Averaging for switched impulsive systems with pulse width modulation ⋆

Elisa Mostacciuolo [a], Stephan Trenn [b], Francesco Vasca [c]

[a] *Teacher at the Italian Ministry of Education, 82010 San Leucio del Sannio, Benevento, Italy*

[b] *SCAA @ BI(FSE), University of Groningen, Nijenborgh 9, 9747 AG Groningen*

[c] *Department of Engineering, University of Sannio, 82100 Benevento, Italy*

Abstract

Linear switched impulsive systems (SIS) are characterized by ordinary differential equations as modes dynamics and state jumps at the switching time instants. The presence of possible jumps in the state makes nontrivial the application of classical averaging techniques. In this paper we consider SIS with pulse width modulation (PWM) and we propose an averaged model whose solution approximates the moving average of the SIS solution with an error which decreases with the multiple of the switching period and by decreasing the PWM period. The averaging result requires milder assumptions on the system matrices with respect to those needed by the previous averaging techniques for SIS. The interest of the proposed model is strengthened by the fact that it reduces to the classical averaged model for PWM systems when there are no jumps in the state. The theoretical results are verified through numerical results obtained by considering a switched capacitor electrical circuit.

Key words: Averaging, switched impulsive systems, pulse width modulation, differential algebraic equations.

1 Introduction

Switching represents the natural behavior of many systems of practical interest, e.g., mechanical systems [18], electronic circuits [23], piecewise affine systems [7,6,1]. In particular, switched systems with pulse width modulation (PWM) are characterized by a sequence of modes which repeats periodically in time [19]. The "fast" switching behaviour determines oscillations, i.e., the so called *ripple*, of the state variables around a smooth trajectory whose dynamics are typically much slower than the switching period. The main goal of the averaging theory consists of obtaining a smooth model whose solution is able to capture the averaged behaviour of the switched system. The corresponding theoretical objective consists of proving that the error between the solutions of the switched and the averaged systems is of order of the switching period.

Averaging theory has been extensively studied for PWM systems with Lipschitz continuous solutions, see among others [3,17,21,24,25]. On the other hand, there exist practical PWM systems, such as switched capacitor DC/DC converters, which exhibit state jumps at the switching time instants and they still present a sort of averaging behaviour [16,9]. Linear switched impulsive systems (SIS) is a class of these systems where each mode is characterized by a set of linear ordinary differential equations and algebraic constraints which determine the rule of the state jumps at the switching time instants [20]. In this paper we study the application of averaging theory to SIS with PWM.

The presence of state discontinuities makes nontrivial the formal study of switched systems [2] and two aspects are specifically critical for the averaging analysis of SIS. The first issue is related to the fact that the amplitudes of the state discontinuities usually do not reduce by decreasing the switching period. The approach we propose for overcoming this obstacle consists of comparing the averaged solution with the moving average of the SIS solution. Another theoretical challenge is due to the dependence of the SIS solution on the matrices which characterize the state jumps which one would then expect should be included in the averaged model too. This de-

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Email addresses: elisa.mostacciuolo@posta.istruzione.it (Elisa Mostacciuolo), s.trenn@rug.nl (Stephan Trenn), vasca@unisannio.it (Francesco Vasca).

pendence introduces several problems for the analysis which requires nontrivial theoretical arguments in order to be solved.

The averaging analysis for switched systems with state jumps is still at its infancy. An averaged model for homogeneous SIS with two modes was presented in [5] where strict algebraic conditions (commutativity) on the matrices characterizing the state jumps and those describing the modes dynamics were required. These conditions are not assumed in the analysis of this paper. The averaging result in [5] was extended to more than two modes in [4], to the non-autonomous case in [10] and to partial averaging in [11], however the corresponding theoretical findings were still based on the algebraic assumptions on the SIS matrices introduced in [5]. The commutativity condition was relaxed in [12] by using conditions on the kernel and the image of the matrices of the modes. However, there exist practical SIS for which these conditions are not satisfied [16,15].

In this paper we propose a continuous-time averaged model for SIS under milder assumptions with respect to those formerly used in the literature. The averaging property was conjectured by the authors in [13] without providing any formal proof and by taking inspiration from the application of theoretical findings in [14] applied to discrete-time models. In this paper we provide a formal proof for the averaging result by showing that the error between the solution of the averaged model and the moving average of the solution of the SIS decreases exponentially with the number of switching periods and linearly with respect to the period duration. The proposed averaged model is a generalization of the classical averaged model adopted for PWM systems with Lipschitz solution, in the sense that if there are no state jumps the matrices of the proposed model reduce to those of the classical one. A switched capacitor electrical circuit is considered as a motivating practical example and corresponding numerical simulations validate the effectiveness of the proposed model.

The rest of the paper is organized as follows. In Section 2 some preliminary definitions and properties of SIS are recalled and the motivating example is presented. Section 3 describes the structure of the proposed averaged model and Section 4 our main theoretical result (all proofs are reported in the Appendix). In Section 5 numerical verification of the theoretical results is proposed. The synthesis in Section 6 summarizes conclusions and future work.

2 Switched impulsive systems

In this section some preliminaries on notation, the definition of the class of SIS of interest and the theoretical motivation of our work are presented.

2.1 Notation

The following notation is adopted throughout the paper: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^+$ ($\mathbb{R}_0^+$) is the set of positive (nonnegative) real numbers, $\mathbb{R}^n$ is the set of n-dimensional vectors of real numbers, $\mathbb{C}$ is the set of complex numbers, $F \in \mathbb{R}^{m\times n}$ indicate a real matrix with m rows and n columns, $\mathbb{N}_0$ ($\mathbb{N}$) is the set of (positive) natural numbers; $\|x\|$ indicates the Euclidean norm of the vector $x \in \mathbb{R}^n$; $\lfloor x \rfloor$ is the largest integer less or equal than $x \in \mathbb{R}$. A matrix $F \in \mathbb{R}^{n\times n}$ is *bounded* if $\|F^k\| := \|F^k x\| \le \alpha$ for some finite $\alpha \in \mathbb{R}^+$, for any $k \in \mathbb{N}$ and for any $x \in \mathbb{R}^n$; it is *idempotent* if $F^k = F$ for any $k \in \mathbb{N}$; it is *Schur* if all its eigenvalues have magnitude smaller than 1. A pair of matrices $F_i, F_j \in \mathbb{R}^{n\times n}$ is *commutative* if $F_iF_j = F_jF_i$ with $i, j \in \mathbb{N}$. The product of q matrices F_i, $i = 1,\dots,\mathsf{q}$ is defined as (note the order) $\prod_{i=1}^{\mathsf{q}} F_i = F_{\mathsf{q}}F_{\mathsf{q}-1}\cdots F_2F_1$. The following notation is used: $G_i(\xi) = e^{F_i\xi}$ for all $\xi \in \mathbb{R}$ and $G_{i,p} = G_i(d_ip) = e^{F_id_ip}$ for some $d_i \in D = [0, 1)$, $\Sigma = \{1,\dots\mathsf{q}\}$ with $\mathsf{q} \in \mathbb{N}$. A function $u(t) : \mathbb{R}_0^+ \to \mathbb{R}^n$ is a *Bohl function* if it is a linear combination of terms of the form $t^ke^{\lambda t}$ where $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{C}$. A matrix function $G_p : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is said to be an $\mathrm{O}(p^r)$ function as $p \to 0$ for any $r \in \mathbb{N}_0$, ($G_p = \mathrm{O}(p^r)$ for short), if there exist constants $\alpha \in \mathbb{R}^+$ and $\bar{p} \in \mathbb{R}^+$ such that $\|G_p\| \le \alpha p^r$ for all $p \in (0, \bar{p}]$.

