Nonlinear Switched Singular Systems in Discrete-time: The One-step Map and Stability Under Arbitrary Switching Signals

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***Abstract*— The solvability of nonlinear nonswitched and switched singular systems in discrete time is studied. We provide necessary and sufficient conditions for solvability. The one-step map that generates equivalent nonlinear (ordinary) systems for solvable nonlinear singular systems under arbitrary switching signals is introduced. Moreover, the stability is studied by utilizing this one-step map. A sufficient condition for stability is provided in terms of Lyapunov functions.**

I. INTRODUCTION

The study on this paper focuses on the class of switched systems where each mode is a discrete-time nonlinear singular system without input of the form

$$E_{\sigma(k)}x(k+1) = F_{\sigma(k)}(x(k)) \tag{1}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $\sigma : \mathbb{N} \to \{0, 1, 2, ..., \mathtt{p}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k, $E_i \in \mathbb{R}^{n\times n}$ are singular and $F_i : \mathbb{R}^n \to \mathbb{R}^n$ are continuous nonlinear functions. We refer to the pair (E_i, F_i) as the mode-i.

In some references, singular systems are also called descriptor systems, semi-state systems, implicit systems, differential-algebraic equations (in discrete time) or systems with algebraic constraints. Many physical systems can be modelled as a singular system, and this system class has been widely applied to numerous practical applications, such as electrical circuits [1], [2], industrial processes [3], power systems [4], constrained mechanical systems [5], [6], robotics [7], [8], [9], economic systems [10], discretization of partial differential equations [11] and neural networks [12].

Solution theory for nonlinear singular systems, both in continuous and discrete time domains, have been widely studied; however, the existing studies consider both linear and nonlinear terms in the equation, and the nonlinear terms were considered as suitable disturbances such that the solution theory for singular linear systems still applies (see e.g. [13], [14]).

For (ordinary) nonlinear systems i.e. E_i in (1) being invertible for all i, assuming only that F_i is well-defined is sufficient to guarantee the existence of a unique solution for any initial value $x(0) = x_0 \in \mathbb{R}^n$; the solution can be calculated easily via recursive computation for $k = 0, 1, \ldots$. However, for the singular system (1), there may not be a solution; furthermore, if a solution exists, it may not be unique; this still applies even though a stricter assumption of considering the same subsystem/pair $(E, F) = (E_i, F_i)$ for all modes is considered i.e. the system being nonswitched (see the following example).

Example 1.1: Consider system (1) with

$$(E, F(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1^2 \\ x_2^2 + 4 \end{bmatrix} \right).$$

This system has no solutions since the pure singular subsystem, or the algebraic constraint $x_2^2 + 4 = 0$ has no (real) solutions.

Replacing the second row of F with $x_2^2 - 4$ clearly makes the system solvable for any initial value $x(0) \in \mathcal{S} = \{x \in \mathbb{R}^n : \left(\begin{smallmatrix} x_1^2 \\ x_2^2-4 \end{smallmatrix}\right) \in \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\} = \{x \in \mathbb{R}^2 | x_2^2 - 4 = 0\}$. However, the solution is not unique since at every time instant k, $x_2(k)$ could be -2 or 2. //

In this paper, we study the solution theory for system (1) under arbitrary switching signals, and introduce the one-step map that generates equivalent (ordinary) singular systems. Furthermore, by utilizing these equivalent systems, we formulate a necessary and sufficient condition for stability in terms of Lyapunov functions.

II. PRELIMINARIES

We recall some notations and basic results about generalized inverse, preimage and projector, which will be used in the one-step map formulation in the subsequent sections.

Definition 2.1 (Generalized inverse, cf. [15]): For a matrix $M \in \mathbb{R}^{m\times n}$, a generalized inverse of M is defined as a matrix $M^+ \in \mathbb{R}^{n\times m}$ that satisfies $MM^+M = M$.

A generalized inverse always exists but is not unique in general [16]. Let $M^{-1}\mathcal{X}$ be the preimage of a (in general singular) matrix $M \in \mathbb{R}^{n\times n}$ over a set $\mathcal{X}$, i.e. $M^{-1}\mathcal{X} = \{\xi \in \mathbb{R}^n : M\xi \in \mathcal{X}\}$. The following lemma utilizes the generalized inverse to represent the preimage.

Lemma 2.2 ([17]): For any matrix $M \in \mathbb{R}^{n\times n}$ and $x \in \operatorname{im} M$, we have that

$$M^{-1}\{x\} = \{M^+x\} + \ker M$$

where M^+ is a generalized inverse of M.

In addition, the following lemma regarding an intersection that results in a singleton will be used in formulating the solvability conditions, and moreover, the second part of the lemma will be used in formulating the one-step map.

