Reachability and Controllability Characterizations for Linear Switched Systems in Discrete Time: A Geometric Approach

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***Abstract*— This article presents the reachability and controllability characterizations for discrete-time linear switched systems under a fixed and known switching signal. A geometric approach is used, and we are able to provide alternative conditions which are more computationally friendly compared to existing results by utilizing the solution formula at switching times. Furthermore, the proposed conditions make it easier to study the dependency of the reachability and controllability on the switching times and the mode sequences; this is a new result currently not investigated in the literature. Some academic examples are provided to illustrate the novel features found in this study.**

I. Introduction

We study the reachability and controllability of a class of switched systems where each mode is a discrete-time linear system of the form

$$x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1}$$

where $k \in \mathbb{N}$ is the time instant, $x(k) \in \mathbb{R}^n$ is the state, $u(k) \in \mathbb{R}^m, m \in \mathbb{N}$ is the input, $\sigma : \mathbb{N} \to \{0, 1, 2, ..., \mathtt{p}\}, \mathtt{p} \in \mathbb{N}$, is the switching signal determining which mode $\sigma(k)$ is active at a time instant k, $A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$. The switching signal σ is assumed to be fully known and fixed. Furthermore, it is a time-only-triggered switching signal. In other words, neither state nor input trigger the switching. This means that system (1) can be seen as a time-varying linear system with a special structure. In applications, there are many physical systems which can be modelled in this framework (see e.g. [1], [2], [3], [4], [5], [6]), and this system class has been attracting many researchers to study its properties such as stability [7], [8], [9], [10] and control designs [11], [12], [13], [14], [15].

We focus in this study with the state reachability notion in the sense of the ability to reach a certain state from a certain initial value under a fixed and known switching signal. Study for reachability of system class (1) was initially introduced in [16] where the set of points reachable from the origin were investigated; this set was called controllable set in this study. It has been pointed out that this set is a subspace under some hypothesis. This study was extended in [17] where the reachable set was formulated as the union of its maximal components; some structural properties, such as the bound of the number of time steps bound on the number of iterations necessary to reach a state, were also presented. Nevertheless, no necessary and sufficient condition for reachability characterization was formulated in those reports. Another study related to reachability was reported in [18] for the case that a zero-nonzero structure of the matrices A and B is known.

Necessary and sufficient conditions for reachability and controllability of (1) were proposed in [19] using geometric approach under arbitrary switching signal. The reachable set was presented as a subspace derived from a calculation using the system's matrices obtained from each time step for the whole time interval of the characterization, which may demand high computational resources. In fact, the condition can be simplified; this is the first contribution of our study. Furthermore, the characterizations in [19] are under arbitrary switching signals, and we need to find a switching signal which yields reachability. When we have a fixed switching signal, we can extract the condition directly from the condition for systems under arbitrary switching signals. In general, the reachability depends on the switching times, meaning that a system may not be reachable for a certain switching signal but unreachable for another switching signal, see the forthcoming Example 3.7. Now, a new question arises i.e. when the reachability characterization stays the same if we change the switching times but keep the mode sequence; this is our next contribution in this paper, which has not been studied yet in any literature. It is important if we can know whether the system is always reachable or unreachable for any choice of switching times so that only one characterization with certain switching times is needed to perform.

This paper is structured as follows. In Section II, we first present some preliminaries for recalling the explicit solution of linear switched systems, for introducing the switching signal forms used in the study, and for introducing the solution formulas at switching times including the state transition matrix, which plays an important role in the reachability and controllability characterizations. We present the main results regarding the reachability and controllability characterizations in Section III. Finally, the main results regarding the study of the dependencies of those properties on the switching times are presented in Section IV. Some (counter-)examples are also presented within each section to illustrate the results.

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II. Preliminaries

Recall the Inhomogeneous Linear Switched System (InLSS) (1). Its solutions satisfy $\forall k, h \in \mathbb{N}$ with $k \geq h$,

$$x(k) = \phi(k,h)x(h) + \sum_{j=h}^{k-1} \phi(k, j+1) B_{\sigma(j)} u(j),$$

where $\phi_\sigma(k,h) = A_{\sigma(k-1)} A_{\sigma(k-2)} \cdots A_{\sigma(h)}$ is the so called state transition matrix. With initial value $x(0) = x_0 \in \mathbb{R}^n$, system (1) has the unique solution at any time instant $k \in \mathbb{N}$

$$x(k) = \phi(k,0)x(0) + \sum_{j=0}^{k-1} \phi(k, j+1) B_{\sigma(j)} u(j). \quad (2)$$

For the rest of this paper, we restrict the switching signal to a fixed and known switching signal given via a fixed mode sequence $(\sigma_j)_{j \in \mathbb{N}}$ as follows (see also Fig. 1)

$$\sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s),\ j \in \{0, 1, 2, ..., J\}, \quad (3)$$

where $k_j^s \in \mathbb{N}$ denote the switching times with $k_0^s = 0$, $k_{J+1}^s = K$, and $\sigma_j \in \{0, 1, ..., \mathtt{p}\}$. Here $K > 0$ denotes the final time of interest and $J > 0$ is the number of switches on the time interval $[0, K]$. Furthermore, note that the solution at $k = K$ is obtained from the last mode equation $x(K) = A_{\sigma_J} x(K-1) + B_{\sigma_J} u(K-1)$, i.e. we only consider the equation (1) up to $k = K - 1$.