2.2 SIS with pulse width modulation

The class of SIS considered in our analysis is now introduced. It is characterized by a PWM with $\mathsf{q} \in \mathbb{N}$ modes and a switching period $p \in \mathbb{R}^+$. The sequence of modes is assumed to be fixed. At each $t_k = kp$, $k \in \mathbb{N}_0$, the mode $i = 1$ is activated and it remains active since $t_k + d_1p$ where $d_1 \in D$ is the duty cycle of the first mode. Then the system commutes from the mode $(i-1)$-th to the mode i-th, $i = 2,\dots,\mathsf{q}$, at the time instants $s_{k,i} := t_k + \sum_{j=1}^{i-1} d_jp$, $k \in \mathbb{N}_0$ where $d_i \in D$, is the duty cycle of the i-th mode; in particular, $\sum_{i=1}^{\mathsf{q}} d_i = 1$, see Fig. 1.

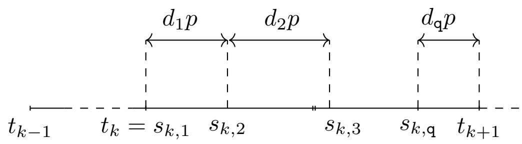

Fig. 1. Illustration of the switching times notation for $k \ge 1$.

The continuous-time switched impulsive system can be represented as follows

$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{1a}$$
$$\dot{x}(t) = F_ix(t), \quad t \in (s_{k,i}, s_{k,i+1}) \tag{1b}$$

with $x(0^-) = x_0 \in \mathbb{R}^n$, for $k \in \mathbb{N}_0$, $i \in \Sigma$, where $s_{k,q+1} := t_{k+1} = s_{k+1,1}$, the state variable is the same for each mode and $x(s_{k,i}^-)$ ($x(s_{k,i}^+)$) is the state at the end (beginning) of the $(i-1)$-th (i-th) mode at the k-th period. The nonzero flow matrix $F_i \in \mathbb{R}^{n\times n}$, $i \in \Sigma$, characterizes the dynamics of the i-th mode and the matrix $\Pi_i \in \mathbb{R}^{n\times n}$, $i \in \Sigma$, called consistency projector in the differential algebraic equations terminology, determines the possible jumps of the state variables at the switching time instants. The switched impulsive system (1) includes several practical systems and, among them, switched descriptor systems which can be represented in the form of homogeneous switched differential algebraic equations with regular matrix pairs [14].

The solution of (1) can be written by cascading the solutions of the different modes and by considering the jumps at the switching time instants. In particular, at the switching time instants one can write

$$x(s_{k,i}^+) = \Pi_i x(s_{k,i}^-) \tag{2a}$$
$$x(s_{k,i+1}^-) = G_{i,p} x(s_{k,i}^+), \tag{2b}$$

where $G_{i,p} = e^{F_i d_i p}$, for $k \in \mathbb{N}_0$, $i \in \Sigma$. By combining (2), one obtains that the left solution of (1) at the time instants multiple of the switching period must satisfy the following iterative equation

$$x_{k+1}^- = \Theta_p x_k^- \tag{3}$$

for all $k \in \mathbb{N}_0$ where $x_k^- := x(t_k^-)$, $x_0^- = x_0$ and

$$\Theta_p = \prod_{j=1}^{\mathtt{q}} G_{j,p}\Pi_j. \tag{4}$$

By iteratively applying (3), the left solution of (1) at the time instants multiple of p can be written as

$$x_k^- = \Theta_p^k x_0^- \tag{5}$$

for all $k \in \mathbb{N}_0$.

Remark 1 *The model* (1) *is autonomous, however the analysis presented below can be easily applied to the case of non-autonomous systems whose inputs are Bohl functions by extending the state space.*

2.3 Basics on averaging for SIS

Averaging theory has been widely studied for SIS. In what follows we briefly recall the existing theoretical result in order to motivate the proposed analysis and the novelties of our results.

In [5] a SIS model (1) with $\mathtt{q} = 2$ was considered and the following averaged model

$$\dot{\xi}(t) = A_{\text{av}}\xi(t), \quad t \in \mathbb{R}_0^+ \tag{6}$$

with $\xi(0) = \Pi x_0$, $A_{\text{av}} = \Pi(F_1 d_1 + F_2 d_2)\Pi$, $\Pi = \Pi_2\Pi_1$, was introduced. In particular, it was proved that if the matrices Π_1 and Π_2 are commutative and idempotent, and the conditions

$$\Pi_i F_i = F_i \Pi_i = F_i \tag{7}$$

hold for all $i \in \Sigma$ then for any finite $\bar{t} \in \mathbb{R}^+$ the error between the solution of (1) and that of (6) is decreasing with the same order of the switching period, i.e.

$$x(t) - \xi(t) = \mathrm{O}(p) \tag{8}$$

for all $t \in (0, \bar{t}]$. This result was extended to more than two modes in [5], to the non autonomous case in [10] and to partial averaging in [11].

The commutativity condition was relaxed in [12] by introducing the following conditions on the kernel and the image of the matrices of the system (1):

$$\operatorname{im}\Pi \subseteq \operatorname{im}\Pi_i, \tag{9a}$$
$$\ker\Pi \supseteq \ker\Pi_i, \tag{9b}$$

for all $i \in \Sigma$, where the matrix $\Pi \in \mathbb{R}^{n\times n}$ is given by

$$\Pi = \prod_{i=1}^{\mathtt{q}} \Pi_i. \tag{10}$$

Condition (7) and the assumption that Π_i, $i \in \Sigma$, are idempotent were still required in order to obtain the averaging result in [12]. It should be noticed that the commutativity conditions imply (9) also if Π_i, $i \in \Sigma$, are not idempotent. In general, if the matrices Π_i, $i \in \Sigma$, are idempotent, the matrix Π may not be. However, if (9) hold and Π_i, $i \in \Sigma$, are idempotent then Π is idempotent.

Unfortunately, it arises that several practical electrical circuits do not satisfy (9), even if they present a sort of averaging behaviour [16]. The averaging result presented in this paper considers SIS (1) where the matrices Π_i, $i \in \Sigma$, do not commute and the conditions (7) and (9) are not required.

2.4 A motivating example

As a motivating example, let us consider the switched capacitor electrical circuit shown in Fig. 2. The circuit represents the typical elementary cell of a ladder step-up switched capacitor and it consists of two capacitors and four switches that are controlled in a complementary

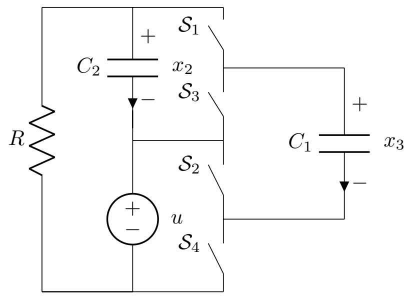

Fig. 2. Elementary cell of a ladder step-up switched capacitor converter.

way. Then the modes of the system are two, $i = 1$ in (1) corresponding to the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off and $i = 2$ in (1) for the reverse conduction of the switches pairs. By considering as input a constant voltage source $u = x_1$, the circuit can be modeled with x_2 and x_3 being the state variables corresponding to the voltages on the capacitors C_1 and C_2, respectively. Then the matrices pairs of (1) are:

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix}, \quad F_1 = -\frac{\rho}{R}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \tag{11a}$$

$$\Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad F_2 = -\frac{1}{RC_2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{11b}$$

where $\rho = \frac{1}{C_1 + C_2}$.

