Stability of switched systems with multiple equilibria: a mixed stable-unstable subsystem case

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Abstract

This paper studies stability of switched systems that are composed of a mixture of stable and unstable modes with multiple equilibria. The main results of this paper include some sufficient conditions concerning set convergence of switched nonlinear systems. We show that under suitable dwell-time and leave-time switching laws, trajectories converge to an initial set and then stay in a convergent set. Based on these conditions, LMI conditions are derived that allow for numerical validation of practical stability of switched affine systems, which include those with all unstable modes. Two examples are provided to verify the theoretical results.

1. Introduction

Many complex engineering systems operate as finite-state machines with different mode of operations and functions. The modes can correspond to the multitude of tasks designed for these systems and to the adaptability of these systems in dealing with dynamic environment. In this regards, these systems can be modeled as switched systems, which have received attention in the past decades. Some well-known examples of such engineering systems described by switched systems are aircraft systems in [1], power electronics in [2], and electrical circuits in [3].

Typically, a switched system is described by a finite set of continuous-time or discrete-time dynamic subsystems/modes and a switching law/signal that determines which subsystem/mode is active at any given moment or time. Such switching law can depend on particular state values, time events, or an external state as a memory.

In the time-dependent switching signal, the dwell-time (DT) as studied in [4] provides an important notion that gives us the minimal time where the switched systems must remain in a subsystem before switching to another one. Correspondingly, a significant amount of literature has been directed towards the stability of switched systems [4; 5; 6; 7; 8; 9; 10; 11]. In [5; 6] the common Lyapunov function and multiple Lyapunov function techniques are used to analyze the stability of switched systems with all stable subsystems. In recent years, some results have also been reported on switched systems with both stable and unstable subsystems [7; 8]. The main idea of these studies is to make the dwell-time of the stable subsystems large enough, while shortening the dwell time of the unstable subsystems to offset the divergent trajectory of the unstable subsystem. This approach of having a trade-off between stable and unstable subsystems is no longer applicable when all subsystems are unstable. In [9; 10; 11], a discretized Lyapunov function technique is presented that can be used to analyze the stability of switched systems with all unstable subsystems. The switched systems considered in these papers all share a common equilibrium point and they provide analysis on the convergence of the trajectories to the common equilibrium.

On the contrary, in some engineering applications, there may not be a common equilibrium between subsystems. Some well-known examples are neural networks [12] and bipedal walking robots [13]. In these systems, it has been shown that the trajectories converge to a set rather than to a specific equilibrium point. The property of convergent sets has been studied and estimated in [14; 15; 16; 17]. When all subsystems are stable, dwell-time criteria was investigated in [14] to guarantee that the trajectories converge globally to a superset and remain in such superset. This work was extended to switched systems satisfying input-to-state stability property with bounded disturbance in [15], and to switched discrete systems in [15; 16]. Another extension of [14] was presented in [17], which allows each subsystem to have multiple stable equilibria. However, the studies in [14; 15; 16; 17] have not yet considered the case where the switched systems can contain unstable subsystems.

Inspired by [14], we study in this paper the set convergence property of switched systems with distinct equilibria in a more general case. The switched systems can contain both stable and unstable subsystems. Such situation can be found, for instance, in game theoretic setting [18; 19; 20], where each game's Nash equilibria may be different and unstable.

This paper provides theoretical tools relevant to ensuring set stability for such systems. Firstly, we define compact sets $N_q(k)$, $N^\alpha(k)$ and $L(k)$ parametrized by the level set constant k, all of which are derived from the level set of multiple time-dependent Lyapunov functions of the switched systems. Based on the fulfillment of multiple Lyapunov conditions on the state-space outside the compact sets $N_q(k)$ around the equilibrium point of all modes q, we obtain conditions for the switching signals such that the set $L(k)$ is attractive for all initial states in $N^\alpha(k)$. In this regards,

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instead of finding multiple Lyapunov functions for the entire state space as studied recently in [14], we only need to find multiple Lyapunov-like functions in the state space outside some compact set $N(k)$, which is less conservative than the former. Consequently we have enlarged the region-of-attraction set to that reported in [14], e.g. from $N(k)$ in [14] to $N^{\alpha}(k)$ as defined later in Section 3. The generalization allows us as well to consider switched systems with all unstable subsystems.

Related works on the study of switched systems with multiple equilibria is the practical stability analysis of switched systems in [21; 22; 23]. In these studies, they analyze switched affine systems with all stable subsystems [21], or with stable switching condition among subsystems [22; 23]. Related to this, we present the practical stability analysis for mixed stable-unstable switched affine systems that relaxes these restrictions. Based on the obtained sufficient conditions for set convergence, we present a numerical construction of such multiple Lyapunov function using time-dependent multiple quadratic Lyapunov functions. It leads to Linear Matrix Inequality (LMI) conditions that can be numerically implemented.

The paper is organized as follows. In Section 2, we present preliminaries and problem formulation. The construction of convergent set and some sufficient conditions for the set convergence property of switched systems are presented in Sections 3 along with an example. Application of such sufficient conditions to the practical stability analysis of switched affine systems that include examples with all unstable subsystems are provided in Section 4. Finally, we present the conclusion in Section 5.

Notation. The symbols $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, and $\mathbb{N}$ denote the set of real, non-negative real, and natural numbers, respectively. Correspondingly, the n-dimensional Euclidean space is denoted by $\mathbb{R}^n$. Given any matrix A, A^T refers to the transpose of A, and $\lambda(A)$ refers the eigenvalue of A. For symmetric matrices B and C, the inequality $B > 0$ (or $B \geq 0$) means that B is positive definite (or positive semidefinite) and $B < 0$ (or $B \leq 0$) refers to B being negative definite (or negative semidefinite). The inequality $B < C$ ($B \leq C$) means that $B - C < 0$ ($B - C \leq 0$). The symbol $\|\cdot\|$ is the Euclidean vector norm on $\mathbb{R}^n$. For a given set N, the sets ∂N and $\overline{N}$ denote the boundary of N and the complement of N, respectively. Finally, whenever it is clear from the context, the symbol "$*$" inside a matrix stands for the symmetric elements in a symmetric matrix.