Lemma 2.3 (cf. Lemma 3.4 in [18]): Consider set $\mathcal{U} \subseteq \mathbb{R}^n$ and two subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$, then $\mathcal{V} \cap (\{u\} + \mathcal{W})$

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is a singleton for all $u \in \mathcal{U}$ if, and only if, $\mathcal{U} \subseteq \mathcal{V} \oplus \mathcal{W}$. In that case

$$\mathcal{V} \cap (\{u\} + \mathcal{W}) = \{\Pi_{\mathcal{V}}^{\mathcal{W}} u\}, \tag{2}$$

where $\Pi_{\mathcal{V}}^{\mathcal{W}} : \mathcal{V} \oplus \mathcal{W} \to \mathcal{V}$ is the canonical projector from $\mathcal{V} \oplus \mathcal{W}$ to $\mathcal{V}$.

III. (Nonswitched) Nonlinear Singular Systems

We present in this section the solution theory and stability for nonswitched cases of (1) of the form

$$Ex(k+1) = F(x(k)),\ k = 0, 1, ... \tag{3}$$

where E is singular with $\operatorname{rank} E = r < n$. Define $\mathcal{S} = \{x \in \mathbb{R}^n : F(x) \in \operatorname{im} E\}$.

A. Solution Theory

We consider the following solvability notion in establishing the one-step map for system (3).

Definition 3.1: We call (3) *locally uniquely solvable* (for short just *solvable*) if, for all $k \in \mathbb{N}$ and for all $x_0 \in \mathcal{S}$ there exists a unique solution on $[0, k]$ of (3) considered on $[0, k]$ with $x(0) = x_0$.

The solvability notion above requires the existence of a unique solution on any finite time interval $[0, k_1]$, which in particular means that the final value at k_1 does not depend on the values $x(k)$ for $k > k_1$. This solvability notion is stronger compared to the common solvability notion as for ordinary systems where the unique solution is required on $[0, \infty)$ for all (consistent) initial values. However, having the former solvability notion will guarantee the existence of the one-step map for system (3), and it is not always possible to have a one-step map for the latter solvability notion (see the forthcoming Remark 3.8). Furthermore, note that every non-singular system (i.e. E is non-singular) is locally solvable, in fact, solutions are already uniquely determined on $[0, k]$ by only considering (3) on $[0, k-1]$. This is in contrast to the singular case, where the algebraic constraints at k are usually needed to determine uniquely the value of $x(k)$.

From basic algebra, there exist invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that $SET = \left[\begin{smallmatrix} I_r & 0 \\ 0 & 0 \end{smallmatrix}\right]$. By using the state transformation $T^{-1}x(k) = \left(\begin{smallmatrix} v(k) \\ w(k) \end{smallmatrix}\right)$, $v \in \mathbb{R}^r$, $w \in \mathbb{R}^{n-r}$, system (3) can be rewritten as

$$\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \left[\begin{smallmatrix} v(k+1) \\ w(k+1) \end{smallmatrix}\right] = SF\left(T \left[\begin{smallmatrix} v(k) \\ w(k) \end{smallmatrix}\right]\right) =: \left[\begin{smallmatrix} G(v(k), w(k)) \\ H(v(k), w(k)) \end{smallmatrix}\right]. \tag{4}$$

The representation above decouples (1) into pure ordinary subsystem in v and pure singular subsystem or algebraic constraint in w. In this decoupled representation, system (3) can be illustrated by Fig. 1.

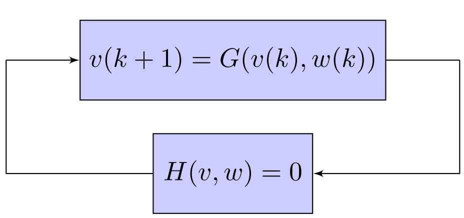

Fig. 1: Block diagram of (4)

To be able to ensure existence and solutions of solution of the switched system (3), we will make the following assumption.

Assumption 3.2: The set $\mathcal{S} := \{x \in \mathbb{R}^n : F(x) \in \operatorname{im} E\}$ of (3) is a linear subspace in $\mathbb{R}^n$.

On a first glance this assumption looks rather restrictive, however, in general the set $\mathcal{S}$ is a differentiable manifold at least locally and then a (local) nonlinear coordinate transformation can be applied to obtain a linear subspace $\mathcal{S}$.

Remark 3.3: The algebraic constraint $H(v, w) = 0$ in (4) is, in general, nonlinear even if $\mathcal{S}$ is a subspace. However, if $\mathcal{S}$ is a subspace in $\mathbb{R}^n$, then $\left(\begin{smallmatrix} v \\ w \end{smallmatrix}\right) \in \ker S^{\perp}$, where $S^{\perp}$ is a matrix whose columns span the orthogonal complement of $\mathcal{S}$, i.e. there exist matrices $P \in \mathbb{R}^{(n-r)\times r}, Q \in \mathbb{R}^{(n-r)\times(n-r)}$ such that $H(v, w) = 0$ if, and only if, $Pv + Qw = 0$. Thus, for every $k \in \mathbb{N}$, the nonlinear algebraic constraint $H(v(k), w(k)) = 0$ can be replaced by the linear algebraic constraint

$$0 = Pv(k) + Qw(k). \tag{5}$$

As a consequence, the nonlinearity appears now only on $G(v, w)$. $\diamond$

The following lemma provides two characterizations for solvability of system (3) under Assumption 3.2.