It is easy to verify that Π_1 and Π_2 are not commutative and also (9) are not satisfied by (11).

In this paper we propose a continuous-time averaged model for the switched impulsive system (1) under milder assumptions with respect to (9). The averaging property was conjectured by the authors in [13] without providing any formal proof and by taking inspiration from the application of theoretical findings in [14] applied to discrete-time models. In this paper we provide a formal proof for the averaging result based on new conditions on the system matrices which can be easily checked.

3 Continuous-time averaged model

The proposed continuous-time averaged model has the following structure

$$\dot{\xi}(t) = A_p \xi(t), \quad t \in \mathbb{R}_0^+ \tag{12a}$$
$$\mu(t) = \Gamma \xi(t) \tag{12b}$$

with $\xi(0) = x_0 \in \mathbb{R}^n$ initial condition, the dynamic matrix function $A_p : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ is given by

$$A_p = \tfrac{1}{p}\left(\Phi_p - I\right) \tag{13}$$

with the matrix function $\Phi_p : \mathbb{R}^+ \to \mathbb{R}^{n \times n}$ and the matrix $\Gamma \in \mathbb{R}^{n \times n}$ given by

$$\Phi_p = \Pi + \Lambda p \tag{14a}$$
$$\Gamma = \sum_{j=1}^{\mathtt{q}} \left(\prod_{h=1}^{j} \Pi_h \right) d_j \tag{14b}$$

where $\Pi \in \mathbb{R}^{n \times n}$ is given by (10) and the matrix $\Lambda \in \mathbb{R}^{n \times n}$ given by

$$\Lambda = \sum_{j=1}^{\mathtt{q}} \left(\prod_{h=j+1}^{\mathtt{q}} \Pi_h F_j \prod_{h=1}^{j} \Pi_h \right) d_j. \tag{15}$$

where $\prod_{h=j+1}^{\mathtt{q}} \Pi_h$ for $j = \mathtt{q}$ is assumed to be the identity matrix.

The output $\mu \in \mathbb{R}^n$ of the model (12) is intended to be an approximation of the moving average of the solution of the impulsive systems (1). The dependence of (13) on the switching period is a crucial aspect in order to obtain a good approximation [14], which is an analogous dependence used in the well known result for the classical averaging technique applied to switched systems with modes represented by ordinary differential equations, i.e., by excluding jumps in the state [8].

It should be noticed that in the case of a switched ordinary differential equations, the matrices Π_i, $i \in \Sigma$, are equal to the identity matrix and the matrix Λ reduces to the dynamic matrices of the classical continuous-time averaged model of pulse width modulated systems with $\mathtt{q}$ modes, i.e. $\sum_{j=1}^{\mathtt{q}} F_j d_j$.

A further motivation for the choice of the matrices in (12) can be obtained by discretizing the model (12) with the forward Euler method and a step size p which leads to the following discrete-time model

$$z_{k+1} = \Phi_p z_k, \quad k \in \mathbb{N}_0 \tag{16a}$$
$$\mu_k = \Gamma z_k \tag{16b}$$

with $z_0 = x_0$. The solution of (16) can be written as

$$z_k = \Phi_p^k z_0 \tag{17}$$

for all $k \in \mathbb{N}_0$. In the sequel it will show that $x_k^- = z_k + \mathrm{O}(p)$ for any k, which motivates the choice (13) with (14a).

The choice of the matrix Γ in the output equation (12b) can be motivated by considering the continuous-time moving average of the solution of (1), which is defined as

$$m(t) = \frac{1}{p}\int_t^{t+p} x(\tau)\mathtt{d}\tau \tag{18}$$

for any $t \in (0, \bar{t}-p]$ with $\bar{t} > p$, where $x(t)$ is the solution of (1). We will show that the $m(t_k) = \mu_k + \mathrm{O}(p)$ which motivates the choice (12b) with (14b).

The main result is proved starting from two basic assumptions. The first one can be expressed as follows.

Assumption 1 *Given the matrix function Φ_p expressed by* (14a)*, there exists a constant $\alpha \in \mathbb{R}_0^+$ and a matrix norm such that*

$$\|\Phi_p\| \leq 1 + \alpha p. \tag{19}$$

Assumption 1 can be verified through the feasibility of a suitable set of linear matrix inequalities [13, Lemma 4].

Remark 2 *Under the situation that all Π_i, $i \in \Sigma$, are idempotent, condition* (9) *imply that Π is idempotent and then Assumption 1 holds. On the other hand, Assumption 1 is also verified if all powers of the matrix Π given by* (10) *are bounded, without requiring that Π is idempotent. This fact can be easily proved by using the Barabanov norm [22]. Therefore, the results presented in next section which are based only on Assumption 1 are proved under milder conditions with respect to the former averaging results which start from* (9)*.*

An important result related to Assumption 1 is the following lemma which has been proved in [13].

Lemma 3 *Consider a Lipschitz continuous matrix function $\Phi_p : \mathbb{R}^+ \to \mathbb{R}^{n\times n}$. Assume there exists a constant $\alpha \in \mathbb{R}_0^+$ such that* (19) *holds. Then, for any Lipschitz continuous matrix function $M_p : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ such that $M_p = \mathrm{O}(p^2)$, it is*

$$\Phi_p^k = \mathrm{O}(1) \tag{20a}$$

$$(\Phi_p + M_p)^k = \Phi_p^k + \mathrm{O}(p). \tag{20b}$$

for all $k \in \{0, \ldots, \ell_p\}$ with $\ell_p = \lfloor \bar{t}/p \rfloor$ and any finite $\bar{t} \in \mathbb{R}^+$.

A further technical assumption required in order to obtain our averaging results is the following.

Assumption 2 *Given the matrices Π and Λ expressed by* (10) *and* (15)*, respectively, there exists a coordinate transformation $T \in \mathbb{R}^{n\times n}$ such that*

$$T\Pi T^{-1} = \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \tag{21a}$$

$$T\Lambda T^{-1} = \begin{bmatrix} \Lambda_1 & 0 \\ \Lambda_3 & \Lambda_2 \end{bmatrix} \tag{21b}$$

where V is Schur, with V and Λ_2 square matrices of the same dimension.

Consider the case that Π_i, $i \in \Sigma$, are idempotent. It can be easily shown that (9) implies (21a), with $V = 0$ but the opposite is not true in general. Indeed, (11) do not satisfy (9) but Assumption 2 holds for these matrices as it can be verified by considering, for instance, the following coordinate transformation

$$T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{C_1+C_2}{C_2} & 1 & \frac{C_1}{C_2} \\ -1 & 0 & 1 \end{bmatrix}. \tag{22}$$

Note that (21a) together with the Schur condition of matrix V, also if Π_i, $i \in \Sigma$, are not idempotent, implies that $\lim_{k\to\infty}[(T\Pi T^{-1})^{k+1} - (T\Pi T^{-1})^k] = 0$ which means that the transformed matrix $T\Pi T^{-1}$ converges to an idempotent matrix when k goes to infinity.

