# On contraction analysis of switched systems with mixed contracting-noncontracting modes via mode-dependent average dwell time

Hao Yin, Bayu Jayawardhana, and Stephan Trenn
Jan C. Willems Center for Systems and Control, University of Groningen, 9747 AG Groningen, The Netherlands.
hao.yin@rug.nl, b.jayawardhana@rug.nl and s.trenn@rug.nl

## 1 Introduction

This paper studies contraction analysis of switched systems that are composed of a mixture of contracting and noncontracting modes. The first result pertains to the equivalence of the contraction of a switched system and the uniform global exponential stability of its variational system. Based on this equivalence property, sufficient conditions for a mode-dependent average dwell-time based switching law to be contractive are established. Correspondingly, LMI conditions are derived that allow for numerical validation of contraction property of switched linear systems, which include those with all unstable modes.

## 2 Problem formulation

Consider switched systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t),t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathscr{X} \subseteq \mathbb{R}^n$ is the state vector, $t_0 \in \mathbb{R}$ is the initial time and $x_0 \in \mathscr{X}$ is the initial value. For a switched system given by (1), it is called contracting if there exists positive numbers $c$ and $\alpha$ such that for all solutions $x_1(t)$, $x_2(t)$ of (1) we have $\|x_1(t) - x_2(t)\| \leq ce^{-\alpha t}\|x_1(t_0) - x_2(t_0)\|$. The objective of this paper is to propose a sufficient condition that guarantees the switched system (1) is contracting with respect to switching law $\sigma(t)$ when not all modes of (1) are contracting, including the case where none of the modes is contracting.

## 3 Main Result

The family of (time-varying) linear switched system

$$\dot{\xi}(t) = F_{\sigma(t)}(x(t),t)\xi(t), \quad \xi(t_0) = \xi_0 \in \mathbb{R}^n \tag{2}$$

with $F_p(x(t),t) = \nabla_x f_p(x(t),t)$ and $x(\cdot)$ be any given solution trajectory of (1) is called uniformly globally exponentially stable (UGES), if there exist positive numbers $c$, $\alpha$ (independently of the chosen solution $x(\cdot)$) such that for every solution $\xi(t)$ of (2) the following inequality holds $\|\xi(t)\| \leq ce^{-\alpha t}\|\xi(t_0)\|$.

**Proposition 1** *For a given switching signal $\sigma(t)$, the system* (1) *is contracting if, and only if, the family of systems* (2) *is UGES.*

**Theorem 1** *Consider switched nonlinear system* (1) *with switching signal $\sigma : [0,\infty) \to \mathscr{M}$ and corresponding switching times $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots\}$. Assume that we can classify each mode $p$ as being either stable or unstable, i.e. assume $\mathscr{M} = \mathscr{S} \dot{\cup} \mathscr{U}$ and, correspondingly, assume the switching signal $\sigma$ has a MDADT $\tau_{ap} > 0$ for each stable mode $p \in \mathscr{S}$ and a MDALT $\tau_{ap} > 0$ for each unstable mode $p \in \mathscr{U}$. Furthermore, assume that for each mode $p \in \mathscr{M}$ there exist $\overline{\nu}_p \geq \underline{\nu}_p \geq 0$ and a continuously differentiable function $V_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that for all $(x,\xi,t)$, $\underline{\nu}_p\|\xi\|_2^2 \leq V_p(x,\xi,t) \leq \overline{\nu}_p\|\xi\|_2^2, \forall p \in \mathscr{M}$, and $\dot{V}_p(x,\xi,t) \leq \eta_p V_p(x,\xi,t), \forall p \in \mathscr{M}$, with $\eta_p \geq 0$ if $p \in \mathscr{U}$ or $\eta_p < 0$ otherwise. Finally, assume that for every $p \in \mathscr{M}$, there exists $\mu_p > 0$ such that $V_{\sigma(t_i)}(x,\xi,t_i) \leq \mu_{\sigma(t_i^-)} V_{\sigma(t_i^-)}(x,\xi,t_i), \forall t_i \in \mathscr{S}$. Then the switched nonlinear system* (1) *is contracting if*

$$\left.\begin{aligned} \tau_{ap} &> \underline{\tau}_{ap} := -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathscr{S}, \\ \tau_{ap} &< \overline{\tau}_{ap} := -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathscr{U}. \end{aligned}\right\} \tag{3}$$

## 4 Simulation Results

In this numerical example, we apply our main results to the synchronization problem of one-way coupled identical oscillators, whose dynamics take the form

$$\dot{w} = f(w(t),t), \tag{4}$$

$$\dot{x} = f(x(t),t) + u_{\sigma(t)}(w(t)) - u_{\sigma(t)}(x(t)), \tag{5}$$

where $w(t), x(t) \in \mathbb{R}^n$ is the state vector, $f(w(t),t)$ is the dynamics of the uncoupled oscillators, and $u_{\sigma(t)}(w(t)) - u_{\sigma(t)}(x(t))$ is the switched coupling force.