Lemma 3.4: The following are equivalent:

- (i) System (3) under Assumption 3.2 is solvable in the sense of the Definition 3.1
- (ii) Q is nonsingular
- (iii) $\mathcal{T} \subseteq \mathcal{S} \oplus \ker E$
  where $\mathcal{T} = \{E^+F(\varsigma) \mid \varsigma \in \mathcal{S}\}$, i.e. $\mathcal{T}$ is the range of $\tau : \mathcal{S} \to \mathbb{R}^n$ with $\tau(\varsigma) = E^+F(\varsigma)$.

Proof: (i) $\Rightarrow$ (ii): The set $\mathcal{S}$ being a subspace implies the existence of the equivalent linear algebraic constraint of the form (5), hence system (3) can now equivalently be rewritten as

$$\begin{cases} v(k+1) = G(v(k), w(k)),\ k = 0, 1, \ldots \\ \qquad\quad 0 = Pv(k) + Qw(k) \end{cases}$$

Consider this system on $[0, 1]$, then it reads

$$\begin{array}{r|r} v(1) = G(v(0), w(0)) & v(2) = G(v(1), w(1)) \\ 0 = Pv(0) + Qw(0) & 0 = Pv(1) + Qw(1) \end{array}$$

where $(v(0), w(0))$ is given, and thus $v(1)$ is also given. Seeking a contradiction assume that the square matrix Q is singular. Then it is first of all not guaranteed anymore that for the specific $v(1)$ a solution $w(1)$ exists with $0 = P(v(1) + Qw(1)$. If $w(1)$ exists at all it is not unique because Q has a nontrivial kernel. Hence we have non-existence or non-uniquess of solutions of (3) considered on the interval $[0, 1]$, contradicting (i).
(ii) $\Rightarrow$ (i): Nonsingularity of Q implies that the algebraic constraints are equivalent to $w(k) = Q^{-1}Pv(k)$, which then leads to the uniquely solvable nonsingular system $v(k+1) = \bar{G}(v(k))$ with $\bar{G}(v) = G(v, Q^{-1}Pv)$. Transforming this unique solution back to its original coordinates provides a unique solution x on any interval $[0, k]$.

(i) $\Rightarrow$ (iii): By assumption for any initial value x_0 there exists a unique solution on $[0,1]$, in particular $x(1)$ is uniquely determined by considering (3) for $k=0$ and $k=1$. By Lemma 2.2 applied to (3) for $k=1$ the value $x(1)$ satisfies

$$x(1) \in E^{-1}(F(x_0)) = \{E^+F(x_0)\} + \ker E. \qquad (6)$$

On the other hand, considering (3) at $k=1$ (not making any assumptions about the unknown $x(2)$), $x(1)$ must satisfy

$$x(1) \in \{x \in \mathbb{R}^n | F(x) \in \operatorname{im} E\} = \mathcal{S}. \qquad (7)$$

Hence $x(1)$ is uniquely determined for all $x_0 \in \mathcal{S}$ if, and only if,

$$\mathcal{S} \cap (\{E^+F(x_0)\} + \ker E) \text{ is a singleton.}$$

Using Lemma 2.3 with $\mathcal{U} = \mathcal{T}$, $\mathcal{V} = \mathcal{S}$ and $\mathcal{W} = \ker E$ we conclude (iii).
(iii) $\Rightarrow$ (i): We prove inductively, that if for any $x_0 \in \mathcal{S}$ there exists a unique solution on $[0,k]$, then there also exist a unique solution on $[0,k+1]$. This together with the trivial observation that $x(0)=x_0$ is the unique solution of (3), $x(0)=x_0$, considered only for $k=0$ will prove (i). Now, given $x(k)$, we choose $x(k+1) \in \mathcal{S} \cap (\{E^+F(x(k))\} + \ker E)$ which is possible due to Lemma 2.3. Then $x(k+1) \in \{E^+F(x(k))\} + \ker E$ implies that $Ex(k+1) = EE^+F(x(k))$. Since $x(k) \in \mathcal{S}$ (because x is a solution on $[0,k]$), it follows that $F(x(k)) \in \operatorname{im} E$, i.e. there exists v such that $F(x(k)) = Ev$. Hence $Ex(k+1) = EE^+Ev = Ev = F(x(k)$ which shows that $x(k+1)$ satisfies (3). Furthermore, $x(k+1)$ also satisfies (3) for $k+1$ because $x(k+1) \in \mathcal{S}$. This shows that x is indeed a solution of (3) on $[0,k+1]$. Uniqueness follows from the fact, that by Lemma 2.3 the set $\mathcal{S} \cap (\{E^+F(x(k))\} + \ker E)$ is a singleton. ∎