Remark 4 *Assumption 2 allows one to obtain a useful transformation for the matrix function Φ_p. Indeed, by using Assumption 2 one can write*

$$T\Phi_p T^{-1} = \begin{bmatrix} I + \Lambda_1 p & 0 \\ \Lambda_3 p & V + \Lambda_2 p \end{bmatrix}. \tag{23}$$

For sufficiently small p the matrices $I + \Lambda_1 p$ and $V + \Lambda_2 p$ have no common eigenvalues, hence there is a unique solution R_p of the Sylvester equation

$$R_p(I + \Lambda_1 p) - (V + \Lambda_2 p)R_p = -\Lambda_3 p \tag{24}$$

such that

$$T_p\Phi_p T_p^{-1} = \begin{bmatrix} I + \Lambda_1 p & 0 \\ 0 & V + \Lambda_2 p \end{bmatrix} \tag{25}$$

with

$$T_p := \begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} T. \tag{26}$$

Note that $R_p = 0$ *is the solution of* (24) *for* $p = 0$ *and* (24) *can be written as*

$$(M + p\,\Delta M)\mathrm{vec}(R_p) = -p\,\mathrm{vec}(\Lambda_3)$$

where $\mathrm{vec}(\cdot) : \mathbb{R}^{r\times r} \to \mathbb{R}^{r^2}$ *is the standard vectorization operator,* $M := I\otimes V - I\otimes I$ *and* $\Delta M := I\otimes\Lambda_2 - \Lambda_1^\top \otimes I$. *Hence a standard perturbation analysis shows that*

$$\|R_p\| \le \frac{\|M^{-1}\|\|\Lambda_3\|}{1-\|M^{-1}\|\|\Delta M\|p}p = O(p).$$

Remark 4 will be used for obtaining the main result of the paper which shows that if Assumptions 1 and 2 hold then there exist constants $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}_0^+$, $\varepsilon \in (0,1)$, $\overline{p}_\varepsilon \in \mathbb{R}^+$ such that the following condition

$$\boxed{\|m(t) - \mu(t)\| \le \alpha p + \beta\varepsilon^k} \tag{27}$$

with $k = \lfloor t/p \rfloor$, holds for all $p \in (0, \overline{p}_\varepsilon]$ and $t \in (0, \bar{t} - p]$, for any $\bar{t} \in \mathbb{R}^+$. In (27) the moving average $m(t)$ is given by (18) with $x(t)$ being a solution of the SIS (1), and $\mu(t)$ is the output of the averaged model (12).

It is interesting to compare (8) and (27). First of all the approximation result (8) involves the solution $x(t)$ of the impulsive system while in (27) the corresponding moving average $m(t)$ is considered. The variables $\xi(t)$ and $\mu(t)$ do not present jumps, so as $m(t)$. The reason why $x(t)$ can be used in (8), is that the amplitudes of the state jumps converge to zero with p if (9) holds, which is not assumed in our main averaging result. Instead, if Assumptions 1 and 2 hold it is still possible to have nontrivial jumps when p decreases. On the other hand, the inequality (27) says that the error $m(t) - \mu(t)$ decreases with the multiple of the switching period and by decreasing the PWM period.

4 Averaging results

In order to prove (27) some preliminary steps are required. We first prove that the difference between the solution of the SIS (1) evaluated at the multiple of the switching period and the solution of the discrete-time system (16), is of order of the switching period.

Lemma 5 *Consider the continuous-time SIS* (1) *with initial condition* x_0*, over a time interval* $t \in [0, \bar{t}]$ *with some* $\bar{t} \in \mathbb{R}^+$ *and the discrete-time model* (16) *with* $k = \lfloor t/p \rfloor$ *and initial condition* $z_0 = x_0$*. If Assumption 1 is satisfied, then the following condition*

$$x_k^- = z_k + \mathrm{O}(p) \tag{28}$$

where x_k^- *is given by* (5) *and* z_k *is given by* (17)*, holds for all* $k \in \{0, \dots, \ell_p\}$*,* $\ell_p = \lfloor \bar{t}/p \rfloor$.

By using Lemma 5 one can prove that the difference between the moving average (18) evaluated at the multiples of the switching period and the output of the discrete-time model (16), is of order of the switching period. As for Lemma 5, also the following result requires Assumption 1 but not Assumption 2.

Lemma 6 *Consider the continuous-time SIS* (1) *with initial condition* x_0*, over a time interval* $t \in [0, \bar{t}]$ *with some* $\bar{t} \in \mathbb{R}^+$*, the moving average of its solution given by* (18) *evaluated at* t_k *for* $k \in \{0, \dots, \ell_p - 1\}$*,* $\ell_p = \lfloor \bar{t}/p \rfloor$*, and the discrete time model* (16) *with* $k = \lfloor t/p \rfloor$ *and initial condition* $z_0 = x_0$*. If Assumption 1 is satisfied then the following condition*

$$m(t_k) = \mu_k + \mathrm{O}(p) \tag{29}$$

holds for all $k \in \{0, \dots, \ell_p - 1\}$.

A further step towards the proof of our main result consists of considering the error between the moving average $m(t)$ expressed by (18) and the values obtained by sampling $m(t)$ at the multiple of p, i.e., $m(t_k)$ where $t_k = kp$ and $k = \lfloor t/p \rfloor$. In particular, by using Assumption 2 one can prove the following result.

Lemma 7 *Consider the continuous-time SIS* (1) *with initial condition* x_0*, over a time interval* $t \in [0, \bar{t}]$ *with some* $\bar{t} \in \mathbb{R}^+$*, the moving average* $m(t)$ *of its solution given by* (18)*. If Assumptions 1 and 2 are satisfied then there exist constants* $\alpha \in \mathbb{R}^+$*,* $\beta \in \mathbb{R}_0^+$*,* $\varepsilon \in (0,1)$ *and* $\bar{p}_\varepsilon \in \mathbb{R}^+$ *such that the following condition*

$$\|m(t) - m(t_k)\| \le \alpha p + \beta\varepsilon^k \tag{30}$$

with $t_k = kp$*,* $k = \lfloor t/p \rfloor$*, holds for any* $t \in (0, \bar{t} - p]$ *and any* $p \in (0, \bar{p}_\varepsilon]$.

Lemma 7 allows one to conclude that the approximation result is valid for any backward δ-shifted version of (18) defined as

$$m_\delta(t) = \frac{1}{p}\int_{t-\delta}^{t-\delta+p} x(\tau)\,\mathtt{d}\tau \tag{31}$$

with $\delta \in [0, p)$, where $x(t)$ is the solution of (1). Indeed, since (18) is defined as a p-forward moving average, it is easy to verify that a condition similar to (30) holds for m_δ, i.e., $\|m_\delta(t) - m(t_k)\| \le \alpha p + \beta\varepsilon^k$ for any $\delta \in [0, p)$, $t \in (\delta, \bar{t} + \delta - p]$ with $\bar{t} > p$. In the following for the sake of simplicity we consider the case $\delta = 0$.

By using the lemmas above, we can prove the following theorem which synthesizes our main result.

Theorem 8 *Consider the continuous-time SIS* (1) *with initial condition* x_0*, over a time interval* $t \in [0, \bar{t}]$ *with some* $\bar{t} \in \mathbb{R}^+$*, the corresponding moving average* $m(t)$ *given by* (18) *and the output* $\mu(t)$ *of the continuous-time* *model* (12) *with initial condition* $\xi(0) = x_0$*. If Assumptions 1 and 2 hold, then there exist constants* $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}_0^+$, $\varepsilon \in (0,1)$*, and* $\overline{p}_\varepsilon \in \mathbb{R}^+$ *such that* (27) *with* $k = \lfloor t/p \rfloor$ *holds for all* $p \in (0, \overline{p}_\varepsilon]$ *and* $t \in (0, \overline{t} - p]$.

The averaging approximation expressed by (27) and proved in Theorem 8 shows that the error between the moving average $m(t)$ of the SIS solution and the output of the averaged model depends on p and k too. In other words, it is not enough to let the switching period going to zero in order to reduce the error of the averaging process, but some periods must elapse too. This is due to the fact that the algebraic conditions on the modes matrices have been relaxed. The following theorem shows that under more restrictive conditions on the modes matrices, one can recover the classical $\mathrm{O}(p)$ averaging result.