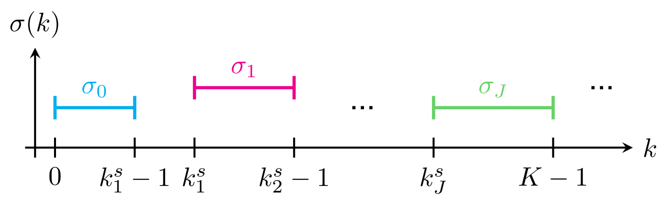


Fig. 1: Mode sequence (3)

The switching times k_j^s are assumed to be strictly increasing, i.e. $k_{j+1}^s > k_j^s$. Under a fixed switching signal we can write the solution at any switching time k_j^s by using the formula given in the following lemma.

Lemma 2.1: Under a fixed switching signal (3), the solution of linear switched systems (1) at any switching time k_j^s is given by

$$\begin{aligned} x(k_j^s) = {} & \psi_\sigma(j,0)x(0) + \psi_\sigma(j,1) R_{\sigma_0}(k_1^s - k_0^s) \begin{bmatrix} u(k_1^s - 1) \\ \vdots \\ u(0) \end{bmatrix} \\ & + \psi_\sigma(j,2) R_{\sigma_1}(k_2^s - k_1^s) \begin{bmatrix} u(k_2^s - 1) \\ \vdots \\ u(k_1^s) \end{bmatrix} + \cdots \\ & + \psi_\sigma(j,j) R_{\sigma_{j-1}}(k_j^s - k_{j-1}^s) \begin{bmatrix} u(k_j^s - 1) \\ \vdots \\ u(k_{j-1}^s) \end{bmatrix} \end{aligned} \quad (4)$$

where for $i, j, h \in \mathbb{N},\ j \geq h \geq 0$,

$$R_i(k) = [B_i, A_i B_i, \cdots, A_i^{k-1} B_i] \quad (5)$$

$$\psi_\sigma(j,h) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s} A_{\sigma_{j-2}}^{k_{j-1}^s - k_{j-2}^s} \cdots A_{\sigma_h}^{k_{h+1}^s - k_h^s} \quad (6)$$

Proof: Extracting the solution from (2) at the switching time k_j^s yields (4). ■

We refer to matrix (6) as the state transition matrix from the switching time k_h^s to the switching time k_j^s. Moreover, (6) can be rewritten in a recursive form as

$$\psi_\sigma(j,h) = A_{\sigma_{j-1}}^{k_j^s - k_{j-1}^s} \psi_\sigma(j-1, h) \quad (7)$$

with $\psi_\sigma(h,h) = I_n$, which is more computational-friendly than (6).

III. Reachability and Controllability

A. Definitions

In this paper, we study reachability (and also controllability) for a finite time interval $[0, K]$. We adopt the reachability definition in the sense of reaching a certain final state within a finite number of time instants. For controllability, we use the notion of bringing the initial state to zero. The following definitions precisely introduce the reachability and controllability notions we use in this study.

Definition 3.1 (Reachability from Zero): A state $x_f \in \mathbb{R}^n$ of the InLSS (1) is called *reachable from zero* on $[0, K]$ w.r.t. the switching signal σ if with $x(0) = 0$ there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K) = x_f$. //

Definition 3.2: The set of all final states $x_f \in \mathbb{R}^n$ that are reachable from zero on $[0, K]$ w.r.t. the switching signal σ is called the *reachable set from zero* and denoted by $\mathcal{R}_{[0,K]}^\sigma$. Furthermore, the InLSS (1) is called *completely reachable from zero* on $[0, K]$ w.r.t. the switching signal σ if $\mathcal{R}_{[0,K]}^\sigma = \mathbb{R}^n$. //

Definition 3.3 (Controllability to Zero): An initial state $x_0 \in \mathbb{R}^n$ of (1) is called *controllable to zero* on $[0, K]$ w.r.t. the switching signal σ if with $x(0) = x_0$ there exists an input sequence $(u(0), u(1), .., u(K-1))$ such that $x(K) = 0$. //

Definition 3.4: The controllable set of system (1) on $[0, K]$ w.r.t. the switching signal σ is the set of all initial states $x_0 \in \mathbb{R}^n$ that are controllable to zero on $[0, K]$ under σ and denoted as $\mathcal{C}_{[0,K]}^\sigma$. Furthermore, the InLSS (1) is called *completely controllable to zero* on $[0, K]$ w.r.t. σ if $\mathcal{C}_{[0,K]}^\sigma = \mathbb{R}^n$. //

In addition, we introduce here the notion of deadbeat controllable i.e. a InLSS is called *deadbeat controllable* if it is completely controllable on $[0, 1]$ i.e. within one time step. In other words, deadbeat controllable means that a single control action is enough to make the state zero. Note that using the solution from the initial mode is enough to characterize deadbeat controllability; thus, it is independent from the switching signal. Clearly, deadbeat controllable implies completely controllable by setting the input to zero for the subsequent time steps. If we know that a switched system is deadbeat controllable, then it is completely controllable on any time interval and for any switching signal.