Lemma 3.4 provides two alternatives for checking whether system (3) is solvable in the sense of Definition 3.1. The condition (ii) requires, first, transforming the original system into (4)'s form, and then finding Q by using Remark 3.3. Meanwhile, the condition (iii) uses data from the original system directly, which requires less computation steps. In particular, using Lemma 2.3 and the same arguments as in the proof for Lemma 3.4 we arrive at the following one-step map that allows to to obtain an equivalently "surrogate" ordinary system for (3):

Corollary 3.5: Consider system (3) under Assumption 3.2. If solvable, its solution satisfies

$$x(k+1) = \Phi(x(k)) = \Pi_{\mathcal{S}}^{\ker E} E^+ F(x(k)) \; \forall k \in \mathbb{N}. \qquad (8)$$

where E^+ is a generalized inverse of E and $\Pi_{\mathcal{S}}^{\ker E}$ is the canonical projector from $\mathcal{S} \oplus \ker E$ to $\mathcal{S}$. Furthermore, any solution of (8) with $x(0) \in \mathcal{S}$ also solves (3).

Remark 3.6 (The nonuniqueness of generalized inverses): Note that the generalized inverse matrix E^+, in general, is not unique, and thus applying different E^+ could provide different $\mathcal{T}$ in Lemma 2.3 and different one-step maps. However, condition (iii) in Lemma 3.4 as well as the restriction of Φ on $\mathcal{S}$ will give the same results regardless of the choice of E^+ used in the calculation, i.e. the nonuniqueness of E^+ has no effect on the solution charactetization/formula; in fact the same arguments from the linear case (see Remark 3.12 in [17]) also apply here.

Now it is possible to write the explicit solution of (3) i.e.

$$x(k) = \underbrace{(\Phi \circ \Phi \circ \cdots \circ \Phi)}_{k \text{ times}}(x_0)$$

where $\Phi(\cdot)$ is as given in (8). The following example illustrates the above solution theory.

Example 3.7: Consider system (3) with

$$(E, F(x)) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} \\ x_1^{\frac{1}{3}} - x_2^{\frac{1}{3}} \end{bmatrix} \right).$$

Simple computations provide $\ker E = \operatorname{span}\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ and $\mathcal{S} = \left\{ x \in \mathbb{R}^n : \begin{pmatrix} x_1^{\frac{1}{3}} + x_2^{\frac{1}{3}} \\ x_1^{\frac{1}{3}} - x_2^{\frac{1}{3}} \end{pmatrix} \in \operatorname{span}\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right) \right\} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Since $\mathcal{S} \oplus \ker E = \mathbb{R}^n$, the condition (iii) in Lemma 3.4 is satisfied (independently of what $\mathcal{T}$ is), and thus this system is solvable and has a unique solution for every initial value $x_0 \in \mathcal{S} = \operatorname{span}\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$. Furthermore, it is easily seen that $\Pi_{\mathcal{S}}^{\ker E} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $E^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$, hence the the one-step map is given by $\Phi(x) = \begin{pmatrix} x_1^{\frac{1}{3}} \\ 0 \end{pmatrix}$ and each solution satisfies

$$x(k+1) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} F(x(k)) = \begin{pmatrix} x_1(k)^{\frac{1}{3}} \\ 0 \end{pmatrix}. \qquad //$$

Remark 3.8: It is not always possible to establish a one-step map for system (3) if only global solvability on $[0,\infty)$ in assumed instead of the local solvability in the sense of Definition 4.1. This is illustrated by the following "counter example":

$$\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right] x(k+1) = \begin{pmatrix} x_1(k)^{\frac{1}{3}} \\ x_2(k)^{\frac{1}{3}} \end{pmatrix}, \; k = 0, 1, ... \qquad (9)$$

with $\mathcal{S} = \operatorname{im}\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. For this system, considered on $[0,\infty)$, the unique solution is given by $x(k) = 0$ for all $k > 0$, because $x_2(k) = 0$ for all k and $x_1(k) = x_2(k+1) = 0$ for all k. However, if we consider the system on $[0,1]$, the system has a non-unique local solution because $x_1(1)$ can be arbitrary. This, in particular, shows that the solvability on $[0,\infty)$ does not imply the solvability in the sense of Definition 3.1, however, the converse is clearly true. Now, since $x_1(1)$ is free, we cannot determine it only from the current and past information, and thus the one-step map, which depends only on the current and past information, cannot exist. Therefore, the solvability notion given in Definition 3.1 is necessary for the existence of the one-step map, which in turn is needed to study switched systems (where at a given time k it may not be clear yet what the mode at $k+1$ will be.