Remark 9 *It is easy to show by checking the proof of Theorem 8 that, under Assumptions 1 and 2, if all* Π_i, $i \in \Sigma$*, are idempotent and* (9) *hold than there exist constants* $\alpha_m, \alpha_\mu \in \mathbb{R}^+$ *and* $\overline{p} \in \mathbb{R}^+$ *such that*

$$\|m(t) - m(t_k)\| \leq \alpha_m p \tag{32a}$$

$$\|m(t) - \mu(t)\| \leq \alpha_\mu p \tag{32b}$$

with $t_k = kp$, $k = \lfloor t/p \rfloor$ *holds for all* $p \in (0, \overline{p}]$ *and* $t \in (0, \overline{t} - p]$.

5 Simulation results

The electrical circuit in Fig. 2 and a numerical example with their respective simulations are analyzed to validate the effectiveness of the results presented in the previous section.

Example 10 *Let us go back to the motivating example of our analysis shown in Fig. 2.*

By considering (11) *it follows that*

$$\Pi = \Pi_2\Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 1 & 0 & 0 \end{bmatrix} \tag{33}$$

being $\rho = \frac{1}{C_1+C_2}$*. It can be easily verified that the matrix* Π *is bounded and then Assumption 1 holds independently of the circuit parameters. Moreover, Assumption 2 holds by considering the transformation matrix* (22)*. The matrix* (15) *can be written as*

$$\Lambda = \Pi_2 F_1 \Pi_1 d_1 + F_2 \Pi_2 \Pi_1 d_2 = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{RC_2} d_2 & -\frac{\rho}{R} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

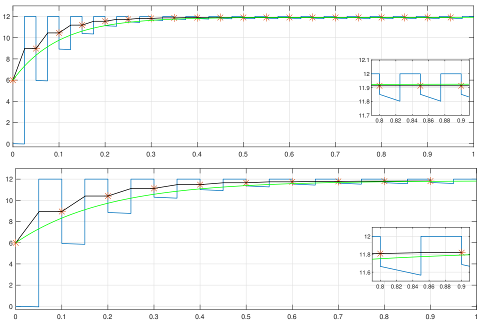

Fig. 3. Time evolution of the state variable x_2 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (12) (green lines), discrete-time model (16) (red stars), moving average (18) (dark lines).

and the matrix (14b) *is given by*

$$\Gamma = \Pi_1 d_1 + \Pi_2 \Pi_1 d_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ d_2 & C_2\rho d_1 & C_1\rho d_1 \end{bmatrix}$$

where we used the condition $d_1 + d_2 = 1$.

It is easy to verify that during the transient the solution of the "classical" averaged model (6) *has a completely different behavior with respect to the solution of SIS* (1).

We now compare the solutions of the SIS (1)*, the averaged model* (12) *proposed in this paper and the discrete-time model* (16) *together with the moving average* (18)*. Let us consider* $C_1 = C_2 = 120\,\mu\mathrm{F}$, $R = 10\,\mathrm{k}\Omega$, $u = 12\,\mathrm{V}$, $d_1 = d_2 = 0.5$ *and null initial conditions. Fig. 3 and Fig. 4 show the dynamics of the state variables* x_2 *and* x_3*, respectively, for different values of the switching period, over a time interval of* $1\,\mathrm{s}$*. Figure 5 shows the left hand side of* (27) *for different values of the switching period. For* $k \geq 12$ *the error between the moving average and the solution of the averaged model decreases by reducing the switching period. The fact that the matrices* Π_i, $i \in \Sigma$ *are not idempotent neither commutative implies that a certain number of switching periods is required in order to obtain an error below a desired finite bound, so as expressed by* (27).

Example 11 *Let us consider the following numerical example where the matrices* F_i *and* Π_i*, with* $i \in \{1, 2\}$

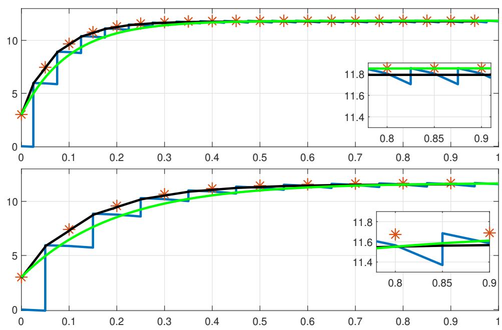

Fig. 4. Time evolution of the state variable x_3 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (12) (green lines), discrete-time model (16) (red stars), moving average (18) (dark lines).

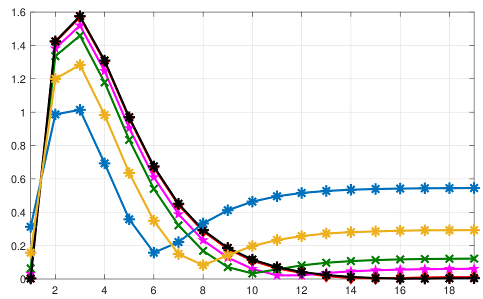

Fig. 5. Error $\|m(t_k) - \mu(t_k)\|$ versus the multiples of p for different values of the switching period: $p = 0.5\,\mathrm{s}$ (blue line), $p = 0.25\,\mathrm{s}$ (orange line), $p = 0.1\,\mathrm{s}$ (green line), $p = 0.05\,\mathrm{s}$ (purple line), $p = 0.01\,\mathrm{s}$ (red line) and $p = 0.005\,\mathrm{s}$ (black line).

are given by

$$\Pi_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_1 = \begin{bmatrix} -4 & -1 & -4 \\ -1 & 4 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad F_2 = \begin{bmatrix} -10 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to verify the matrices Π_1 *and* Π_2 *satisfy conditions* (9)*. Then according to Remark 9 the error between the moving average* $m(t)$ *of the solution of this system and its samples* $m(t_k)$ *is* $\mathrm{O}(p)$*. By considering* $d_1 = d_2 = 0.5$ *and the following matrices*

$$\Lambda = \begin{bmatrix} -7 & -1 & -7 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \Gamma = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

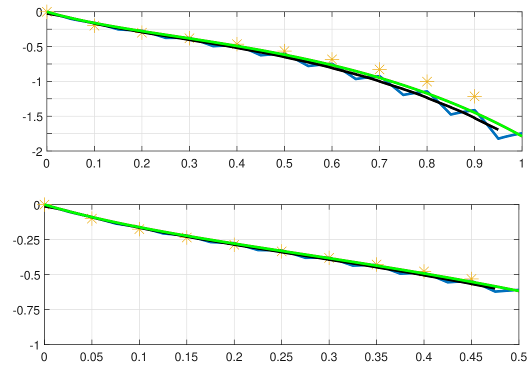

Fig. 6. Time evolution of the state variable x_1 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (12) (green lines), discrete-time model (16) (red stars), moving average (31) with $\delta = p/2$ (black lines).