Remark 3.5: Completely reachable from zero on $[0, K]$ is equivalent to *completely reachable* on $[0, K]$ i.e. any $x_f \in \mathcal{R}(K)$ is reachable from any initial value $x_0 \in \mathbb{R}^n$. This can be seen from the fact that also including the term for the nonzero initial value x_0 i.e. $\psi(J+1, 0)x_0$ yields the same condition. The same phenomenon as in nonswitched systems happens here i.e. reachability and controllability are equivalent if all of A_is are nonsingular; in general, reachability only implies controllability because the zero final state $x_f = 0$ is always reachable from any initial value, but controllability does not imply reachability. The latter is illustrated by the following simple single switch switched system:

$$\begin{array}{r|l} k < k_1^s: & k \geq k_1^s: \\ x_1(k+1) = x_1(k) & x_1(k+1) = 0 \\ x_2(k+1) = u(k) & x_2(k+1) = x_2(k) \end{array}$$

which is easily seen to be controllable (by setting $u(k_1^s-1) = 0$) but not reachable on $[0, K]$ for any $K > k_1^s$. This example also illustrates that for controllability of the overall switched system it is not necessary, that any of the individual modes is controllable. Moreover, by slightly changing the system above to:

$$\begin{array}{r|l} k < k_1^s: & k \geq k_1^s: \\ x_1(k+1) = x_1(k) & x_1(k+1) = u(k) \\ x_2(k+1) = u(k) & x_2(k+1) = x_2(k) \end{array}$$

we obtain a reachable switched system on $[0, K]$ composed of unreachable modes. ◇

B. Characterizations

We arrive at our main result for the reachability characterization given in the following theorem with the following subspaces notations. Let $\mathcal{R}_i(k) = \operatorname{im} R_i(k) = \operatorname{im}[B_i, A_iB_i, \cdots, A_i^{k-1}B_i]$ be the "local" or "individual" reachable space of mode-i on $[0, k]$, and define the following sequence of subspaces for $j = 1, 2, \ldots, J$

$$\mathcal{M}_0 = \mathcal{R}_{\sigma_0}(k_1^s), \tag{8a}$$

$$\mathcal{M}_j = A_{\sigma_j}^{k_{j+1}^s - k_j^s}\mathcal{M}_{j-1} + \mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s). \tag{8b}$$

The following theorem reveals that in fact, the reachable space on $[0, K]$ is equal to the subspace $\mathcal{M}_J$ defined above.

Theorem 3.6: Consider the InLSS (1) under a fixed and known switching signal σ of the form (3). Let $\mathcal{R}_{[0,K]}^\sigma$ be its *reachable space* on $[0, K]$ w.r.t. σ. Then

$$\mathcal{M}_J = \mathcal{R}_{[0,K]}^\sigma \tag{9}$$

where $\mathcal{M}_J$ is given by (8). In particular, (1) is completely reachable if, and only if, $\mathcal{M}_J = \mathbb{R}^n$.

Proof: First, note that $\mathcal{M}_J$ can be rewritten as

$$\mathcal{M}_J = \psi_\sigma(J+1, 1)\mathcal{R}_0(k_1^s) + \psi_\sigma(J+1, 2)\mathcal{R}_1(k_2^s - k_1^s) + \cdots + \mathcal{R}_J(k_{J+1}^s - k_J^s). \tag{10}$$

For any reachable state $x(K) \in \mathcal{R}_{[0,K]}^\sigma$, there exists an input sequence $(u(0), ..., u(K-1))$ such that (4) is satisfied with $x(k_{J+1}^s) = x(K)$ i.e. $x(K) \in \mathcal{M}_J$ and thus $\mathcal{R}_{[0,K]}^\sigma \subseteq \mathcal{M}_J$. From (10), any vector $x_f \in \mathcal{M}_J$ can be rewritten as the summation of vectors of the form (4) with $x_f = x(k_{J+1}^s) = x(K)$ i.e. there exists an input sequence $(u(0), ..., u(K-1))$ such that $x_f = x(K)$ and thus x_f is reachable from zero. Hence, $\mathcal{M}_J \subseteq \mathcal{R}_{[0,K]}^\sigma$. ∎

Example 3.7: Consider InLSS (1) composed of two modes with

$$(A_0, B_0) = \left(\left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]\right)$$
$$(A_1, B_1) = \left(\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]\right).$$

As individual systems, both modes are unreachable with their corresponding reachable spaces $\mathcal{R}_0 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathcal{R}_1 = \{0\}$ respectively. We characterize the switched systems under switching signals with the mode sequence $(\sigma) = (0, 1, 0)$ on time interval $[0, K]$ with $K = 12$ and with switching times $1 \leq k_1^s \leq k_2^s - 1$ and $k_1^s + 1 \leq k_2^s \leq K - 1$. The switched systems are reachable when $k_2^s - k_1^s$ is odd, however, they are unreachable when $k_2^s - k_1^s$ is even; this is explained as follows. The local reachable space corresponds to mode-0 and mode-1 is $\mathcal{R}_0(k) = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \ \forall k \in \mathbb{N}$ and $\mathcal{R}_1(k) = \{0\}, \ \forall k \in \mathbb{N}$ respectively. The sequence of subspaces (8) for the switched systems under the mode sequence $(0, 1, 0)$ is then given by

$$\begin{aligned} \mathcal{M}_0 &= \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \ \mathcal{M}_1 = A_1^{k_1^s}\operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \\ \mathcal{M}_2 &= A_0^{K-k_1^s}\mathcal{M}_1 + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] \\ &= \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]^{K-k_2^s}\left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s-k_1^s}\operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]. \end{aligned}$$

If $k_2^s - k_1^s$ is odd, then

$$\mathcal{M}_2 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]\operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \mathbb{R}^2,$$

i.e. the InLSS is (completely) reachable. If $k_2^s - k_1^s$ is even, then

$$\mathcal{M}_2 = \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$$

i.e. the InLSS is non-reachable. In particular, this example also shows that even though all individual modes are unreachable, switched systems composed of those modes under certain switching signals can be reachable. //

Theorem 3.6 can also be used to characterize the subspace of points from a nonzero initial values as follows.

