internal states can completely be recovered
- Dirac impulses in output needed for observability

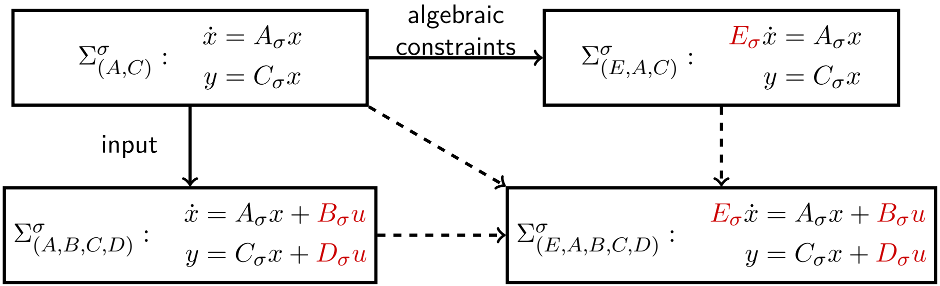
The observability problem



Observability questions

- Is there a unique x_0 for any given σ, u, y? $\to$ (t.v.) observability ✓
- Is there a unique (x_0, σ) for any given u and y?
  $\to$ (x, σ)**-observability**
- Is there a unique σ for any given u, y and unknown x_0?
  $\to$ σ**-observability** $=$ fault detectability (+isolation)
- Is there a unique set $\{t_S\}$ of switching times for any u, y?
  $\to$ t_S**-observability** $=$ fault detectability

System classes



Future work: Nonlinear versions thereof ...

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The simplest system class $\Sigma^\sigma_{(A,C)}$: $\begin{array}{l} \dot{x} = A_\sigma x \\ y = C_\sigma x \end{array}$

Formal Definition: (x,σ)-/σ-Observability

$\Sigma^\sigma_{(A,C)}$ **(x,σ)-observable** $:\Leftrightarrow \forall \sigma, \widehat{\sigma}\quad \forall$ sol. $x, \widehat{x}$ with $(x,\widehat{x}) \neq (0,0)$:

$$(x,\sigma) \neq (\widehat{x},\widehat{\sigma}) \implies y \neq \widehat{y}$$

$\Sigma^\sigma_{(A,C)}$ **σ-observable** $:\Leftrightarrow \forall \sigma, \widehat{\sigma}\quad \forall$ sol. $x, \widehat{x}$ with $(x,\widehat{x}) \neq (0,0)$:

$$\sigma \neq \widehat{\sigma} \implies y \neq \widehat{y}$$

First (surprising?) result for $\Sigma^\sigma_{(A,C)}$

(x,σ)-observability $\iff$ σ-observability

State-observability of each mode

In the context auf fault detection/isolation we have:

fault detection and isolation

$\Longleftrightarrow$

mode detectability

$\Longleftrightarrow$

σ-observability

$\Longleftrightarrow$

(x, σ)-observability

$\Longrightarrow$

(x, σ)-observability with constant σ

$\Longrightarrow$

(A_i, C_i) is observable for each mode i

Assuming (state-)observability for all faulty modes is not realistic.

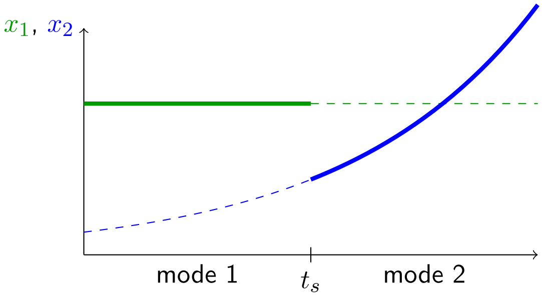
Weaker observability notion

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x$$

$$y = C_\sigma x$$

with

$$\begin{aligned} C_1 &= [1, 0] \\ C_2 &= [0, 1] \end{aligned} \rightarrow \text{ not observable}$$



Switch observability ((x, σ_1)-/σ_1-observability)

Recover (x and) σ from u and y, if at least one switch occurs

Again: σ_1-observability $\iff$ (x, σ_1)-observability

Obs. characterizations for $\Sigma^{\sigma}_{(A,C)}$: $\begin{aligned} \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}$

Kalman observability matrix of mode k: $\quad \mathcal{O}_k := \begin{bmatrix} C_k \\ C_k A_k \\ C_k A_k^2 \\ \vdots \end{bmatrix}$