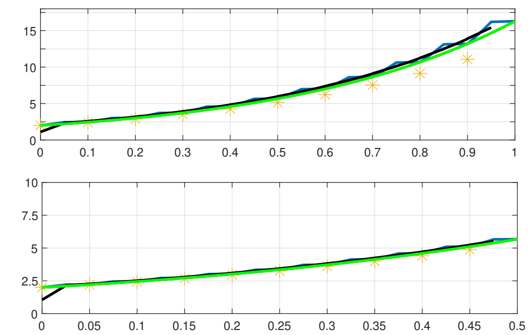

Fig. 7. Time evolution of the state variable x_2 with $p = 0.05\,\mathrm{s}$ (top) and $p = 0.1\,\mathrm{s}$ (bottom): SIS (1) (blue lines), averaged model (12) (green lines), discrete-time model (16) (red stars), moving average (31) with $\delta = p/2$ (black lines).

the dynamic matrix (13) *is given by*

$$A_p = \begin{bmatrix} -7 & -1 & -(7p-1)/p \\ -1 & 2 & -1 \\ 0 & 0 & -1/p \end{bmatrix}$$

where p *is the switching period. Let us compare the solutions of the SIS* (1)*, the averaged model* (12) *and the discrete-time model* (16) *together with the moving average* (18)*. Figure 6 and Figure 7 shows the dynamics of the state variables* x_1 *and* x_2*, respectively, for different values of the switching period, over a time interval of* $0.5\,\mathrm{s}$*. It is evident that the error between the output* $\mu(t)$ *and the moving average* $m(t)$ *is of order* $\mathrm{O}(p)$*, i. e. it is enough to let the switching period going to zero without needing some periods to elapse.*

It is remarkable to make a comparison between the averaged model (6) *presented in our previous studies and* *the proposed model* (12). *Let us consider the averaged dynamic matrix of the continuous averaged model* (6)

$$A_{\rm av} = \Pi(F_1 d_1 + F_2 d_2)\Pi = \begin{bmatrix} -7 & -1 & -7 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

It is easy to see that $\Gamma A_p = A_{\rm av}$. *Moreover the initial condition for* (6) *and* (13) *are the same, indeed* $\Gamma x_0 = \Pi x_0$. *Then the behaviour of the solutions of* (6) *and* (13) *are comparable. Future work will discuss deeper on this result.*

6 Conclusion

A new averaged model for SIS with PWM has been presented. The proposed model generalizes the classical averaged model widely adopted for the analysis of switched PWM systems which do not present state jumps at the switching time instants. The averaging result requires milder assumptions on the system matrices with respect to previous averaging analyses for SIS. A switched capacitor electrical circuit has been used to validate the results and to motivate their practical usefulness.

Future work will be dedicated to the study of scenarios with time-varying and state-dependent duty cycles.

Appendix

6.1 Proof of Lemma 5

Proof. Consider (3)–(4). By using the Taylor approximation one can write

$$G_{j,p} = e^{F_j d_j p} = I + F_j d_j p + \mathrm{O}(p^2) = I + \mathrm{O}(p) \quad (34)$$

for all $j \in \Sigma$, where I is the identity matrix. By using (34) in (4) one obtains

$$\Theta_p = \prod_{j=1}^{\mathtt{q}} G_{j,p}\Pi_j = \Pi + \Lambda p + \mathrm{O}(p^2) = \Phi_p + \mathrm{O}(p^2) \quad (35)$$

where Π is given by (10), Λ by (15) and Φ_p by (14a). By applying Lemma 3 with Assumption 1, from (20b) it follows

$$\Theta_p^k = \Phi_p^k + \mathrm{O}(p) \quad (36)$$

for all $k \in \{0, \ldots, \ell_p\}$. By subtracting (17) to (5) one obtains

$$\begin{aligned} x_k^- &= z_k + \Theta_p^k x_0^- - \Phi_p^k z_0 \\ &\overset{a}{=} z_k + \Phi_p^k (x_0^- - z_0) + \mathrm{O}(p) \\ &= z_k + \mathrm{O}(p) \end{aligned} \quad (37)$$

where in $\overset{a}{=}$ we used (36). ■

6.2 Proof of Lemma 6

Proof. Consider (18). By solving (1) and by using (2) one can write

$$\begin{aligned} pm(t_k) &= \int_{kp}^{(k+1)p} x(t)\mathtt{d}t \\ &= \sum_{i=1}^{\mathtt{q}} \int_0^{d_i p} G_i(\psi)\Pi_i x(s_{k,i}^-)\mathtt{d}\psi \\ &= \sum_{i=1}^{\mathtt{q}} \int_0^{d_i p} G_i(\psi)\Pi_i \prod_{h=1}^{i-1} G_{h,p}\Pi_h x_k^- \mathtt{d}\psi \end{aligned} \quad (38)$$

for all $k \in \{0, \ldots, \ell_p - 1\}$. Then, from (38) by using (34) and by noticing the presence of the integral one can write:

$$\begin{aligned} pm(t_k) &= \sum_{i=1}^{\mathtt{q}} \Pi_i \prod_{h=1}^{i-1} \Pi_h x_k^- d_i p + \mathrm{O}(p^2) \\ &= \sum_{i=1}^{\mathtt{q}} \prod_{h=1}^{i} \Pi_h x_k^- d_i p + \mathrm{O}(p^2) \\ &= \Gamma p x_k^- + \mathrm{O}(p^2) \overset{a}{=} \Gamma p z_k + \mathrm{O}(p^2) \\ &= \mu_k p + \mathrm{O}(p^2) \end{aligned} \quad (39)$$

where Γ is given by (14b), in $\overset{a}{=}$ we used Lemma 5 with Assumption 1 and μ_k is defined by (16b). By dividing both sides of (39) by p it follows that (29) holds. ■

6.3 Proof of Lemma 7

Proof. By definition it is $m(t) = m(t_k)$ for any $t = t_k = kp$, $k \in \{0, \ldots, \ell_p - 1\}$ and then in the time instants multiple of the switching period the condition (30) is trivially satisfied.

Let us consider the moving average over a time interval of length p which starts in i-th mode. For any $t \in [s_{k,i}, s_{k,i+1}]$, $k \in \{0, \ldots, \ell_p - 1\}$, $\tau_i = t - s_{k,i}$, i.e. $\tau_i \in [0, d_i p]$, by substituting the solution of SIS (1) in (18) and by reminding that the duty cycles are constant, one can write

$$
\begin{aligned}
pm(t) =& pm(s_{k,i}+\tau_i) = \int_{\tau_i}^{d_i p} G_i(\psi)\Pi_i x(s_{k,i}^-)\mathtt{d}\psi \\
&+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j x(s_{k,j}^-)\mathtt{d}\psi \\
&+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j x(s_{k+1,j}^-)\mathtt{d}\psi \\
&+ \int_0^{\tau_i} G_i(\psi)\Pi_i x(s_{k+1,i}^-)\mathtt{d}\psi.
\end{aligned}
$$

By using (2)–(4) it follows

$$
\begin{aligned}
pm(t) =& \int_{\tau_i}^{d_i p} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\
&+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\
&+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_{k+1}^- \mathtt{d}\psi \\
&+ \int_0^{\tau_i} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_{k+1}^- \mathtt{d}\psi. \qquad (40)
\end{aligned}
$$

Let us rewrite (38) as follows

$$
\begin{aligned}
pm(t_k) =& \sum_{j=1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\
=& \int_0^{d_i p} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\
&+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi \\
&+ \sum_{j=i+1}^{\mathtt{q}} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w x_k^- \mathtt{d}\psi. \quad (41)
\end{aligned}
$$

By taking the difference between (41) and (40) one obtains

$$
\begin{aligned}
p(m(t)-m(t_k)) =& \int_0^{\tau_i} G_i(\psi)\Pi_i \prod_{w=1}^{i-1} G_{w,p}\Pi_w (x_{k+1}^- - x_k^-)\mathtt{d}\psi \\
&+ \sum_{j=1}^{i-1} \int_0^{d_j p} G_j(\psi)\Pi_j \prod_{w=1}^{j-1} G_{w,p}\Pi_w (x_{k+1}^- - x_k^-)\mathtt{d}\psi. \qquad (42)
\end{aligned}
$$