Proposition 3.8: Consider the InLSS (1) with $x(0) = x_0 \in \mathbb{R}^n$ under a fixed and known switching signal σ of the form (3). Let $\mathcal{R}_{[0,K]}^\sigma(x_0)$ be its reachable space on $[0, K]$ from x_0 w.r.t. σ. Then the following structure applies:

$$\mathcal{R}_{[0,K]}^\sigma(x_0) = \psi_\sigma(J+1, 0)x_0 + \mathcal{M}_J. \tag{11}$$

Proof: Let $x(K, x_0, u(\cdot))$ be the solution of (1) at time step $k = K$ with $x(0) = x_0$ and input sequence $u(\cdot) = (u(0), ..., u(K-1))$. Then, from (4)

$$x(K, x_0, u(\cdot)) = x(K, x_0, 0) + x(K, 0, u(\cdot))$$

i.e. $x(K, x_0, u(\cdot))$ can be decomposed as the sum of the solution with zero inputs and the solution with zero initial value. Since $x(K, x_0, 0) = \psi_\sigma(J+1, 0)x_0$ and $x(K, 0, u(\cdot)) \in \mathcal{M}_J$ then we have $\mathcal{R}_{[0,K]}^\sigma(x_0) = \psi_\sigma(J+1, 0)x_0 + \mathcal{M}_J$. ∎

We now present the main result for the controllability characterization. Define the sequence of subspaces, for $j = J, J-1, \ldots, 0$

$$\mathcal{N}_{J+1} = \{0\}, \tag{12a}$$

$$\mathcal{N}_j = \left[A_{\sigma_j}^{k_{j+1}^s - k_j^s}\right]^{-1}\left[\mathcal{N}_{j+1} + \mathcal{R}_j(k_{j+1}^s - k_j^s)\right] \tag{12b}$$

where $P^{-1}[\mathcal{P}]$ for some matrix $P \in \mathbb{R}^{n \times n}$ and some subspace $\mathcal{P}$ denotes the preimage of $\mathcal{P}$ under P, i.e. $P^{-1}[\mathcal{P}] = \{x \in \mathbb{R}^n : Px \in \mathcal{P}\}$.

Theorem 3.9: Consider the InLSS (1) under a fixed and known switching signal σ of the form (3). Let $\mathcal{C}_{[0,K]}^\sigma$ be its *controllable space* on $[0, K]$ w.r.t. σ. Then

$$\mathcal{N}_0 = \mathcal{C}_{[0,K]}^\sigma \tag{13}$$

where $\mathcal{N}_j$ is defined by (12). In particular, (1) is completely controllable if, and only if, $\mathcal{N}_0 = \mathbb{R}^n$.

Proof: For any controllable initial state $x_0 \in \mathcal{C}_{[0,K]}^\sigma$, there exists an input sequence $\bar{u} = (u(0), ..., u(K-1)$ such that with $x(0) = x_0$ we obtain $x(K) = 0$. Thus, by backward iteration we have that the solution $x(k_j^s),\ j = J, J-1, ..., 0$ satisfies

$$x(k_J^s) \in [A_{\sigma_J}^{K-k_J^s}]^{-1}[\{0\} + \mathcal{R}_{\sigma_J}(K - k_J^s)] = \mathcal{N}_J,$$

$$x(k_{J-1}^s) \in \left[A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s}\right]^{-1} [\{x(k_J^s)\} + \mathcal{R}_{\sigma_{J-1}}(k_J^s - k_{J-1}^s)]$$

$$\subseteq \left[A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s}\right]^{-1} [\mathcal{N}_J + \mathcal{R}_{\sigma_{J-1}}(k_J^s - k_{J-1}^s)] = \mathcal{N}_{J-1},$$

$$\vdots$$

$$x(0) \in \left[A_{\sigma_0}^{k_1^s}\right]^{-1} [\{x(k_1^s\} + \mathcal{R}_{\sigma_0}(k_1^s)]$$

$$\subseteq [A_{\sigma_0}^{k_1^s}]^{-1}[\mathcal{N}_1 + \mathcal{R}_{\sigma_0}(k_1^s)] = \mathcal{N}_0$$

i.e. $x_0 \in \mathcal{N}_0$. Hence, $\mathcal{C}_{[0,K]}^{\sigma} \subseteq \mathcal{N}_0$. Now, for any vector $n_0 \in \mathcal{N}_0 = [A_{\sigma_0}^{k_1^s}]^{-1}[\mathcal{N}_1 + \mathcal{R}_0(k_1^s)]$, there exist $n_1 \in \mathcal{N}_1$ and $r_0 \in \mathcal{R}_0(k_1^s)$ satisfying

$$n_1 = A_{\sigma_0}^{k_1^s} n_0 + r_0.$$

In particular there exists an input sequence $(u(0), u(1), ..., u(k_1^s - 1)$ such that with $x(0) = n_0$ we have $x(k_1^s) = n_1$. By forward iteration, there exist $n_J \in \mathcal{N}_J$ and $r_J \in \mathcal{R}_{\sigma_J}(K - k_J^s)$ which corresponds to an input sequence $(u(k_J^s), u(k_J^s + 1), ..., u(K-1)$ satisfying

$$0 = A_{\sigma_J}^{K-k_J^s} n_J + r_J = x(K).$$

Altogether, we have found an input sequence, which controls the initial value $n_0 \in \mathcal{N}_0$ to zero on the time interval $[0, K]$, hence $\mathcal{N}_0 \subseteq \mathcal{C}_{[0,K]}^{\sigma}$. ∎