Theorem (cf. Küsters & Trenn, Automatica 2018)

$$\sigma\text{-}\textit{observability} \quad \Longleftrightarrow \quad \forall i \neq j: \ \operatorname{rank}[\mathcal{O}_i \ \mathcal{O}_j] = 2n$$

$$\sigma_1\text{-}\textit{observability} \quad \Longleftrightarrow \quad \forall i \neq j, p \neq q, (i,j) \neq (p,q) : \operatorname{rank} \begin{bmatrix} \mathcal{O}_i & \mathcal{O}_p \\ \mathcal{O}_j & \mathcal{O}_q \end{bmatrix} = 2n$$

$$t_S\text{-}\textit{observability} \quad \Longleftrightarrow \quad \forall i \neq j: \ \operatorname{rank}[\mathcal{O}_i - \mathcal{O}_j] = n$$

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Adding inputs

$$\dot{x} = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x + D_\sigma u$$

Input-depending observability

$\Sigma(A_\sigma, C_\sigma)$ σ-observable $\not\Leftrightarrow$ $\Sigma(A_\sigma, B_\sigma, C_\sigma, D_\sigma)$ σ-observable

Strong vs. weak observability

observable for all u $\not\Leftrightarrow$ observable for some/almost all u

Further technicalities

Analytic vs. smooth inputs and equivalent switching signals

Strong obs. for $\Sigma^\sigma_{(A,B,C,D)}$:

$$\begin{aligned}\dot{x} &= A_\sigma x + B_\sigma u\\ y &= C_\sigma x + D_\sigma u\end{aligned}$$

Definition

$\Sigma^\sigma_{(A,B,C,D)}$ is strongly (x,σ)-/ σ-/ (x,σ_1)- /σ_1-/t_S-observable $:\Leftrightarrow$
$\forall u$: $\Sigma^\sigma_{(A,B,C,D)}$ is (x,σ)-/ σ-/ (x,σ_1)- /σ_1-/t_S-observable

Again it holds:

strong (x,σ)-observability $\iff$ strong σ-observability
strong (x,σ_1)-observability $\iff$ strong σ_1-observability

Zero-state problem

Property

$$x \equiv 0 \quad \iff \quad \exists t_0 \in \mathbb{R} : x(t_0) = 0$$

not valid anymore

Avoiding zero-state-problem, variant 1

Additional assumptions

(A1) u is real analytic

(A2) $\ker \begin{bmatrix} B_i \\ B_j \\ D_i - D_j \end{bmatrix} = \{0\} \ \forall i \neq j$

Notation:

$$\Gamma_k = \begin{bmatrix} D_k & & & & \\ C_k B_k & D_k & & & \\ C_k A_k B_k & C_k B_k & D_k & & \\ C_k A_k^2 B_k & C_k A_k B_k & C_k B_k & D_k & \\ \vdots & & & & \ddots \end{bmatrix}$$

Theorem (cf. Lou and Si 2009)

$\Sigma^\sigma_{(A,B,C,D)}$ *with (A1), (A2) is strongly σ-observable* $\Leftrightarrow$

$$\operatorname{rank}[\mathcal{O}_i \quad \mathcal{O}_j \quad \Gamma_i - \Gamma_j] = 2n + \operatorname{rank}(\Gamma_i - \Gamma_j) \quad \forall i \neq j$$

Relationship to ui-observability

Theorem (see e.g. Kratz (1995) or Hautus (1983))

$$\operatorname{rank}[\mathcal{O}_i\ \mathcal{O}_j\ \Gamma_i - \Gamma_j] = 2n + \operatorname{rank}(\Gamma_i - \Gamma_j)$$

$$\iff$$

$$\Sigma_{ij} : \boxed{\begin{aligned} \dot{\xi} &= \begin{bmatrix} A_i & 0 \\ 0 & A_j \end{bmatrix} + \begin{bmatrix} B_i \\ B_j \end{bmatrix} u \\ y_{\Delta_{i,j}} &= [C_i\ \ -C_j]\xi + (D_i - D_j)u \end{aligned}}$$

is *unknown-input (ui-) observable*

Strong t_S-/σ_1-observability (under (A1), (A2))

Theorem (Küsters and T. 2018)