By using (3)–(4)

$$
x_{k+1}^- = \Theta_p x_k^- = \prod_{i=1}^{\mathtt{q}} \Pi_i x_k^- + \mathrm{O}(p) = \Pi x_k^- + \mathrm{O}(p), \quad (43)
$$

together with $G_i(\psi) = I + \mathrm{O}(p)$ and $G_{w,p} = I + \mathrm{O}(p)$, from (42) one can write

$$
\begin{aligned}
p(m(t)-m(t_k)) =& \tau_i \prod_{w=1}^{i} \Pi_w (\Pi - I)x_k^- \\
&+ \sum_{j=1}^{i-1} d_j p \prod_{w=1}^{j} \Pi_w (\Pi - I) x_k^- + \mathrm{O}(p^2). \qquad (44)
\end{aligned}
$$

By using (5) and (36) in Lemma 5, the expression (44) can be rewritten as

$$
\begin{aligned}
&p(m(t)-m(t_k)) = \\
&\left(\tau_i \prod_{w=1}^{i} \Pi_w + \sum_{j=1}^{i-1} d_j p \prod_{w=1}^{j} \Pi_w\right)(\Pi - I)\Phi_p^k x_0^- + \mathrm{O}(p^2). \qquad (45)
\end{aligned}
$$

where $\tau_i = t - s_{k,i}$ and $k = \lfloor t/p \rfloor$.

From Assumption 2 and Remark 4 there exists a matrix T_p such that (25) holds and then one has

$$
\begin{aligned}
T_p(\Pi - I)\Phi_p^k T_p^{-1} &= T_p(\Pi - I)T_p^{-1} T_p \Phi_p^k T_p^{-1} \\
&= \left(\begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} T\Pi T^{-1} \begin{bmatrix} I & 0 \\ -R_p & I \end{bmatrix} - T_p T_p^{-1} \right) T_p \Phi_p^k T_p^{-1} \\
&= \left(\begin{bmatrix} I & 0 \\ R_p & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & V \end{bmatrix} \begin{bmatrix} I & 0 \\ -R_p & I \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) T_p \Phi_p^k T_p^{-1} \\
&= \begin{bmatrix} 0 & 0 \\ & V - I \end{bmatrix} \begin{bmatrix} (I + \Lambda_1 p)^k & 0 \\ 0 & (V + \Lambda_2 p)^k \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & (V - I)(V + \Lambda_2 p)^k \end{bmatrix}. \qquad (46)
\end{aligned}
$$

Since V is Schur it follows there exist constants $\beta_1 \in \mathbb{R}_0^+$, $\varepsilon \in (0,1)$ and $\bar{p}_\varepsilon \in \mathbb{R}^+$ such that, by taking the norms on both side of (46) it is

$$
\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\| \le \beta_1 \varepsilon^k \qquad (47)
$$

for all $p \in (0, \bar{p}_\varepsilon]$. Moreover one can write

$$\begin{aligned}\|(\Pi - I)\Phi_p^k\| &= \|T_p^{-1}T_p(\Pi - I)\Phi_p^k T_p^{-1}T_p\| \\ &\le \|T_p^{-1}\|\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\|\|T_p\| \\ &\le \beta_0\|T_p(\Pi - I)\Phi_p^k T_p^{-1}\| \le \beta_0\beta_1\varepsilon^k, \quad (48)\end{aligned}$$

where $\beta_0 \in \mathbb{R}^+$ is such that

$$\|T_p\|\|T_p^{-1}\| \le \beta_0 \qquad (49)$$

which exists for sufficiently small p because R_p in (26) is $O(p)$.

Then, by dividing both sides of (45) by p, by considering that $\tau_i = \mathrm{O}(p)$, by taking the norms on both sides, given the initial condition x_0^- and by using (48), it follows that there exists an $\alpha_i \in \mathbb{R}^+$ such that the following condition

$$\begin{aligned}\|m(t) - m(t_k)\| \le& \| \left(\frac{\tau_i}{p}\prod_{w=1}^{i}\Pi_w + \sum_{j=1}^{i-1} d_j \prod_{w=1}^{j}\Pi_w\right) \\ &(\Pi - I)\Phi_p^k x_0^-\| + \alpha_i p \\ \le& \left(\|\prod_{w=1}^{i}\Pi_w\| + \sum_{j=1}^{i-1}\|\prod_{w=1}^{j}\Pi_w\|\right) \\ &\|(\Pi - I)\Phi_p^k\|\|x_0^-\| + \alpha_i p \\ \le& \sum_{j=1}^{i}\|\prod_{w=1}^{j}\Pi_w\|\beta_0\beta_1\varepsilon^k\|x_0^-\| + \alpha_i p \\ \le& \beta\varepsilon^k + \alpha_i p \qquad (50)\end{aligned}$$

is satisfied for any $t \in [s_{k,i}, s_{k,i+1})$, $\tau_i = t - s_{k,i}$, for all $k \in \{0, \ldots, \ell_p - 1\}$ and $p \in (0, \bar{p}_\varepsilon]$, where

$$\beta = \beta_0\beta_1\|x_0^-\|\sum_{j=1}^{\mathtt{q}}\|\prod_{w=1}^{j}\Pi_w\|.$$

By considering (50) for all $i \in \Sigma$ it follows that (30) holds for all $t \in (0, \bar{t} - p]$ and any $p \in (0, \bar{p}_\varepsilon]$ with $\alpha = \max_{i\in\Sigma}\alpha_i$. ■

6.4 Proof of Theorem 8

Proof. Let us consider (12) and (18). By taking the norm of the difference one can write

$$\begin{aligned}\|m(t) - \mu(t)\| &= \|m(t) - m(t_k) + m(t_k) - \mu(t)\| \\ &\overset{(a)}{\le} \alpha_1 p + \beta_1\varepsilon_1^k + \|m(t_k) - \mu_k + \mu_k - \mu(t)\| \\ &\overset{(b)}{\le} \alpha_3 p + \beta_1\varepsilon_1^k + \|\mu_k - \mu(t)\| \\ &\overset{(c)}{\le} \alpha_3 p + \beta_1\varepsilon_1^k + \|\Gamma\|\,\|z_k - \xi(t)\| \\ &\le \alpha_3 p + \beta_1\varepsilon_1^k + \|\Gamma\|\,\|z_k - \xi(kp)\| \\ &+ \|\Gamma\|\,\|\xi(kp) - \xi(t)\| \qquad (51)\end{aligned}$$

holds for all $p \in (0, \bar{p}_{\varepsilon_1}]$, $t \in (0, \bar{t}]$, $k = \lfloor t/p \rfloor$, where in (a) we used Lemma 7 with α called α_1, β called β_1 and ε called ε_1, in (b) we used Lemma 6 which allows one to write (29) as $\|m(t_k) - \mu_k\| \le \alpha_2 p$ and we defined $\alpha_3 = 2\max\{\alpha_1, \alpha_2\}$, in (c) we used (16b) and (12b).