B. Stability Based on Lyapunov Function

We can now study the nonlinear singular system (3) for further analysis, stability in this paper, by utilizing its "surrogate" ordinary system (8). Suppose $\Phi(0) = 0$ i.e. $x = 0$ is an equilibrium point for (8). This can also be generalized for a nonzero equilibrium: when $x = x_e \neq 0$

is the equilibrium point we are investigating, the new state $\widehat{x} = x - x_e$ provides 0 as an equilibrium point in $\widehat{x}$ coordinate. However, this coordinate transformation is not needed if $F(0) = 0$ since it directly implies that $\Phi(0) = 0$. To be precise, we present the stability notion used in this study in the following.

Definition 3.9: The equilibrium $x = 0$ of system (8) (or system (3)) is

- *stable* if for each $\epsilon > 0$ there is $\delta = \delta(\epsilon)$ such that for all solutions x of (3)
$$||x(0)|| < \delta \Longrightarrow ||x(k)|| < \epsilon \quad \forall k \geq 0$$
- *asymptotically stable* if it is stable and δ can be chosen such that for all solutions x of (3)
$$||x(0)|| < \delta \Longrightarrow \lim_{k\to\infty} x(k) = 0$$
- *unstable* if it is not stable.

Since the "surrogate" system (8) can be seen as an ordinary system, we can utilize the stability theory for ordinary systems. The following corollary for stability of 0 of (8) is a simple consequence from the classical stability theorem for ordinary systems in [19].

Corollary 3.10: Consider the solvable singular system (3) via its surrogate ordinary system (8). Assume $\Phi : \mathcal{S} \to \mathbb{R}^n$ is continuous on $\mathcal{S} \subset \mathbb{R}^n$. If there exists a continuous function $V : \mathcal{S} \to \mathbb{R}$ such that

$$V(0) = 0, V(x) > 0 \ \forall x \in \mathcal{S} - \{0\}, \text{ and} \tag{10}$$
$$V(\Phi(x)) - V(x) \leq 0 \ \forall x \in \mathcal{S} \tag{11}$$

then $x = 0$ is stable for (3). Furthermore, if

$$V(\Phi(x)) - V(x) < 0 \quad \forall x \in \mathcal{S} - \{0\} \tag{12}$$

then $x = 0$ is asymptotically stable for (3).

IV. Switched Nonlinear Singular Systems

A. Solution Theory

Recall system (1) and define

$$\mathcal{S}_i := \{x \in \mathbb{R}^n | F_i(x) \in \operatorname{im} E_i\}. \tag{13}$$

We first generalize the solvability notion for nonswitched systems in Definition 3.1 to the following solvability notion for switched systems.

Definition 4.1: We call (1) *locally uniquely solvable* (for short just *solvable*) w.r.t. to a given switching signal $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{p}\}$ if, for all $k_0, k_1 \in \mathbb{N}$, $k_1 > k_0$ and for all $x_{k_0} \in \mathcal{S}_{\sigma(k_0)}$ there exists a unique solution of (1) considered on $[k_0, k_1]$ with $x(k_0) = x_{k_0}$.

Note that the solvability notion above requires the existence of a unique solution considered on any time interval with any arbitrary initial time and, furthermore, for any consistent initial value at that initial time. For similar reasons as discussed in Remark 3.8, we use this solvability notion because it is not always possible to define the one-step map for system (1) with the common solvability notion on $[0, \infty)$.

We extend Assumption 3.2 to the switched case as follows:

Assumption 4.2: Each $\mathcal{S}_i$ given by (13) is a linear subspace in $\mathbb{R}^n$ for each $i \in \{0, 1, ..., \mathtt{p}\}$.

The first important observation for switched systems is that solvability for individual modes is, in general, not sufficient for switched systems composed by those modes to be solvable. This is illustrated by the following Example.

Example 4.3: Consider system (1) with

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{bmatrix} \right),$$
$$(E_1, F_1(x)) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} x_1^2 \\ x_1^2 + x_2^2 \end{bmatrix} \right).$$

Simple computations provide that

$$\begin{aligned} \ker E_0 &= \operatorname{span}\{(0,1)^\top\} \\ \ker E_1 &= \operatorname{span}\{(1,0)^\top\} \\ \mathcal{S}_0 &= \operatorname{span}\{(1,0)^\top\} \\ \mathcal{S}_1 &= \operatorname{span}\{(0,1)^\top\}. \end{aligned}$$