2. Preliminaries and problem formulation

Consider switched systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t), t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathcal{X} \subseteq \mathbb{R}^n$ is the state vector, $t_0 \in \mathbb{R}$ is the initial time and $x_0 \in \mathcal{X}$ is the initial value. Define an index set $\mathcal{Q} := \{1, 2, \cdots, M\}$, where M is the number of modes. The signal $\sigma : [t_0, \infty) \to \mathcal{Q}$ denotes the switching signal, which is assumed to be a piecewise constant function and continuous from the right. The vector field $f_i : \mathcal{X} \times [t_0, \infty) \to \mathbb{R}^n$, $i \in \mathcal{Q}$, is continuous in t and continuously differentiable in x. The switching instants are expressed by a monotonically increasing sequence $\mathscr{S} := \{t_1, t_2, \cdots, t_k, \cdots\}$, where t_k denotes the k-th switching instance. We assume that (1) is forward complete, which means for each $x_0 \in \mathcal{X}$ there exists a unique solution of (1) on $[t_0, \infty)$ and no jump occurs in the state at a switching time.

In this paper, we assume that there is no common equilibria for the switched systems (1). In addition, we allow each subsystem has multiple equilibria. Since the equilibria are different, trajectories will converge to a set rather than a specific point.

The set convergence problem for switched systems with all stable modes has attracted considerable attentions. For example, in [14; 15; 16; 17], a convergent set is constructed by the level sets of multiple Lyapunov functions. Then, convergence can be achieved by activating the stable subsystems for a sufficient long time. However, for unstable subsystems, we can not find such multiple Lyapunov functions that limit the application of results in [14; 15; 16; 17]. Correspondingly, the main objective of this paper is to propose a sufficient condition that guarantees the switched system (1) is set convergent with respect to switching law $\sigma(t)$, which include the case when not all modes of (1) are stable and when none of the modes is stable.

3. Main result

In this section, the sets construction is introduced and some sufficient conditions are given to guarantee the set convergence of the switched system (1).

We denote the subset of modes in $\mathcal{Q}$ that compose of unstable sub-systems by $\mathcal{U}$ and its complement (e.g., the stable ones) by $\mathcal{S}$. Hence $\mathcal{Q} = \mathcal{U} \dot{\cup} \mathcal{S}$. Consider the switched system (1) under a certain switching signal $\sigma(t)$. Suppose that there exists a compact set K such that for each mode $q \in \mathcal{Q}$ there exists a continuously differentiable function $V_q : \mathcal{X} \setminus K \times [0, \tau_{q,\max}) \to \mathbb{R}_{\geq 0}$, where $\tau_{q,\max} \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ represents the maximal local time, such that the following inequality holds for all $\xi \in \mathcal{X} \setminus K$ and $\tau \in [0, \tau_{q,\max})$

$$\dot{V}_q(\xi, \tau) := \frac{\partial V_q(\xi, \tau)}{\partial \xi} f_q(\xi, \tau) + \frac{\partial V_q(\xi, \tau)}{\partial \tau} \leq \eta_q V_q(\xi, \tau), \tag{2}$$

with $\eta_q \geq 0$ if $q \in \mathcal{U}$ or $\eta_q < 0$ otherwise. This mode-dependent locally time-varying Lyapunov function provides us with a mean to describe the stability of the compact set K in a local time-interval whenever the mode q is activated. For facilitating the numerical computation later via LMI conditions, we will use an explicit relation of the compact set K with the Lyapunov function V_q through a parametrized compact set $N(k)$, where the parameter $k > 0$ gives us a degree-of-freedom to check the Lyapunov condition.

We introduce this locally time-varying Lyapunov function in order to relax the requirement of finding a common time-invariant Lyapunov function for switched systems, which may be hard to find. The maximal time of definition

$\tau_{q,\max}$ can be ∞ and we do not exclude the usual time-invariant V_q in this definition by taking $V_q(\xi, \tau)$ to be constant for all $\tau \in [0, \tau_{q,\max})$ with arbitrary $\tau_{q,\max} > 0$. As will be clear later, such maximal time $\tau_{q,\max}$ must necessarily be greater than the usual required dwell-time condition. In our previous work [24], we have shown the applicability of such locally time-varying Lyapunov function in order to set up verifiable LMI conditions for establishing stability of switched systems comprising of (un)stable modes. The function constructed in [24] is based on time interpolation of two constant quadratic Lyapunov function.