Example 3.10: Recall the switched system in Example 3.7. The local controllable sets corresponding to mode-0 and mode-1 are $\mathcal{C}_{[0,K]}^0 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\mathcal{C}_{[0,K]}^1 = \{0\}$ respectively for all $K \in \mathbb{N}$. Thus, both individual systems are not (completely) controllable. The sequence of subspaces (12) for the switched systems under the mode sequence $(0, 1, 0)$ is then given by

$$\mathcal{N}_3 = \{0\}$$

$$\mathcal{N}_2 = \left[A_0^{K-k_2^s}\right]^1 [\mathcal{N}_3 + \mathcal{R}_0(K - k_2^s)] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],$$

$$\mathcal{N}_1 = \left[A_1^{k_2^s - k_1^s}\right]^{-1} [\mathcal{N}_2 + \mathcal{R}_1(k_2^s - k_1^s)]$$

$$= \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s - k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],$$

$$\mathcal{N}_0 = \left[A_0^{k_1^s}\right]^{-1} [\mathcal{N}_1 + \mathcal{R}_0(k_1^s)]$$

$$= \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]^{k_2^s - k_1^s} \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$$

If $k_2^s - k_1^s$ is odd, then

$$\mathcal{N}_0 = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \mathbb{R}^n,$$

i.e. the InLSS is (completely) controllable. If $k_2^s - k_1^s$ is even, then

$$\mathcal{N}_0 = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right] \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right],$$

i.e. the InLSS is not (completely) controllable. Similar phenomenon is derived here as in reachability where even though all individual modes are uncontrollable, the switched system composed of those modes under certain switching signals can be controllable. //

Example 3.11: Consider InLSS (1) composed of two modes with

$$(A_0, B_0) = \left(\left[\begin{smallmatrix}0&-1&0\\1&0&0\\0&0&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\-1\\0\end{smallmatrix}\right]\right),$$

$$(A_0, B_0) = \left(\left[\begin{smallmatrix}1&-1&0\\1&-1&1\\1&-1&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\\0\end{smallmatrix}\right]\right).$$

First, we observe the switched system on $[0, K]$ with $K = 20$ under the mode sequence $(0, 1, 0)$ and switching times $k_1^s = 5$ and $k_2^s = 10$. The sequence of subspaces (12) for this switched system is given by

$$\mathcal{N}_3 = \{0\}$$

$$\mathcal{N}_2 = \left[A_0^{10}\right]^{-1} [\mathcal{N}_3 + \mathcal{R}_0(10)] = \operatorname{im}\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right],$$

$$\mathcal{N}_1 = \left[A_1^{5}\right]^{-1} [\mathcal{N}_2 + \mathcal{R}_1(5)] = \operatorname{im}\left[\begin{smallmatrix}1&0\\1&0\\0&1\end{smallmatrix}\right],$$

$$\mathcal{N}_0 = \left[A_0^{5}\right]^{-1} [\mathcal{N}_1 + \mathcal{R}_0(5)] = \mathbb{R}^3,$$

i.e. the switched system is (completely) controllable. If the second switching time is changed with $k_2^s = 11$, then we have $\mathcal{N}_0 = \operatorname{im}\left[\begin{smallmatrix}1&0\\0&1\\0&0\end{smallmatrix}\right]$ i.e. the switched system is uncontrollable. The characterization results including the reachability for all possible switching times within the time interval $[0, 20]$ are shown in Fig. 2. It can be seen that in general, the switched system is reachable and controllable for some switching times, however, it is unreachable and uncontrollable for some other switching times. //

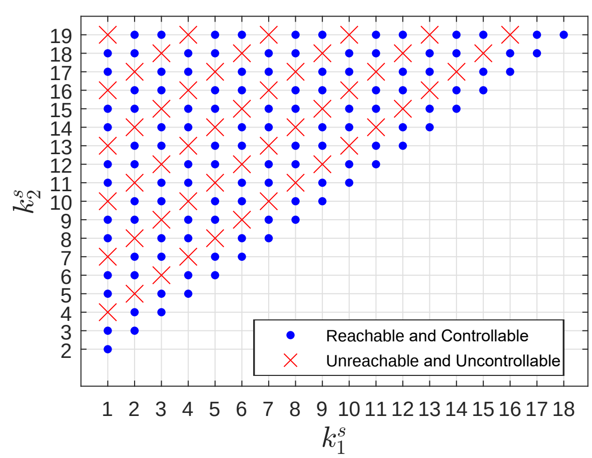


Fig. 2: Switching times vs reachability/controllability Example 3.11

Finally, the condition for deadbeat controllable can be derived from the characterization of controllability given in Theorem 3.9 considered for $K = 1$:

Corollary 3.12: The InLSS (1) is *deadbeat controllable* if, and only if, $A_{\sigma_0}^{-1}[\operatorname{im} B_{\sigma_0}] = \mathbb{R}^n$.

Note that $A_{\sigma_0}^{-1}[\operatorname{im} B_{\sigma_0}] = \mathbb{R}^n$ is equivalent to $\operatorname{im} A_{\sigma_0} \subseteq \operatorname{im} B_{\sigma_0}$, which just means that the input can compensate any value $A_{\sigma_0} x_0$ resulting from an arbitrary initial value x_0.