For each pair, as an individual system, we have that $\ker E_i \oplus \mathcal{S}_i = \mathbb{R}^n, \ i = 0, 1$ i.e. individual system is solvable. Their solutions are $\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} x_{10}^{\frac{1}{3^k}} \\ 0 \end{pmatrix}, \ k = 1, 2, ...$ and $\begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} = \begin{pmatrix} 0 \\ x_{20}^{2k} \end{pmatrix}, \ k = 1, 2, ...$, respectively. In particular, when the switched system under the switching signal $\sigma(k) = 0$ for $k < k^s$ and 1 for $k \geq k^s$ reads:

$$\begin{aligned} &k < k^s : & &k \geq k^s : \\ x_1(k+1) &= x_1^{1/3}(k), & 0 &= x_1^2(k), \\ 0 &= x_2^{1/3}(k), & x_2(k+1) &= x_2^2(k). \end{aligned}$$

From this it is clear that once the switch occurs at $k = k^s$, the only solution for x_1 is $x_1(k) = 0$ also before the switch, although x_1 was not restricted for $k < k^s$. Furthermore, $x_2(k^s)$ is not restricted by the above equations and hence uniqueness of solutions with respect to $x(0)$ is not satisfied. //

We generalize the solvability condition for nonwswitched systems to the condition for switched systems in the following theorem, which provides a characterization of solvability of system (1). Furthermore, the one-step map for switched systems is also presented in this theorem.

Theorem 4.4: System (1) under Assumption 4.2 is solvable (in the sense of Definition 4.1) for all switching signals $\sigma : \mathbb{N} \to \{1, 2, \ldots, \mathtt{p}\}$ if, and only if,

$$\mathcal{T}_j \subseteq \mathcal{S}_i \oplus \ker E_j \ \forall i, j \in \{0, 1, ..., \mathtt{p}\}, \tag{14}$$

where $\mathcal{T}_i = \{E_i^+ F_i(\varsigma) | \varsigma \in \mathcal{S}_i\}$. Moreover, if solvable, its solution satisfies

$$x(k+1) = \Phi_{\sigma(k+1),\sigma(k)}(x(k)), \ \forall k \in \mathbb{N} \tag{15}$$

where $\Phi_{i,j}$ is the one-step map from mode-j to mode-i given by

$$\Phi_{i,j}(x) := \Pi_{\mathcal{S}_i}^{\ker E_j} E_j^+ F_j(x) \tag{16}$$

where E_j^+ is a generalized inverse of E_j and $\Pi_{\mathcal{S}_i}^{\ker E_j}$ is the canonical projector from $\mathcal{S}_i \oplus \ker E_j$ to $\mathcal{S}_i$.

Proof: Step 1: Solvability
Necessity: We consider a solution on some interval $[k, k+1]$ where $\sigma(k) = i$ and $\sigma(k+1) = j$. For a given $x(k) \in \mathcal{S}_i$, in order to have a unique $x(k+1)$ for any switching signal, the following system of equations must have a unique solution for $x(k+1)$:

$$E_i x(k+1) = F_i(x(k)), \tag{17a}$$
$$E_j x(k+2) = F_j(x(k+1)), \tag{17b}$$

Equation (17a) is equivalent to $x(k+1) \in E_i^{-1}[F_i(x(k))$ which by Lemma 2.2 is equivalent to

$$x(k+1) \in \{E_i^+ F_i(x(k))\} + \ker E_i. \tag{18}$$

Since we only consider a solution on $[k, k+1]$, the value $x(k+2)$ in (17b) is arbitrary, hence Equation (17b) is equivalent to

$$x(k+1) \in \{x \in \mathbb{R}^n : F_j(x) \in \operatorname{im} E_j\} = \mathcal{S}_j \tag{19}$$

By applying $\mathcal{U} = \mathcal{T}_i, \mathcal{V} = \mathcal{S}_j$ and $\mathcal{W} = \ker E_i$ to Lemma 2.3, the uniqueness of $x(k+1)$ implies $\mathcal{T}_i \subseteq \mathcal{S}_j \oplus \ker E_i$. Since arbitrary switching signals can be considered, this condition mus hold for all $\forall i, j \in \{0, 1, ..., \mathtt{p}\}$.
Sufficiency: Identical arguments as for the non-switched case allow us to inductively extend any solution x on $[0, k]$ uniquely to a solution on $[0, k+1]$ if (15) holds.
Step 2: One-step map
By applying formula (2) in Lemma 2.3 to (18) and (19) with $\mathcal{U} = \{E^+_{\sigma(k)} F_{\sigma(k)}(x(k))\}$, $\mathcal{V} = \mathcal{S}_{\sigma(k+1)}$ and $\mathcal{W} = \ker E_{\sigma(k)}$, the solution $x(k+1)$ satisfies the one-step map formula (8). ∎

Regarding the nonuniqueness of the generalized inverse matrix E_i^+, the same phenomenon discussed in Remark 3.6 also applies i.e. the nonuniqness of E_i^+ has no effect to the solution or the formula (15).

The following example illustrates the solution of (1) calculated by using the one-step map formula introduced in Theorem 4.4.