$\Sigma^\sigma_{(A,B,C,D)}$ *is* ***t_S-observable*** $\iff$ $\forall i \neq j$:

$$\operatorname{rank}[\mathcal{O}_i - \mathcal{O}_j \quad \Gamma_i - \Gamma_j] = n + \operatorname{rank}(\Gamma_i - \Gamma_j)$$

and

$$\mathcal{R}(\Sigma_{ij}) = \{0\}$$

$\Sigma^\sigma_{(A,B,C,D)}$ *is* ***σ_1-observable*** $\iff$ $\forall i \neq j,\ p \neq q,\ (i,j) \neq (p,q)$:

$$\operatorname{rank}\begin{bmatrix} \mathcal{O}_i & \mathcal{O}_p & \Gamma_i - \Gamma_p \\ \mathcal{O}_j & \mathcal{O}_q & \Gamma_j - \Gamma_q \end{bmatrix} = 2n + \operatorname{rank}\begin{bmatrix} \Gamma_i - \Gamma_p \\ \Gamma_j - \Gamma_q \end{bmatrix}$$

and

$$\mathcal{R}(\Sigma_{ij}) = \{0\}$$

$\mathcal{R}(\Sigma_{ij})$ is set of initial values which are controllable to zero while output is zero.

Avoiding (A1) and (A2)

Definition (Equivalent switching signal, c.f. Kaba (2014))

For $\Sigma^\sigma_{(A,B,C,D)}$, initial value $x_0 \in \mathbb{R}^0$, input u

$$\sigma \overset{x_0,u}{\sim} \widetilde{\sigma} \quad :\Leftrightarrow \quad \begin{array}{c} x \equiv \widetilde{x},\ y \equiv \widetilde{y} \text{ and } \sigma(t) = \widetilde{\sigma}(t) \text{ expect on} \\ \text{intervals where the state is identically zero} \end{array}$$

Corresponding equivalence class: $[\sigma_{x_0,u}] := \left\{ \widetilde{\sigma} \;\middle|\; \sigma \overset{x_0,u}{\sim} \widetilde{\sigma} \right\}$

Definition

$\Sigma^\sigma_{(A,B,C,D)}$ is called $(x,[\sigma])$-, $[\sigma]$-, $(x,[\sigma_1])$, $[\sigma_1]$-, and $[t_S]$-observable $:\Leftrightarrow$ replace in previos definitions $\sigma \neq \widehat{\sigma}$ by $[\sigma_{x_0,u}] \neq [\widehat{\sigma}_{x_0,u}]$

Exactly the same rank-conditions as before!

Overview for $\Sigma^\sigma_{(A,B,C,D)}$: $\begin{aligned} \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}$

$u = 0$	u analytical $\wedge$ (A2)	equivalence classes for σ, u smooth
(x,σ)-observability $=$ **σ-observability**	strong (x,σ)-observability $=$ **strong σ-observability**	strong $(x,[\sigma])$-observability $=$ **strong $[\sigma]$-observability**
$\operatorname{rank}\begin{bmatrix}\mathcal{O}_i & \mathcal{O}_j\end{bmatrix} = 2n$	$\operatorname{rank}\begin{bmatrix}\mathcal{O}_i & \mathcal{O}_j & \Gamma_i - \Gamma_j\end{bmatrix} = 2n + \operatorname{rank}\left(\Gamma_i - \Gamma_j\right)$	
(x,σ_1)-observability $=$ **σ_1-observability**	strong (x,σ_1)-observability $=$ **strong σ_1-observability**	strong $(x,[\sigma_1])$-observability $=$ **strong $[\sigma_1]$-observability**
$\operatorname{rank}\begin{bmatrix}\mathcal{O}_i & \mathcal{O}_p \\ \mathcal{O}_j & \mathcal{O}_q\end{bmatrix} = 2n$	$\mathcal{R}\left(\Sigma_{i,j}\right) = \{0\} \wedge \operatorname{rank}\begin{bmatrix}\mathcal{O}_i & \mathcal{O}_p & \Gamma_i - \Gamma_p \\ \mathcal{O}_j & \mathcal{O}_q & \Gamma_j - \Gamma_q\end{bmatrix} = 2n + \operatorname{rank}\begin{bmatrix}\Gamma_i - \Gamma_p \\ \Gamma_j - \Gamma_q\end{bmatrix}$	
t_S-observability	**strong t_S-obervability**	**strong $[t_S]$-observability**
$\operatorname{rank}\left(\mathcal{O}_i - \mathcal{O}_j\right) = n$	$\mathcal{R}\left(\Sigma_{i,j}\right) = \{0\} \wedge \operatorname{rank}\begin{bmatrix}\mathcal{O}_i - \mathcal{O}_j & \Gamma_i - \Gamma_j\end{bmatrix} = n + \operatorname{rank}\left(\Gamma_i - \Gamma_j\right)$	

Contents

Switch-observability for switched DAEs

$$\Sigma^{\sigma}_{(E,A,B,C,D)}: \quad \boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned}}$$

After quite a bit of new notations, theory and definitions ...