Remark 3.13 (Dwell time simplication): By Cayley-Hamilton theorem, if all modes are active for at least n time steps (dwell time), then the local/individual reachable space can be simplified as $\mathcal{R}_i = \operatorname{im}[B_i, A_i B_i, \cdots, A_i^{n-1} B_i]$. In this case, the constructions for reachable and controllable sets (8) and (12) can be simplified as

$$\begin{cases}\mathcal{M}_0 = \mathcal{R}_{\sigma_0},\\ \mathcal{M}_j = A_{\sigma_j}^{k_{j+1}^s-k_j^s}\mathcal{M}_{j-1}+\mathcal{R}_{\sigma_j}\end{cases}$$

and

$$\begin{cases}\mathcal{N}_{J+1} = \{0\},\\ \quad\mathcal{N}_j = \left[A_{\sigma_j}^{k_{j+1}^s-k_j^s}\right]^{-1}[\mathcal{N}_{j+1}+\mathcal{R}_{\sigma_j}]\end{cases}$$

respectively. If fact, the reachability space under the above dwell time condition can then also be written as

$$\mathcal{R}_{[0,K]}^{\sigma} = \sum_{j=0}^{J}\psi_\sigma(J+1,j+1)\mathcal{R}_{\sigma_j}. \tag{14}$$

The controllability space can however not be written in a similar way, but we can still conclude that

$$\mathcal{N}_{[0,K]}^{\sigma} \supseteq \sum_{j=0}^{J}\psi_\sigma(j+1,0)^{-1}\mathcal{R}_{\sigma_j}.$$

This difference occurs because for general matrices M and subspaces $\mathcal{P}$ and $\mathcal{Q}$ we have $M(\mathcal{P}+\mathcal{Q}) = M\mathcal{P}+M\mathcal{Q}$ but only $M^{-1}(\mathcal{P}+\mathcal{Q}) \supseteq M^{-1}\mathcal{P}+M^{-1}\mathcal{Q}$.

Remark 3.14: It is well known that for nonswitched systems, the following nice inclusion holds:

$$\mathcal{R}_{[0,1]} \subseteq \mathcal{R}_{[0,2]} \subseteq \cdots \subseteq \mathcal{R}_{[0,n]} = \mathcal{R}_{[0,n+1]} = \cdots. \tag{15}$$

The subspace inclusion also holds for the controllability spaces, because once an initial state is controllable to zero it can also be controlled to zero on a larger time interval (by simply choosing $u(k) = 0$ after zero was reached) and this property remains true also for the switched case, i.e.

$$\mathcal{C}_{[0,k_1]}^{\sigma} \subseteq \mathcal{C}_{[0,k_2]}^{\sigma} \quad 0 < k_1 < k_2 < K,$$

which implies that (complete) controllability on $[0,K]$ implies (complete) controllability on $[0,k], \forall k > K$. In contrast, for the reachibility spaces of switched systems, there is no general relationship between $\mathcal{R}_{[0,k_1]}^{\sigma}$ and $\mathcal{R}_{[0,k_2]}^{\sigma}$ with two different time steps k_1 and k_2 that correspond to two different mode activation time intervals i.e. $k_1 \in [k_i^s, k_i^s+1)$ and $k_2 \in [k_j^s, k_j^s+1)$ with $i \neq j$; a simple example for this is when we have $\mathcal{R}_{[0,k_i^s]}^{\sigma} = \mathbb{R}^n$ and the system switches at $k = k_i^s$ to the mode $(A_{i+1}, B_{i+1}) = (0,0)$ which yields $\mathcal{R}_{[0,k]}^{\sigma} = \{0\}$ for all $k > k_i^s$.

The inclusion (15) may even not be valid within a single mode interval; this is illustrated by the following single switch system:

$k < k_1^s$:	$k \geq k_1^s$:
$x(k+1) = x(k) + \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]u(k)$	$x(k+1) = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]x(k)$

where $\mathcal{R}_{[0,K]}$ with $K > k_1^s$ is

$$\mathcal{R}_{[0,K]} = \begin{cases}\operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] & \text{if } K-k_1^s \text{ is odd}\\ \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] & \text{if } K-k_1^s \text{ is even.}\end{cases}$$

However, the following nice observation still applies after the mode-j is active for at least n time steps: (complete) (un)reachability on $[0,k_1], k_1 \in (k_j^s, k_{j+1}^s]$ with $k_1 - k_j^s \geq n$ implies (complete) (un)reachability on $[0, k_1 + k], \forall k \leq k_{j+1}^s - k_1$; however, it does not imply (complete) (un)reachability on $[0,k_2],\ k_2 > k_{j+1}^s$ i.e. once the switched system is (un)reachable at some time step after the current mode is active for at least n time steps, it stays (un)reachable within the current mode time interval, and once it switches to another mode, it may be no longer (un)reachable. This is explained as follows; w.l.o.g., we consider here a single switch system with the mode sequence $(0,1)$ and switching time k_1^s. From (15), for $k > k_1^s$ we have that $\mathcal{R}_1(1) \subseteq \mathcal{R}_j(2) \subseteq \cdots \subseteq \mathcal{R}_j(n) = \mathcal{R}_j(n+1) = \cdots$, and from (8)

$$\mathcal{R}_{[0,k]}^{\sigma} = A_1^{k-k_1^s}\mathcal{R}_0 + \mathcal{R}_1,$$
$$\mathcal{R}_{[0,k+1]}^{\sigma} = A_1^{k+1-k_1^s}\mathcal{R}_0 + \mathcal{R}_1.$$

Note that $\mathcal{R}_1$ is invariant under A_1, then

$$\begin{aligned}\mathcal{R}_{[0,k+1]}^{\sigma} &= A_1^{k+1-k_1^s}\mathcal{R}_0 + A_1\mathcal{R}_1\\ &= A_1(A_1^{k-k_1^s}\mathcal{R}_0 + \mathcal{R}_1)\\ &= A_1\mathcal{R}_{[0,k]}^{\sigma}.\end{aligned}$$