In the following, we will define $N(k)$, $N^\alpha(k)$, $L(k)$, which are a subset of $\mathcal{X}$ and parametrized by positive constant $k > 0$. These sets will be used in our main result to define the attractive invariant set of the switched systems. For a given positive constant $k > 0$ and for any given mode $q \in \mathcal{Q}$, we define $N_q(k)$ as a level set of $V_q(\xi, \tau)$ given by

$$N_q(k) := \{\xi \in \mathcal{X} : V_q(\xi, \tau) \leq k, \forall \tau \in [0, \tau_{q,\max})\}. \quad (3)$$

The superset $N(k)$ is defined by the union of $N_q(k)$ over all modes $q \in \mathcal{Q}$ as follows

$$N(k) := \bigcup_{q \in \mathcal{Q}} N_q(k). \quad (4)$$

Since $N(k)$ is generally larger than any of the individual $N_q(k)$, let us define the maximum range of V_q in $N(k)$ by

$$\alpha_q(k) := \max_{\substack{\xi \in N(k) \\ \tau \in [0, \tau_{\max})}} V_q(\xi, \tau), \quad (5)$$

and

$$\alpha(k) := \max_{q \in \mathcal{Q}} \alpha_q(k). \quad (6)$$

For every $q \in \mathcal{Q}$, we define a level set $N_q^\alpha(k)$ by

$$N_q^\alpha(k) := N_q(\alpha(k)), \quad (7)$$

and $N^\alpha(k)$ by

$$N^\alpha(k) := \bigcap_{q \in \mathcal{Q}} N_q^\alpha(k). \quad (8)$$

Note that $N^\alpha(k) \neq N(\alpha(k))$, because the former is the intersection, while the latter is the union of all $N_q(\alpha(k))$.

Now, using the above notions of V_q and the sets $\mathcal{U}, \mathcal{S}$, and $N(k)$, we will consider the following locally time-varying Lyapunov characterisation for establishing the set stability of (1). For each mode $q \in \mathcal{Q}$, we assume that (2) holds for $K = N(k)$, i.e.

$$\dot{V}_q(\xi, \tau) \leq \eta_q V_q(\xi, \tau), \forall \xi \in \mathcal{X} \setminus N(k), \forall \tau \in [0, \tau_{q,\max}) \quad (9)$$

with $\eta_q > 0$ if $q \in \mathcal{U}$ or $\eta_q < 0$ otherwise, and with $\tau_{q,\max} > 0$. Additionally, we assume that the mode-dependent functions V_q are bounded by each other as follows: there exists $0 < \mu_q < 1$ if $q \in \mathcal{U}$ or $\mu_q > 1$ otherwise, such that

$$V_p(\xi, 0) \leq \mu_q V_q(\xi, \tau), \forall \xi \in \mathcal{X} \setminus N(k)$$
$$\forall p, q \in \mathcal{Q}, \forall \tau \in [\tau_{q,\min}, \tau_{q,\max}), \quad (10)$$

with $\tau_{q,\min} > 0$.

Finally, let us introduce the set $L(k)$, where the trajectories will eventually remain in. Firstly, we denote for all $q \in \mathcal{Q}$

$$\beta_q(k) := \alpha(k) \cdot \max\left\{\frac{1}{\mu_q}, 1\right\}, \quad (11)$$

$$M_q(k) := \{x \in \mathcal{X} : V_q(\xi, \tau) \leq \beta_q(k),$$
$$\forall \tau \in [0, \tau_{q,\max})\}, \quad (12)$$

Accordingly, we define $L(k)$ by

$$L(k) := \bigcup_{q \in \mathcal{Q}} M_q(k). \quad (13)$$

For an unstable sub-system, it follows from (11) that $\beta_q(k) = \frac{1}{\mu_q}\alpha(k) \geq \alpha(k)$ which implies that $N_q^\alpha(k) \subseteq M_q(k)$.

In the following the relations between the above defined sets are discussed and illustrated. Suppose $\xi \in N(k)$, according to (3)-(6), we have $V_q(\xi, \tau) \leq \alpha_q(k) \leq \alpha(k)$, then $x \in N(\alpha)$. In addition, according to (12), (13), we have $V_q(\xi, \tau) \leq \beta_q(k)$, then $\xi \in \bigcap_{q \in \mathcal{Q}} M_q(k) \subseteq L(k)$. It can be checked that $N(k) \subseteq N^\alpha(k) \subseteq \bigcap_{q \in \mathcal{Q}} M_q(k) \subseteq L(k)$. An illustration of this construction for two modes can be seen in Fig.1. In this illustration, $N_1(k)$ and $N_2(k)$ are disconnected, however, they can also be connected as shown later in Example 3.

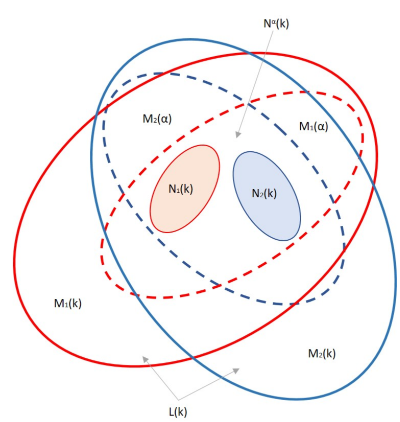

Figure 1: An illustration of the set constructions for two modes.

Remark 3.1. *There are two main differences between our results and those in [14; 15; 16]. Firstly, the results in this paper can include unstable subsystems; moreover, we do not exclude the case of all unstable subsystems. To cater for the presence of unstable subsystems, we use piecewise time-varying Lyapunov functions instead of time-invariant Lyapunov functions as used in [14; 15; 16] with the restriction of* (10)*. Secondly, the time-varying Lyapunov characterization of the sub-systems are applied outside a compact set $N(k)$ instead of the whole state space $\mathcal{X}$ as assumed in [14; 15;*

16]. It will be shown later in Example 1 that checking these Lyapunov conditions outside a compact set in (9) is easier that checking the counterparts in the whole state space $\mathcal{X}$.

Before we present our main result, we introduce the following definition of mode dependent dwell (leave) time.

Definition 3.2. *A constant $\tau_p > 0$ is called mode dependent dwell (leave) time for stable (unstable) mode $p \in \mathcal{Q}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{Q}$, if the time interval between two consecutive switches or jumps being no smaller (or larger) than τ_p.*

We present now the main result of this section for the set convergence of switched systems (1).