Theorem (Dissertation Küsters 2018)

$\Sigma^{\sigma}_{(E,A,B,C,D)}$ *is strongly* $(x, [\sigma_1])$*-observable* $\Leftrightarrow$ $[t_S]$*-observability and*

$$\operatorname{rank} \begin{bmatrix} \mathcal{O}_i^{\text{diff}} & \mathcal{O}_p^{\text{diff}} & \Gamma_i - \Gamma_p \\ \mathcal{O}_j^{\text{diff}} \Pi_i & \mathcal{O}_q^{\text{diff}} \Pi_p & (\Gamma_j - \mathcal{O}^{\text{diff}} M_i^{\text{imp}}) - (\Gamma_q - \mathcal{O}_q^{\text{diff}} M_p^{\text{imp}}) \\ \mathcal{O}_j^{\text{imp}} \Pi_i & \mathcal{O}^{\text{imp}} \Pi_p & \mathcal{O}_j^{\text{imp}} (M_j^{\text{imp}} - M_i^{\text{imp}}) - \mathcal{O}_q (M_q^{\text{imp}} - M_p^{\text{imp}}) \end{bmatrix}$$

$$= \dim \overline{\mathcal{V}^*}_{i,p} - \dim \mathcal{M}_{i,j,p,q} + \operatorname{rank} \left(\begin{bmatrix} \Gamma_i - \Gamma_p \\ \Gamma_j - \Gamma_q \\ \Gamma_j^{\text{imp}} - \Gamma_q^{\text{imp}} \end{bmatrix} Z_{i,p}^2 \right) \quad \begin{array}{l} \forall i \neq j, p \neq q, \\ (i,j) \neq (p,q) \end{array}$$

Contents

“Trivial” observer design for (x, σ)-obs.

Instantenous observability

(x, σ)-observability $\Longrightarrow$ local state and mode observability

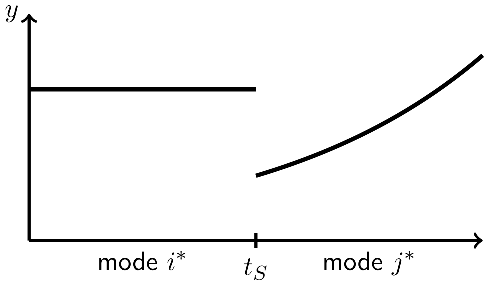
Observer design

1. For each mode run a classical state observer
2. Pick the one which converges $\rightarrow$ mode and state estimation
3. Repeat

Nothing switch specific

Information at the switch (e.g. jumps) not utilized.

Overall observer design



(0. Detect switching time t_S.)

1a. Run partial state observers on $(t_S - \tau, t_S)$ for all modes.

1b. Run partial state observers on $(t_S, t_S + \tau)$ for all modes.

2. Combine partial information to find (i^*, j^*) and state estimation $\widehat{x}(t_S)$

Partial state observer

$$\begin{aligned} \dot{x} &= A_p \dot{x} + B_p u, \\ y &= C_p x + D_p u, \end{aligned}, \qquad \mathcal{O}_p := \begin{bmatrix} C_p \\ C_p A_p \\ \vdots \\ C_p A_p^{n-1} \end{bmatrix} \qquad r_p := \operatorname{rank} \mathcal{O}_p$$

Choose orthogonal $Z_p \in \mathbb{R}^{n \times r_p}$ with $\operatorname{im} Z_p = \operatorname{im} \mathcal{O}_p^\top$, then

$$\begin{aligned} \dot{z}_p &= {Z_p}^\top A_p Z_p z_p + Z_p^\top B_p u \\ y &= C_p Z_p z_p + D_p u \end{aligned}$$

is observable

Definition (Partial state observer)

Any observer for $z_p = Z_p^\top x$ is a partial state observer.

Mode dependence

Z_p and size r_p are mode dependent.

Reasonable modes

Definition (Reasonable modes)

Mode i is reasonable on $(t_S - \tau, t_S)$ $\quad :\Leftrightarrow$

$$\exists x_i^{t_S} : y = C_i x_i + D_i u \quad \text{where } \dot{x}_i = A_i x_i + B_i u, \; x_i(t_S) = x_i^{t_S}$$

In particular, i^* is reasonable on $(t_S - \tau, t_S)$.