If A_1 is nonsingular then clearly $\dim\mathcal{R}_{[0,k+1]}^{\sigma} = \dim\mathcal{R}_{[0,k]}^{\sigma}$ i.e. the system's reachability property remains the same. Now, if A_1 is singular then by applying the state transformation $\widetilde{x} = Px$ to the mode-1 with the nonsingular P and nilpotent N satisfying $PA_1P^{-1} = \left[\begin{smallmatrix}\widetilde{A}_1 & 0\\ 0 & N\end{smallmatrix}\right]$, the reachable space is now

$$\begin{aligned}\mathcal{R}_{[0,k+1]}^{\sigma} &= \begin{bmatrix}\widetilde{A}_1^{k+1-k_1^s} & 0\\ 0 & 0\end{bmatrix}\operatorname{im}\begin{bmatrix}R_0^1\\ R_0^2\end{bmatrix} + \operatorname{im}\begin{bmatrix}\widetilde{R}_1^1\\ \widetilde{R}_1^2\end{bmatrix}\\ &= \operatorname{im}\begin{bmatrix}\widetilde{A}_1^{k+1-k_1^s}R_0^1\\ 0\end{bmatrix} + \operatorname{im}\begin{bmatrix}\widetilde{R}_1^1\\ \widetilde{R}_1^2\end{bmatrix}\end{aligned}$$

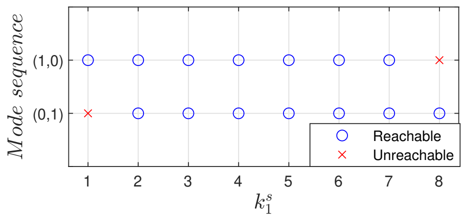
with $\operatorname{im}\left[\begin{smallmatrix}R_0^1\\ R_0^2\end{smallmatrix}\right] = \mathcal{R}_0$ and $\operatorname{im}\left[\begin{smallmatrix}\widetilde{R}_1^1\\ \widetilde{R}_1^2\end{smallmatrix}\right] = \operatorname{im}[PB_1, PA_1P^{-1}PB_1, ..., (PA_1P^{-1})^{n-1}PB_1]$, and thus $\dim\mathcal{R}_{[0,k+1]}^{\sigma} = \dim\mathcal{R}_{[0,k]}^{\sigma}$. ⋄

From the reachable and controllable sets constructions (8) and (12), and the confirmation inferred from Example 3.7 and Example 3.10, the reachability and controllability properties depend on the switching times and on how long the individual modes are active. In those examples, this dependency happens due to the rotation matrix A_1 which rotates the reachable/controllable set so that the reachable/controllable set of the switched system is equal to the whole space for some switching times and is unequal to the whole space for some other switching times. Moreover, in those examples, the dependency happens both with switching times that are closed and far away from the time interval boundaries. However, under single switch switching signals, this dependency seems happen only with switching times that are closed enough to the time interval boundaries as illustrated by the forthcoming Example 3.15. We will show that this dependency indeed only happens when the switching times are closed enough to the time interval boundaries, see the forthcoming Proposition 4.3.

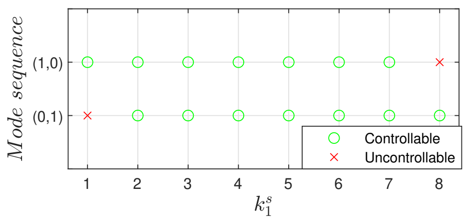
Example 3.15: Consider the InLSS (1) composed of two modes with

$$(A_0, B_0) = \left(\begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ -1 & 1 & 1\end{bmatrix}, \begin{bmatrix}-1\\ -1\\ 0\end{bmatrix}\right)$$
$$(A_1, B_1) = \left(\begin{bmatrix}0 & 0 & 1\\ 1 & 1 & 1\\ -1 & -1 & 1\end{bmatrix}, \begin{bmatrix}1\\ -1\\ -1\end{bmatrix}\right).$$

We observe here the switched system with mode sequences $(0,1)$ and $(1,0)$ on the time interval $[0,K]$ with $K = 9$. With short enough (only one time step) activation time for the first/second mode i.e. when $k_1^s = 1$ or 8, the switched system is unreachable and uncontrollable, however, with longer activation times, the switched system is always reachable and controllable, see Fig. 3. //



(a) Reachability



(b) Controllability

Fig. 3: Reachability and controllability characterizations for Example 3.15

IV. Dependencies on the Switching Times

To study the dependency of the reachability and controllability properties on the switching times, we will use the dwell-time term which was initially introduced in [20]. For a positive constant τ_D, let $\mathbb{S}^{[\tau_D]}$ be the set of all fixed and known mode sequences given by (3) where the difference between two consecutive switching times is not smaller than τ_D. We refer to τ_D as the (fixed) dwell-time. Furthermore, by $\mathbb{S}_{[0,K]}^{[\tau_D]}$ we denote the set of all switching signals $\sigma \in \mathbb{S}^{[\tau_D]}$ defined on the time interval $[0, K]$.

Definition 4.1 (Constant Reachability/Controllability): Consider the InLSS (1). Its reachability (controllability) on $[0, K]$ is called *constant* w.r.t. the dwell time $\tau_D > 0$ if it is either reachable (controllable) for all $\sigma \in \mathbb{S}_{[0,K]}^{\tau_D}$ or unreachable (uncontrollable) for all $\sigma \in \mathbb{S}_{[0,K]}^{[\tau_D]}$. //

We now present some specific situations where the reachability and controllability properties are constant w.r.t. the dwell time n. The first situation, which is rather obvious, is that when A_i are idempotent. Indeed, this will make powers to A_i matrices in the reachability and controllability conditions do not affect the characterization results. The second situation is when the state is one-dimensional as proved in the following proposition, which is a property also seen for observability/determinability, see [21].