Theorem 3.3. *Suppose that for every $q \in \mathcal{Q}$ there exists $V_q : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_+$ satisfying (9) and (10) with a given η_q, μ_q and $k > 0$. Then for every switching signal $\sigma : \mathbb{R}_+ \to \mathcal{Q}$ satisfying the following dwell and leave time condition*

$$\left.\begin{array}{ll} \tau_q > \max\left\{-\frac{\ln \mu_q}{\eta_q}, \tau_{q,\min}\right\} & \forall q \in \mathcal{S}, \text{ and} \\ \tau_q < \min\left\{-\frac{\ln \mu_q}{\eta_q}, \tau_{q,\max}\right\}, & \forall q \in \mathcal{U}. \end{array}\right\} \quad (14)$$

the following properties hold for the state trajectory of the switched system (1):

(i) *there exists $T = T(x_0) > 0$ such that $x(T) \in N(k)$;*

(ii) *for any time $t \in [T, +\infty)$, the trajectory will stay in $L(k)$, i.e. $x(t) \in L(k)$;*

(iii) *for all starting points $x_0 \in N^\alpha(k)$, the trajectory of switched system (1) remains in the set $L(k)$, i.e. $x(t) \in L(k)$.*

Proof. Let us consider a given switching signal σ satisfying the hypotheses of the theorem with the switching time $\{t_0, t_1, t_2, \dots\}$. For such switching signal σ, we can construct a piecewise time-varying Lyapunov function, by piecing all V_q together, as follows

$$V(x(t), t) = V_q(x(t), t - t_i),$$

where t_i is the latest switching moment before time t and q is the current mode.

Proof of part (i): Trivially, if $x_0 \in N(k)$ then $T = 0$. Let us now consider $x_0 \in \mathcal{X} \setminus N(k)$. In the following, we will show that under the condition (14), the function $V(x(t), t)$ will converge to an arbitrarily small constant. It implies that there exists a time $T > 0$ s.t. the trajectory enters $N(k)$, i.e. $V(x(T), T) \le k$.

Firstly, for any $t \in [t_i, t_{i+1})$ and $q := \sigma(t_i)$, according to (2), we have

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} V(x(t), t) &= \frac{\mathrm{d}}{\mathrm{d}t} V_q(x(t), t - t_i) \qquad t \in [t_i, t_{i+1}) \\ &= \frac{\partial V_q(x(t), \tau)}{\partial x} f_q(x(t)) + \frac{\partial V_q(x(t), \tau)}{\partial \tau} \qquad \tau \in [0, t_{i+1} - t_i) \\ &\le \eta_q V_q(x(t), \tau) \end{aligned}$$

where we have introduced a time transformation of $\tau = t - t_i$ in the second equality. The comparison lemma implies

$$V(x(t), t) \le e^{\eta_q (t - t_i)} V(x(t_i), t_i) \quad (15)$$

for all $t \in [t_i, t_{i+1})$. Using (9) and (10), and by denoting now $p := \sigma(t_i)$ and $q := \sigma(t_{i-1})$,

$$\begin{aligned} V(x(t_i), t_i) &= V_p(x(t_i), 0) \\ &\le \mu_q V_q(x(t), t - t_{i-1}) \qquad t \in [t_{i-1} + \tau_{p,\min}, t_i) \\ &\le \mu_q e^{\eta_q (t_i - t_{i-1})} V_q(x(t_{i-1}), 0), \end{aligned}$$

where the last inequality follows a similar line as in (15). By recursively computing the inequality bound down to t_0, for $t = t_{i+1}$ we arrive at the following inequality

$$\begin{aligned} V(x(t_{i+1}), t_{i+1}) &= V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \\ &\le V_{\sigma(t_0)}(x(t_0), 0) \prod_{j=0}^{i} \mu_{\sigma(t_j)} \exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j)\right) \\ &= V_{\sigma(t_0)}(x(t_0), 0) \prod_{j=0}^{i} \exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j) + \ln \mu_{\sigma(t_j)}\right). \end{aligned} \quad (16)$$

It follows from the dwell and leave time condition (14) that for all $q \in \mathcal{S} \cup \mathcal{U}$, the inequality $\ln \mu_q + \eta_q \tau_q =: d_q < 0$ holds. This implies immediately for $d := \max_q d_q < 0$ that $\exp\left(\eta_{\sigma(t_j)}(t_{j+1} - t_j) + \ln \mu_{\sigma(t_j)}\right) \le e^d < 1$. If σ has infinitely many switches, it therefore follows from (16) that $V(x(t_i), t_i)$ converges to zero for $i \to \infty$. Together with (15), we conclude that for $t \in [t_i, t_{i+1})$ either $V(x(t), t) \le V(x(t_i), t_i))$ if $\sigma(t_i) \in \mathcal{S}$ or $V(x(t), t) \le e^{\eta_{\max} \tau_{\max}} V(x(t_i), t_i)$ if $\sigma(t_i) \in \mathcal{U}$ and where $\eta_{\max} := \max_{q \in \mathcal{U}} \eta_q$, $\tau_{\max} := \max_{q \in \mathcal{U}} \tau_{q,\max}$. Consequently, $t \mapsto V(x(t), t)$ converges also to zero as $t \to \infty$. In the case, that σ only has finitely many switches, the last mode most be a stable mode (because each unstable mode has a maximal leave time by assumption), hence (15) considered for the last (stable) mode also implies that $t \mapsto V(x(t), t)$ converges to zero.

Particularly, for any given $k > 0$, there exists $T > 0$ and q such that $V(x(T), T) = V_q(x(T), T) \le k$.