Let us consider the term $\|z_k - \xi(kp)\|$ in (51). By solving (12a) and by using (17) one can write

$$\xi(kp) - z_k = \left(e^{(\Phi_p - I)k} - \Phi_p^k\right)x_0. \qquad (52)$$

From (52) and by using arguments similar to (48) it follows

$$\begin{aligned}\|\xi(kp) - z_k\| &\le \|\left(e^{(\Phi_p - I)k} - \Phi_p^k\right)\|\|x_0\| \\ &\le \beta_0\|T_p\left(e^{(\Phi_p - I)k} - \Phi_p^k\right)T_p^{-1}\|\|x_0\| \qquad (53)\end{aligned}$$

where T_p is given by (26) and we used (49). From Remark 4 one can write

$$T_p(\Phi_p - I)T_p^{-1} = \begin{bmatrix}\Lambda_1 p & 0\\ 0 & V - I + \Lambda_2 p\end{bmatrix}, \qquad (54)$$

and then

$$\begin{aligned}&T_p\left(e^{(\Phi_p - I)k} - \Phi_p^k\right)T_p^{-1} \\ &= e^{T_p(\Phi_p - I)T_p^{-1}k} - (T_p\Phi_p T_p^{-1})^k \\ &= e^{\left(\begin{bmatrix}\Lambda_1 p & 0\\ 0 & V - I + \Lambda_2 p\end{bmatrix}\right)k} - \begin{bmatrix}I + \Lambda_1 p & 0\\ 0 & V + \Lambda_2 p\end{bmatrix}^k \\ &= \begin{bmatrix}(e^{\Lambda_1 p})^k & 0\\ 0 & (e^{V - I + \Lambda_2 p})^k\end{bmatrix} - \begin{bmatrix}(I + \Lambda_1 p)^k & 0\\ 0 & (V + \Lambda_2 p)^k\end{bmatrix}. \qquad (55)\end{aligned}$$

Considering the Taylor expansion of the exponential function, we have $e^{\Lambda_1 p} = I + \Lambda_1 p + O(p^2)$ and hence being $k = \lfloor t/p \rfloor$,

$$(e^{\Lambda_1 p})^k = (I + \Lambda_1 p)^k + O(p). \qquad (56)$$

By using (56) in (55) one has

$$T_p\left(e^{(\Phi_p-I)k}-\Phi_p^k\right)T_p^{-1}=\begin{bmatrix}\mathrm{O}(p) & 0\\ 0 & (e^{V-I+\Lambda_2 p})^k-(V+\Lambda_2 p)^k\end{bmatrix}. \tag{57}$$

Since the matrix V is Schur by hypothesis, for sufficiently small p the eigenvalues of $V+\Lambda_2 p$ have magnitude smaller than 1 and $V-I+\Lambda_2 p$ is Hurwitz (and hence the eigenvalues of $e^{V-I+\Lambda_2 p}$ also have magnitude smaller than 1). Consequently, there exist constants $\beta_2,\beta_3\in\mathbb{R}_0^+$, $\varepsilon_2\in(0,1)$ and $\bar{p}_{\varepsilon_2}\in\mathbb{R}^+$ such that

$$\|(e^{V-I+\Lambda_2 p})^k\|\le\beta_2\varepsilon_2^k \tag{58a}$$
$$\|(V+\Lambda_2 p)\|^k\le\beta_3\varepsilon_2^k \tag{58b}$$

for all $p\in(0,\bar{p}_{\varepsilon_2}]$. By taking the norms on both sides of (57) and by using (58) it follows that there exists a constant $\alpha_4\in\mathbb{R}^+$, such that

$$\|T_p\left(e^{(\Phi_p-I)k}-\Phi_p^k\right)T_p^{-1}\|\le\alpha_4 p+\beta_4\varepsilon_2^k. \tag{59}$$

where $\beta_4=2\max\{\beta_2,\beta_3\}$. Then from (53) with (59) the following inequality

$$\|\xi(kp)-z_k\|\le\alpha_5 p+\beta_5\varepsilon_2^k \tag{60}$$

with $\alpha_5=\alpha_4\beta_0\|x_0\|$, $\beta_5=\beta_0\beta_4\|x_0\|$, holds for all $p\in(0,\bar{p}_{\varepsilon_2}]$, $t\in(0,\bar{t}]$, $k=\lfloor t/p\rfloor$.

By substituting (60) in (51) it follows

$$\|m(t)-\mu(t)\|\le\alpha_6 p+\beta_6\varepsilon_3^k+\|\Gamma\|\,\|\xi(kp)-\xi(t)\| \tag{61}$$

with $\alpha_6=2\max\{\alpha_3,\alpha_5\}$, $\beta_6=2\max\{\beta_1,\beta_5\}$ and $\varepsilon_3=\max\{\varepsilon_1,\varepsilon_2\}$ and for all $p\in(0,\bar{p}_{\varepsilon_3}]$ with $\bar{p}_{\varepsilon_3}=\min\{\bar{p}_{\varepsilon_1},\bar{p}_{\varepsilon_2}\}$.

By considering the last term in (61) and the solution of (12a), for any $t\in[kp,kp+p)$ one can write

$$\begin{aligned}\xi(t)-\xi(kp)&=\left(e^{\frac{1}{p}(\Phi_p-I)(t-kp)}-I\right)\xi(kp)\\&=\left(e^{(\Phi_p-I)(\frac{t}{p}-k)}-I\right)\xi(kp).\end{aligned} \tag{62}$$

By using (49) in (62) it follows

$$\begin{aligned}&\|\xi(t)-\xi(kp)\|=\|\left(e^{(\Phi_p-I)(\frac{t}{p}-k)}-I\right)e^{(\Phi_p-I)k}x_0\|\\&=\|\left(e^{(\Phi_p-I)\frac{t}{p}}-e^{(\Phi_p-I)k}\right)x_0\|\\&\le\beta_0\|T_p\left(e^{(\Phi_p-I)\frac{t}{p}}-e^{(\Phi_p-I)k}\right)T_p^{-1}\|\|x_0\|\\&\overset{(a)}{=}\beta_0\|\begin{bmatrix}(e^{\Lambda_1 p})^{\frac{t}{p}}-(e^{\Lambda_1 p})^k & 0\\ 0 & (e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\end{bmatrix}\|\,\|x_0\|\\&=\beta_0\|\begin{bmatrix}e^{\Lambda_1 t}-e^{\Lambda_1 kp} & 0\\ 0 & (e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\end{bmatrix}\|\,\|x_0\|\end{aligned} \tag{63}$$

where in (a) we used arguments similar to those used for (55). By taking the Taylor series one can write

$$e^{\Lambda_1 t}-e^{\Lambda_1 kp}=\Lambda_1(t-kp)+\mathrm{O}(p^2)=\mathrm{O}(p). \tag{64}$$

Since V is Schur then $V-I$ is Hurwitz and there exists a sufficiently small p such that $V-I+\Lambda_2 p$ is Hurwitz and $e^{V-I+\Lambda_2 p}$ is Schur. Then there exists a constant $\beta_7\in\mathbb{R}_0^+$ such that

$$\begin{aligned}&\|(e^{V-I+\Lambda_2 p})^{\frac{t}{p}}-(e^{V-I+\Lambda_2 p})^k\|\\&\qquad\le\|(e^{V-I+\Lambda_2 p})\|^{\frac{t}{p}}+\|(e^{V-I+\Lambda_2 p})\|^k\\&\qquad\le\beta_7\varepsilon_2^k+\beta_2\varepsilon_2^k.\end{aligned} \tag{65}$$

By using (64) and (65) in (63) it follows that there exists a constant $\alpha_7\in\mathbb{R}^+$ such that

$$\|\xi(t)-\xi(kp)\|\le\alpha_7 p+\beta_8\varepsilon_2^k \tag{66}$$

with $\beta_8=2\beta_0\|x_0\|\max\{\beta_2,\beta_7\}$. By substituting (66) in (61), it follows that (27) holds with $\alpha=2\max\{\alpha_6,\alpha_7\}$, $\beta=2\max\{\beta_6,\beta_8\}$ and $\varepsilon=\max\{\varepsilon_2,\varepsilon_3\}$ and for all $p\in(0,\bar{p}_\varepsilon]$ with $\bar{p}_\varepsilon=\min\{\bar{p}_{\varepsilon_2},\bar{p}_{\varepsilon_3}\}$.

■

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