Example 4.5: Consider system (1) with

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} x_1^{\frac{1}{3}} \\ x_2^{\frac{1}{3}} \end{bmatrix} \right),$$

$$(E_1, F_1(x)) = \left(\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} x_1^2 + x_2^2 \\ x_2^2 \end{bmatrix} \right).$$

Simple computations provide that

$$\begin{aligned} \ker E_0 &= \operatorname{span}\{(0,1)^\top\} \\ \ker E_1 &= \operatorname{span}\{(0,1)^\top\} \\ \mathcal{S}_0 &= \operatorname{span}\{(1,0)^\top\} \\ \mathcal{S}_1 &= \operatorname{span}\{(1,0)^\top\}. \end{aligned}$$

Few observations are discussed as follows:

- Since $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n, \ \forall i, j \in \{0, 1\}$, then clearly the condition (14) holds, and thus the system is solvable.
- Choosing $E_0^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $E_1^+ = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix}$ provides the following one-step maps $\Phi_{i,j}$ from mode-j to mode-i

$$\Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{pmatrix} x_1^{\frac{1}{3}}(k) \\ 0 \end{pmatrix}$$
$$\Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k)) = \begin{pmatrix} \frac{1}{2} x_1^2(k) + \frac{3}{2} x_2^2(k) \\ 0 \end{pmatrix}.$$

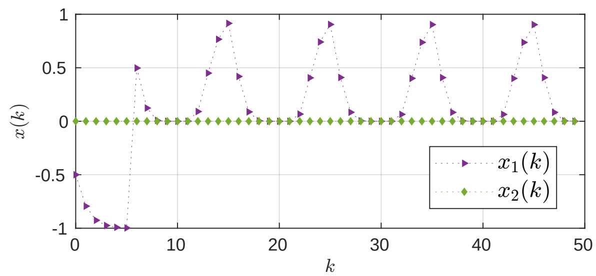

Fig. 2: Solution of Example 4.5

Under the periodic switching signal $\sigma(k) = 1$ for $k \in [0, 5) \cup [10, 15) \cup ...$ and $= 0$ for $k \in [6, 10) \cup [15, 20) \cup ...$, and with $x(0) = (-\frac{1}{2}, 0)^\top$, the solution is shown in Fig. 2. //

B. Stability Theory

We analyze the stability of $x = 0$ of switched system (1) via its "surrogate" system (15) as follows. Suppose $x = 0$ is an equilibrium for (1) i.e. $\Phi_{i,j}(0) = 0 \ \ \forall i, j \in \{0, 1, ..., \mathtt{p}\}$.

First note that requiring each mode to be (asymptotically) stable is not sufficient to make sure that the switched system is (asymptotically) stable; this is a well known challenge in the stability analysis of switched systems, cf. [20].

The first approach that can be used to study the stability of $x = 0$, even though it is conservative, is the common Lyapunov function approach. The following corollary is derived from the common Lyapunov stability theorem for the general time-varying nonlinear systems of the form $x(k+1) = f_k(x(k))$ in [21].

Corollary 4.6 (Common Lyapunov function approach): Consider system (1) under Assumption 4.2 and assume that for all switching signals it is solvable and $x = 0$ is an equilibrium. Then $x = 0$ is asymptotically stable if there is exists a function $V : \mathbb{R}^n \to \mathbb{R}$ such that

- V is a positive–definite and radially unbounded;
- $V(x(k+1)) - V(x(k)) < 0$ for all solutions x of (15) and all switching signals.

Note that in order to check the condition $V(x(k+1)) - V(x(k)) < 0$ one could require that

$$V(\Phi_{i,j}(x)) - V(x) < 0 \quad \forall i, j \in \{1, 2, \ldots, \mathtt{p}\} \forall x \in \mathbb{R}^n.$$

But this means that (15) is considered as a switched system with $\mathtt{p}^2$ independent different modes (one for each *pair* (i, j)). However, this viewpoint is too conservative in our situation, because the mode sequences in (15) are restricted to those where at time $k+1$ the mode pair (i_{k+1}, j_{k+1}) is related to the mode pair (i_k, j_k) at time k via $i_k = j_{k+1}$. This motivates us to introduce the following switched Lyapunov function approach.

Theorem 4.7 (Switched Lyapunov function approach): Consider the singular switched system (1) via its surrogate ordinary switched system (15). Assume for all $i \in \{0, 1, ..., \mathtt{p}\}$, $\Phi_i : \mathcal{S}_i \to \mathbb{R}^n$ is continuous on $\mathcal{S}_i \subset \mathbb{R}^n$ and each mode is (asymptotically) stable with corresponding Lyapunov function V_i. If for all $i, j \in \{0, 1, ..., \mathtt{p}\}$ the following conditions hold,

(i) $V_i(x) = V_j(x) \ \forall x \in \mathcal{S}_i \cap \mathcal{S}_j$ and

(ii) $V_i(\Phi_{i,j}(x)) - V_j(x)(<) \leq 0 \ \forall x \in \mathcal{S}_j - \{0\}$

then $x = 0$ is (asymptotically) stable for arbitrary switching signals.