Proof of part (ii): The proof is decomposed in two steps. In the first step, we show that once the trajectory enters $N(k)$, i.e. $V(x(T), T) \le k$ with the switch time $t_i \le T$, it stays in $L(k)$ before the next switch at t_{i+1}, i.e. $V(x(t), t) \le \beta(k)$ for all $T \le t < t_{i+1}$. Thereafter it stays in $N^\alpha(k)$ after the next switch time t_{i+1}, i.e. $V(x(t_{i+1}), t_{i+1}) \le \alpha(k)$. In the second step, we show that when the trajectory starts in $N^\alpha(k)$, it stays in $L(k)$ for all forward time.

First step: Let us consider the time interval $[t_i, t_{i+1})$, and $T \in [t_i, t_{i+1})$, i.e. the trajectory enters $N(k)$ in $[t_i, t_{i+1})$. Let us first show that during the subsequent switch time t_{i+1}, we have $V(x(t_{i+1}), t_{i+1}) \leq \alpha(k)$. It follows from (5) and (6) that when the trajectory enters $N(k)$ at time T, we have $V(x(T), T) \leq \alpha_{\sigma(t_i)}(k) \leq \alpha(k)$.

We first show for a stable mode, by means of contradiction, that once the trajectory enters $N(k)$ at time T, it will stay in $N(k)$ in the time interval $[T, t_{i+1})$. Let us assume there exists $T'' > T$ such that $x(T'') \notin N(k)$. According to the continuity of the trajectory, there exists $T' > T$ such that $T < T' < T''$ and $x(T') \in \partial N(k)$. According to (9), outside $N(k)$, we have $V(x(T''), T'') = V_q(x(T''), T'' - t_i) \leq e^{\eta_q(T''-T')} V_q(x(T'), T' - t_i)$. Using (14) and $T'' - T' \leq \tau_q$, we have $e^{\eta_q(T''-T')} V_q(x(T'), T' - t_i) \leq e^{\eta_q \tau_q} V_q(x(T'), T' - t_i) \leq e^{\ln \frac{1}{\mu_q}} V_q(x(T'), T' - t_i) = \frac{1}{\mu_q} V_q(x(T'), T' - t_i)$. Since we are in a stable mode with $\mu_q > 1$, it follows that $V_q(x(T''), T'' - t_i) < V_q(x(T'), T' - t_i)$. In other words, $x(T'') \in N(k)$, which is a contradiction. Since $x(t) \in N(k)$, for all $t \in [T, t_{i+1})$, it follows from (5) and (6) that in the subsequent switch time t_{i+1}, we have $V(x(t_{i+1}), t_{i+1}) = V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \leq \alpha_{\sigma(t_{i+1})}(k) \leq \alpha(k)$. This means that the trajectory stays in $N^\alpha(k)$ at the subsequent switch time t_{i+1}.

Now let us consider the other case when an unstable mode is active during the time interval $[T, t_{i+1})$. For this situation, there are two further possible cases: $x(t_{i+1}) \in N(k)$ and $x(t_{i+1}) \notin N(k)$.

For the first case, with $x(t_{i+1}) \in N(k)$, we will show that $x(t) \in L(k)$ for all $t \in [T, t_{i+1})$ and $x(t_{i+1}) \in N^\alpha(k)$. Since it is an unstable mode, in the time interval $[T, t_{i+1})$, there can be a moment $T' > T$ such that $x(T') \notin N(k)$. It follows from (9), (14), and $T' - T < \tau_q$ that $V(x(T'), T') = V_q(x(T'), T' - t_i) \leq e^{\eta_q(T'-T)} V_q(x(T), T - t_i) < e^{\eta_q \tau_q} V_q(x(T), T - t_i) < \frac{1}{\mu_q} V_q(x(T), T - t_i) < \frac{1}{\mu_q} \alpha_q(k) \leq \beta_q(k)$. This inequality implies that $x(T') \in L(k), \forall T' \in [T, t_{i+1})$. Since $x(t_{i+1}) \in N(k)$, according to (5) and (6), after the switching at t_{i+1} we have $V(x(t_{i+1}), t_{i+1}) = V_{\sigma(t_{i+1})}(x(t_{i+1}), 0) \leq \alpha_q(k) \leq \alpha(k)$.

For the second case, with $x(t_{i+1}) \notin N(k)$, following the same arguments as in the first case, we have that $x(t) \in L(k)$, for all $t \in [T, t_{i+1})$. Accordingly, at t_{i+1}, with $x(t_{i+1}) \notin N(k)$, $p := \sigma(t_{i+1})$ and $q := \sigma(t_i)$, we can apply (9)-(10), which gives us

$$
\begin{aligned}
&V(x(t_{i+1}), t_{i+1}) = V_p(x(t_{i+1}), 0) \\
&\leq V_q(x(t_{i+1}), t_{i+1} - t_i) \\
&\leq e^{\ln \mu_q + \eta_q(t_{i+1} - T)} V_q(x(T), T - t_i).
\end{aligned}
$$

Using (14), it follows that $\ln \mu_q < -\eta_q(t_{i+1} - t_i)$. Hence the above inequality can be further upper-bounded by

$$
\begin{aligned}
&V(x(t_{i+1}), t_{i+1}) < e^{-\eta_q(t_{i+1}-t_i)+\eta_q(t_{i+1}-T)} V_q(x(T), T - t_i) \\
&= e^{\eta_q(t_i - T)} V_q(x(T), T - t_i).
\end{aligned}
$$

Since $t_i \leq T$ and $\eta_q > 0$, we have $V(x(t_{i+1}), t_{i+1}) < V_q(x(T), T - t_i) \leq \alpha_q(k) \leq \alpha(k)$. This implies that the trajectory remains in $N^\alpha(k)$.

In summary, for all $t \in [T, t_{i+1})$, the trajectory always stays in $L(k)$, and at t_{i+1}, the trajectory remains in $N^\alpha(k)$.