Proposition 4.2: Reachability and controllability of one-dimensional InLSSs are constant on $[0, K]$ for any $K \in \mathbb{N}$ w.r.t. the dwell time $\tau_D = 1$.

Proof: The local reachable spaces $\mathcal{R}_{\sigma_j}(k_{j+1}^s - k_j^s) = \operatorname{im} B_{\sigma_j}$ are either 0 or $\mathbb{R}$ independently of the switching times and the scalar $A_{\sigma_j}^{k_{j+1}^s - k_j^s}$ is either 0 or a nonzero independently of the switching times. Altogether, the sequences $\mathcal{N}_j$ and $\mathcal{M}_j$ do not depend on the switching times. ■

Another situation where constant reachability/controllability can be shown is the single switch case under a sufficiently large dwell time:

Proposition 4.3: Consider the InLSS (1) under a single switch switching signal defined on $[0, K], K > k_1^s$ with k_1^s as the single switching time. Then, its reachability and controllability are constant w.r.t. the dwell time $\tau_D = n$.

Proof: W.l.o.g. consider the mode sequence $(0, 1)$ with the switching time k_1^s. The reachable space at $K > k_1^s + n$ is

$$\mathcal{R}_{[0,K]}^{\sigma} = A_1^{K-k_1^s} \mathcal{R}_0 + \mathcal{R}_1$$

with $\mathcal{R}_i = \operatorname{im}[B_i, A_i B_i, \cdots, A_i^{n-1} B_i], i = 0, 1$. Clearly, $\mathcal{R}_1$ is fixed no matter the switching time k_1^s, and $\mathcal{R}_1$ is A_1-invariant. A nonconstant reachability happens if $\mathcal{R}_{[0,K]}^{\sigma} = \mathbb{R}^n$ for some k_1^s and $\mathcal{R}_{[0,K]}^{\sigma} \subsetneq \mathbb{R}^n$ for some other k_1^s, or in other words $\dim \mathcal{R}_{[0,K]}^{\sigma} = n$ for some k_1^s and $\dim \mathcal{R}_{[0,K]}^{\sigma} < n$ for some other k_1^s.

Case 1: A_1 is nonsingular. Assume first for $k_1^s = K - n$, $\mathcal{R}_{[0,K]}^{\sigma} = A_1^n \mathcal{R}_0 + \mathcal{R}_1 \subsetneq \mathbb{R}^n$ i.e. unreachable then $A_1^{n+l} \mathcal{R}_0 + \mathcal{R}_1 \subsetneq \mathbb{R}^n$ for any $l \in \mathbb{N}$ i.e. it remains unreachable for any other possible k_1^s since $\dim A_1^{n+l} \mathcal{R}_0$ cannot increase. Now, if for $k_1^s = K - n$, $\mathcal{R}_{[0,K]}^{\sigma} = A_1^n \mathcal{R}_0 + \mathcal{R}_1 = \mathbb{R}^n$ i.e. reachable then from the dimension formula, $\dim A_1^n \mathcal{R}_0 + \dim \mathcal{R}_1 = n$ and $\dim A_1^{n+l} \mathcal{R}_0 + \dim A_1^l \mathcal{R}_1 = n$, and thus $\dim A_1^{n+l} \mathcal{R}_0 + \dim \mathcal{R}_1 = n$ for any $l \in \mathbb{N}$ since $\mathcal{R}_1$ is A_1-invariant. Hence, it stays reachable for any other possible k_1^s.

Case 2: A_1 is singular. Then, there exists a nonsingular matrix P such that

$$P^{-1} A_1 P = \begin{bmatrix} N & 0 \\ 0 & \widetilde{A}_1 \end{bmatrix}$$

where N is a nilpotent with nilpotency index less than n and $\widetilde{A}_1$ is nonsingular. Applying the state transformation $\widetilde{x} = P^{-1} x$, the system's modes can now be written as

$$(\widetilde{A}_0, \widetilde{B}_0) = (P^{-1} A_0 P, P^{-1} B_0) \text{ and}$$
$$(\widetilde{A}_1, \widetilde{B}_1) = \left(\begin{bmatrix} N & 0 \\ 0 & \widetilde{A}_1 \end{bmatrix} \right), P^{-1} B_1).$$

The reachable space for this new tranformed system is

$$\widetilde{\mathcal{R}}_{[0,K]}^{\sigma} = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{A}_1^{K-k^s} \end{bmatrix} \widetilde{\mathcal{R}}_0 + \widetilde{\mathcal{R}}_1$$

with $\widetilde{\mathcal{R}}_i = \operatorname{im}[\widetilde{B}_i, \widetilde{A}_i \widetilde{B}_i, \cdots, \widetilde{A}_i^{n-1} \widetilde{B}_i]$. The same arguments as in Case 1 apply here, and thus the reachability is constant. The constant controllability follows with similar arguments and the details are omitted. ■

V. Summary and Future Works

Necessary and sufficient conditions using a geometric approach for reachability and controllability characterizations have been presented for linear switched systems. Moreover, the notion of constant reachability/controllability has been introduced to study when reachability/controllability does not depend on the switching times. Some specific situations for constant reachability/controllability have been discussed.

Extending the results from this paper to singular linear switched systems where each mode is a singular system is ongoing research and we expect that the same methods will result in similar results. Furthermore, we will investigate sufficient or necessary conditions for reachability/controllability which are independent of the switching times.

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