Crucial property of reasonable modes

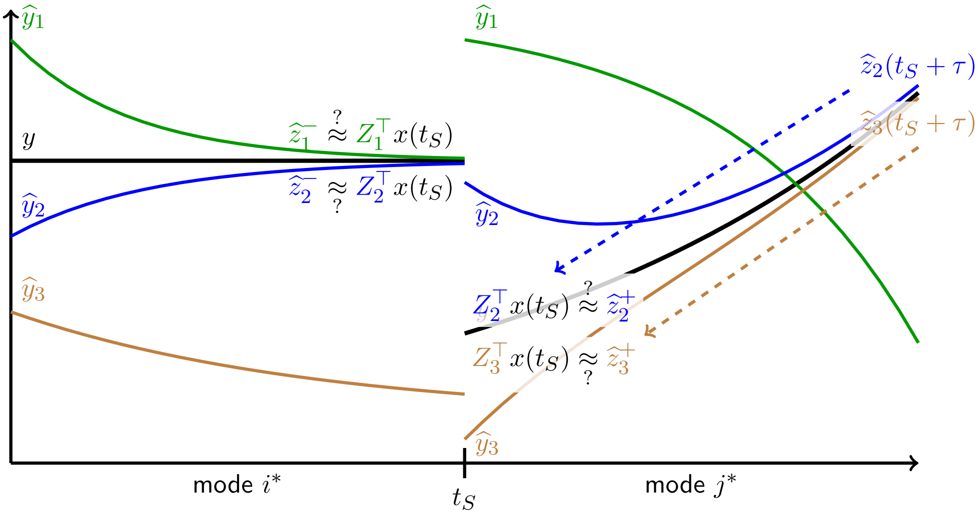
Partial state observers "converge" for all reasonable modes, i.e.

$$y \approx C_i Z_i \widehat{z}_i + D_i u \quad \text{on } (t_S - \varepsilon, t_S) \quad \forall \text{ reasonable } i$$

Analog definition for reasonable modes j on $(t_S, t_S + \tau)$, with

$$y \approx C_j Z_j \widehat{z}_j + D_j u \quad \text{on } (t_S + \tau - \varepsilon, t_S + \tau) \quad \forall \text{ reasonable } j$$

Illustration of Steps 1 and 2



Combining partial state estimations

Question

How to combine the obtained information before and after the switch?

Obvious fact

(x, σ_1)-observability $\implies$ observability for known σ with one switch
$\implies$ $\ker \mathcal{O}_i \cap \ker \mathcal{O}_j = \{0\} \quad \forall i \neq j$
$\implies$ $\operatorname{rank} [Z_i, Z_j] = n \quad \forall i \neq j$

State estimation candidates

For $(i, j) = (i^*, j^*)$ we have

$$\begin{pmatrix} \widehat{z}_i^- \\ \widehat{z}_j^+ \end{pmatrix} \approx \begin{bmatrix} Z_i^\top \\ Z_j^\top \end{bmatrix} x(t_S) \implies x(t_S) \approx \begin{bmatrix} Z_i^\top \\ Z_j^\top \end{bmatrix}^\dagger \begin{pmatrix} \widehat{z}_i^- \\ \widehat{z}_j^+ \end{pmatrix} =: \widehat{x}_{ij}$$

Final step

Theorem (Küsters & T. 2017)

For sufficiently accurate partial observers and for all reasonable (i, j)

$$(i,j) = (i^*, j^*) \quad \implies \quad \begin{bmatrix} Z_i^\top \\ Z_j^\top \end{bmatrix} \widehat{x}_{ij} \approx \begin{bmatrix} \widehat{z_i^-} \\ \widehat{z_j^+} \end{bmatrix}$$

$$(i,j) \neq (i^*, j^*) \quad \implies \quad \begin{bmatrix} Z_i^\top \\ Z_j^\top \end{bmatrix} \widehat{x}_{ij} \not\approx \begin{bmatrix} \widehat{z_i^-} \\ \widehat{z_j^+} \end{bmatrix}$$

Summary

- Classical mode-detection property too restrictive
  - State-observability required for each individual mode
  - Information around switch not utilized
  - Novel concept: switch-observability (σ_1-observability)
- Characterizations in the form of simple rank-tests
- Observer design based on partial state-observers

Future work and topics:

- Extension to nonlinear cases
- Testing in “real” appplications
- Distributed design for large networks
- Using state- and mode-estimations for feedback-control