Second step: Let us now consider the subsequent time interval $[t_{i+1}, t_{i+2})$. Following the previous step, we have established that $V(x(t_{i+1}), t_{i+1}) \leq \alpha(k)$. We will now show that also $x(t) \in L(k)$, for all $t \in [t_{i+1}, t_{i+2})$. On the one hand, if the active mode in $[t_{i+1}, t_{i+2})$ is a stable one, the maximal value of $V(x(t), t)$ occurs at t_{i+1} since $V(x(t), t)$ is non-increasing in $[t_{i+1}, t_{i+2})$. In this case, we have $V(x(t), t) \leq V(x(t_{i+1}), t_{i+1}) \leq \alpha(k)$.

On the other hand, if the active mode in $[t_{i+1}, t_{i+2})$ is an unstable one then the upper bound of $V(x(t), t)$ occurs at t_{i+2}. By denoting $q := \sigma(t_{i+1})$ then for all $t \in [t_{i+1}, t_{i+2})$

$$
\begin{aligned}
V(x(t), t) &= V_q(x(t), t - t_{i+1}) \\
&\leq V_q(x(t_{i+2}), t_{i+2} - t_{i+1}) e^{\eta_q(t - t_{i+1})} \\
&\leq \frac{1}{\mu_q} V_q(x(t_{i+2}), t_{i+2} - t_{i+1}) \\
&\leq \frac{1}{\mu_q} \alpha(k) \leq \beta_q(k),
\end{aligned}
$$

where we have used (14) in the second inequality above to establish that $e^{\eta_q(t - t_{i+1})} < e^{\eta_q \frac{-\ln \mu_q}{\eta_q}} = \frac{1}{\mu_q}$ for all $t \leq t_{i+2} < t_{i+1} + \tau_q$.

It follows from (12) and (13) that $x(t) \in L(k), \forall t \in [t_{i+1}, t_{i+2})$.

Finally, let us consider the trajectory at the switch-time t_{i+2}. When $x(t_{i+2}) \in N(k)$, it immediately holds that $V(x(t_{i+2}), t_{i+2}) \leq \alpha(k)$. Otherwise, using (9), (10) and (14) and by denoting $p := \sigma(t_{i+2})$, we have

$$
\begin{aligned}
&V(x(t_{i+2}), t_{i+2}) = V_p(x(t_{i+2}), 0) \\
&\leq \frac{1}{\mu_q} V_q(x(t_{i+2}), t_{i+2} - t_{i+1}) \\
&\leq e^{-\ln \mu_q + \eta_q(t_{i+2} - t_{i+1})} V_q(x(t_{i+1}), 0) \\
&\leq e^{-\ln \mu_q + \eta_q \tau_q} V_q(x(t_{i+1}), 0) \\
&< V_q(x(t_{i+1}), 0) \leq \alpha(k).
\end{aligned}
$$

Thus the trajectory $x(t)$ remains in $N^\alpha(k)$ at t_{i+2}.

By computing recursively for the subsequent time intervals, we can conclude that the trajectory $x(t)$ remains in $L(k)$ for all $t \geq T$.

Proof of part (iii): The proof of part (iii) follows directly from the second step of the proof of part (ii). □

Remark 3.4. *The results presented in [14], which deals with all stable modes, can be considered as a particular case of our results. In particular, if we assume that the subsystems in Theorem 3.3 are all stable, i.e.* $\mathcal{Q} = \mathcal{S}$, *the trajectory of switched nonlinear system (1) is in* $L(k)$ *after time* T *for any switching signals satisfying* $\tau_q > -\frac{\ln \mu_q}{\eta_q}$. *In this regards, part (i) and (ii) of Theorem 3.3 coincide with [14, Theorem 1] with a common* $\mu = \max \mu_q$ *and a common* $\eta = \max \eta_q$. *For part (iii) of the theorem, we established that for all trajectories starting in* $N^\alpha(k)$, *which is larger than* $N(k)$ *used in [14, Corollary 2], will stay in the same level set* $L(k)$. *This shows that our result is less conservative.*

For switched system (1), if all subsystems are unstable, which represents the worst case scenario, the trajectories will not converge to any of the modes and the divergence can only be compensated by the switching events as shown in the following corollary.

Corollary 3.5. *Assume that $\mathcal{Q} = \mathcal{U}$ (i.e., all modes are unstable). Suppose that for every $q \in \mathcal{Q}$ there exists $V_q(\xi, \tau) : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_+$ satisfying (9) and (10) with a given η_q and μ_q. Then for any trajectory of switched nonlinear system (1) with switching signals σ satisfying*

$$\tau_q < \min\left\{-\frac{\ln \mu_q}{\eta_q}, \tau_{q,\max}\right\}, \forall q \in \mathcal{U}, \tag{17}$$

there exists $T > 0$ such that $x(t)$ remains in $L(k)$ for all $t \geq T$.

Example 1: Let us consider a switched system (1) composed of two scalar subsystems as follows

$$\begin{aligned} q = 1: &\quad \dot{x} = x + 4, \\ q = 2: &\quad \dot{x} = -x(1 + x)^2. \end{aligned} \tag{18}$$

The mode q=1 is an unstable system and the mode q=2 is a stable system with multiple equilibria. Both systems do not have common equilibria. For these sub-systems, we can define V_1 and V_2 that satisfy (9) and (10). Indeed, by using $V_1(x(t), t) = 2x^2$ and $V_2(x(t), t) = \frac{1}{2}x^2$, we have $\mu_1 = \frac{V_2(x(t),t)}{V_1(x(t),t)} = \frac{1}{4}$ and $\mu_2 = \frac{V_1(x(t),t)}{V_2(x(t),t)} = 4$; thus (10) is satisfied globally.