Proof: We construct the following Lyapunov function for (1) from the Lyapunov functions of all individual modes as follows:

$$V : \mathbb{R}^n \to \mathbb{R}, \ V(x) = \begin{cases} V_i(x) & \text{if } x \in \mathcal{S}_i \\ 0 & \text{otherwise.} \end{cases}$$

Note that condition (i) is necessary for having V being well defined. Then for all solutions $x(k)$ of (15) at any $k \in \mathbb{N}$

$$\begin{aligned} & V(x(k+1)) - V(x(k)) \\ & = V_{\sigma(k+1)}(x(k+1)) - V_{\sigma(k)}(x(k)) \\ & = V_{\sigma(k+1)}(\Phi_{\sigma(k+1),\sigma(k)}(x(k)) - V_{\sigma(k)}(x(k)) \leq (<)0. \end{aligned}$$

which by Corollary 4.6 guarantees the (asymptotic) stability of the equilibrium $x = 0$ for arbitrary switching signals. ■

It can be seen that the condition (ii) above is necessary only for certain switches i.e. after $\Phi_{i,j}$, and the condition is checked only for switches to $\Phi_{i,j}$ and not for all switches to any other one-step map matrix. This makes the stability theorem above more relaxed compared to the common Lyapunov approach. The following example illustrates the stability analysis by using the condition provided by the theorem above.

Example 4.8: Consider system (1) composed of the following two modes:

$$(E_0, F_0(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} (x_1+1)^{\frac{1}{3}} - 1 \\ x_2^{\frac{1}{3}} \end{bmatrix} \right),$$

$$(E_1, F_1(x)) = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{pmatrix} (x_2+1)^{\frac{1}{5}} - 1 \\ x_1^{\frac{1}{5}} \end{pmatrix} \right).$$

Basic computations provide

$$\begin{aligned} \ker E_0 &= \operatorname{span}\{(0,1)^\top\} \\ \ker E_1 &= \operatorname{span}\{(0,1)^\top\} \\ \mathcal{S}_0 &= \operatorname{span}\{(1,0)^\top\} \\ \mathcal{S}_1 &= \operatorname{span}\{(1,0)^\top\}. \end{aligned}$$

Since $\ker E_i \oplus \mathcal{S}_j = \mathbb{R}^n, \ \forall i, j \in \{0, 1\}$, clearly the condition (14) holds i.e. the system is solvable for arbitrary switcbing signals. Choosing $E_0^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $E_1^+ = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and with $\Pi_{\mathcal{S}_1}^{\ker E_0} = \Pi_{\mathcal{S}_0}^{\ker E_1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ provide

$$\Phi_0(x(k)) = \Phi_{0,0}(x(k)) = \Phi_{1,0}(x(k)) = \begin{bmatrix} (x_1+1)^{\frac{1}{3}} - 1 \\ 0 \end{bmatrix}$$

and

$$\Phi_1(x(k)) = \Phi_{1,1}(x(k)) = \Phi_{0,1}(x(k)) = \begin{bmatrix} (x_1+1)^{\frac{1}{5}} - 1 \\ 0 \end{bmatrix}.$$

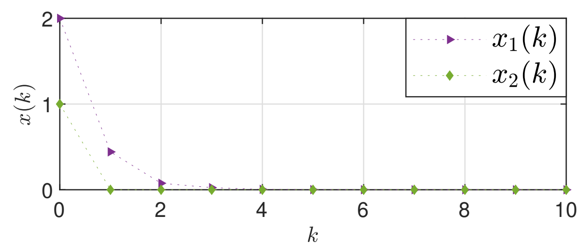

Fig. 3: A solution of the switched system in Example 4.8

As an individual system, $x = 0$ of each mode is stable with Lyapunov function e.g. $V_i(x) = x_1^2 + x_2^2, \ i = 0, 1$. Clearly, the conditions (i) and (ii) in Theorem 4.7 with strict inequality are satisfied, and moreover $V_0(x) = V_1(x)$). Hence, $x = 0$ of the switched system is asymptotically stable for arbitrary switching signals. With $\sigma(k) = 0$ if $k = 0, 2, 4, ...$ and $= 1$ if $k = 1, 3, 5, ...$, the trajectory of the solution is illustrated in Fig. 3. //

V. Summary

The solution theory and stability analysis for nonlinear singular systems in discrete time, both for nonswitched and switched cases, were studied in this paper. Solvability conditions have been proposed, and the corresponding one-step map has been introduced to get the equivalent "surrogate" ordinary system. Moreover, by utilizing the one-step map representation, sufficient conditions for stability have been proposed via common Lyapunov function and switched Lyapunov function. The second stability condition is more convenient than the first which is rather conservative.

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