Let us fix $k = 2$ in (3) so that $N_1(2) = (-1, 1)$, $N_2(2) = (-2, 2)$, $N(2) = (-2, 2)$, and $\mathcal{X} \setminus N(2) = (-\infty, -2] \bigcup [2, +\infty)$. In $\mathcal{X} \setminus N(2)$, one can obtain that (9) holds with $\eta_1 = 6$ and $\eta_2 = -2$. Following the computation in (5), we have $\alpha(2) = 8$ in $N(2)$. Subsequently, using (11), we can obtain that $\beta_1 = 32$ and $\beta_2 = 8$. Consequently, $M_1(2) = [-4, 4]$, $M_2(2) = [-4, 4]$, so that $L(2) = [-4, 4]$. Thus $N(\alpha) = N(k) = [-2, 2]$, $L(k) = [-4, 4]$. According to the main dwell-time condition (14) in Theorem 3.3, the dwell-time for each subsystem is given by $\tau_1 \leq 0.231$, $\tau_2 \geq 0.693$.

For numerical simulation, we consider $\tau_1 = 0.231$ and $\tau_2 = 0.693$, and the switched system is initialized at two different position: $x(0) = -6$, which is outside $L(2)$, and $x(0) = 2$, which is on the boundary of $N^\alpha(k)$. Figure 2 shows the resulting trajectories where the blue line gives the trajectory initialized at -6 while the red one is the trajectory initialized at 2. According to part (i) and part (ii) in Theorem 3.3, there exists $T > 0$ such that the trajectory will enter $N(k)$ and remains in $L(k)$ for all $t \geq T$. As shown in Figure 2, the trajectory that starts at -6 enters $N(k)$ at $T = 0.52s$, and remains in $L(k)$ afterwards. On the other hand, when the state is initialized at 2, which is in $N^\alpha(k)$, the trajectory will remain in $L(k)$ for all $t \geq 0$.

The analysis tools provided by Theorem 3.3 only require us to get Lyapunov characterization for each sub-system outside a given compact set. For instance, the Lyapunov inequality (9) does not need to be fulfilled in the neighborhood of the equilibria. In the example above, one can check for the second subsystem that by using the given Lyapunov function $V_2(x(t), t) = \frac{1}{2}x^2$, we have $\dot{V}_2(x(t), t) = -x^2(1 + x)^2 \leq 0$. However, it is not possible to fulfill the inequality (9) for all $\mathbb{R}$. By letting $k = 2$, $-2(1 + x)^2 \leq -2$ holds for all x in $(-\infty, -2) \bigcup (2, +\infty)$ (which is a state domain outside the compact interval $[-2, 2]$). Thus, in this domain, we have $\dot{V}_2(x(t), t) \leq -2V_2(x(t), t)$ fulfilling (9) with the dissipation rate $\eta_2 = -2$.

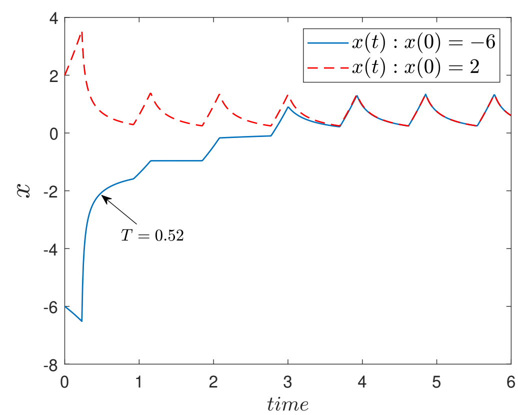

Figure 2: The plot of trajectories of switched system in Example 1 initialized at $x(0) = -6$, $x(0) = 2$, and using a periodic switching signal with $\tau_1 = 0.231$ and $\tau_2 = 0.693$, $x(t) : x(0) = -6$ enters $N(k)$ at 0.52s.

4. Practical stability for the switched affine systems

In this section, we focus on the application of Theorem 3.3 in the practical stability analysis of switched affine systems with mixed stable-unstable subsystems. Let us consider a switched affine system in the form of

$$\dot{x}(t) = A_q x(t) + B_q, \quad \forall q \in \mathcal{Q}, \tag{19}$$

where $x(t)$ and $\sigma(t)$ are as in (1). Here we do not restrict A_q to be stable matrices, nor they have stable matrix combination as pursued in [22; 23].

Following [21], the switched affine system (19) is said to be *practically stable with respect to the sets* $\Omega_1 \subseteq \mathcal{X}$ and $\Omega_2 \subseteq \mathcal{X}$ ($\Omega_1 \subseteq \Omega_2$) for any switching signal $\sigma(t)$ from the given class, if the implication $x(t_0) \in \Omega_1 \Rightarrow x(t) \in \Omega_2$ holds for all $t \geq 0$.

In Theorem 3.3, it is assumed that there exist multiple Lyapunov functions $V_q(\xi, \tau)$ in $\mathcal{X} \setminus N(k) \times [0, \tau_{q,\max})$ satisfying (9) and (10). In general, checking the existence of such Lyapunov functions is not trivial. In the following lemma, we present a sufficient condition that can simplify the construction of such Lyapunov functions.

Lemma 4.1. *Suppose that for each mode $q \in \mathcal{Q}$ there exists a continuously differentiable function $V_q : \mathcal{X} \times [0, \tau_{q,\max}) \to \mathbb{R}_{\geq 0}$ such that the following inequalities*

$$\dot{V}_q(\xi,\tau) \leq \eta_q V_q(\xi,\tau) + \gamma_q\Big(k - V_q(\xi,\tau)\Big), \quad \forall \xi \in \mathcal{X}, \forall \tau \in [0,\tau_{q,\max}) \tag{20}$$

$$V_p(\xi,0) \leq \mu_q V_q(\xi,\tau) + \gamma'_q\Big(k - V_q(\xi,\tau)\Big), \quad \forall \xi \in \mathcal{X}, \forall p,q \in \mathcal{Q}, \forall \tau \in [\tau_{q,\min},\tau_{q,\max}), \tag{21}$$

hold with $0 < \tau_{q,\min} < \tau_{q,\max}$, where $\gamma_q \geq 0$, $\gamma'_q \geq 0$, and the constant k is as used before in (3). Then the statements of Theorem 3.3(i), (ii) and (iii) hold for any switching signals σ satisfying (14).

Proof. It follows from (20) and (21) that $V_q(\xi,\tau) > k \Rightarrow \dot{V}_q(\xi,\tau) \leq \eta_q V_q(\xi,\tau)$ and $V_p(\xi,0) \leq \mu_q V_q(\xi,\tau)$. This implies that $V_q(\xi,\tau)$ and $V_p(\xi,0)$ as given in (20) and (21) satisfy (9) and (10) outside the compact set $N(k)$, i.e. in the set $\bigcap_{q\in\mathcal{Q}}\{\xi \in \mathcal{X} \mid V_q(\xi) \geq k)\}$. In this case, all hypotheses of Theorem 3.3 are satisfied and hence the claim of the lemma follows immediately. Moreover, it holds globally if $k = 0$. □

For switched linear systems, it is common to use a quadratic Lyapunov function $V_q(\xi,\tau) = \xi^\top R_q \xi$, where R_q is a positive definite matrix. The use of such quadratic form may not be suitable, particularly when the systems switch consecutively between unstable modes. For instance, when the system switches from an unstable mode q to another unstable mode p and then back to mode q again, for constant matrices R_q and R_p, (10) becomes $R_p \leq \mu_q R_q \leq \mu_q \mu_p R_p$ with $0 < \mu_q < 1$, $0 < \mu_p < 1$, which cannot be satisfied. This shows that the matrix R_q can not be a constant matrix when switching between unstable modes are admitted, such as the switched systems considered in our main result above. In order to compensate the conservativity brought by the affine term B_q, we construct a shifted time-varying quadratic Lyapunov function given by

$$V_q(\xi,\tau) = (\xi - x^\star)^\top R_q(\tau)(\xi - x^\star), \quad \forall q \in \mathcal{Q}, \tag{22}$$

where $x^\star \in \mathbb{R}^n$ is the centroid of the level set V_q. For the time-varying matrix $R_q(\tau)$, the inequality (10) is not trivial to solve. A well-known technique to solve such problem is the discretized Lyapunov function technique, which is widely used in the stabilization of linear switched systems [9; 10; 11]. The basic idea of the discretized Lyapunov function technique is to linearize $R_q(\tau)$ into the form of $\frac{\tau}{\tau_{q,\min}}P_q + (1-\frac{\tau}{\tau_{q,\min}})Q_q$ for all $\tau \in [0,\tau_{q,\min})$, and $R_q(\tau) = P_q$ elsewhere. In the following, we transform Lemma 4.1 into LMI conditions by using the discretized Lyapunov function technique and coordinate transformation. By defining $\tilde{x}(t) = x(t) - x^\star$, we can rewrite (19) as

$$\dot{\tilde{x}}(t) = A_q \tilde{x}(t) + L_q, \quad \forall q \in \mathcal{Q}, \tag{23}$$

where $L_q = A_q x^\star + B_q$.

Lemma 4.2. *Suppose that for each mode $q \in \mathcal{Q}$ there exist positive symmetric matrices P_q, Q_q, and constants $\mu_q > 0$, $\gamma_q > 0$, $\gamma'_q > 0$, $\tau_{q,\min} > 0$, and $\eta_q \neq 0$, such that the following inequalities*

$$\begin{bmatrix} \Xi_{q,1} & P_q L_q \\ * & -\gamma_q k \end{bmatrix} \leq 0, \quad \forall q \in \mathcal{Q}, \qquad \begin{bmatrix} \Xi_{q,2} & Q_q L_q \\ * & -\gamma_q k \end{bmatrix} \leq 0, \quad \forall q \in \mathcal{Q}, \tag{24}$$

$$\begin{bmatrix} \Xi_{q,3} & P_q L_q \\ * & -\gamma_q k \end{bmatrix} \leq 0, \quad \forall q \in \mathcal{Q}, \tag{25}$$

$$Q_p \leq (\mu_q - \gamma'_q)P_q, \quad \forall q \in \mathcal{Q} \tag{26}$$

hold, where $\Xi_{q,1} = A_q^\top P_q + P_q A_q + \frac{1}{\tau_{q,\min}}(P_q - Q_q) + (\gamma_q - \eta_q)P_q$, $\Xi_{q,2} = A_q^\top Q_q + Q_q A_q - \frac{1}{\tau_{q,\min}}(P_q - Q_q) + (\gamma_q - \eta_q)Q_q$, and $\Xi_{q,3} = A_q^\top P_q + P_q A_q + (\gamma_q - \eta_q)P_q$. Then the statements of Theorem 3.3(i), (ii) and (iii) hold for any switching signals σ satisfying (14).

Proof. We will prove the lemma by constructing the matrix $R_q(\tau)$ used in (22) such that $V_q(\xi,\tau)$ in (22) satisfies the hypotheses in Theorem 3.3. Let us define $R_q(\tau)$ by

$$R_q(\tau) = \begin{cases} \frac{\tau}{\tau_{q,\min}}P_q + (1-\frac{\tau}{\tau_{q,\min}})Q_q & \forall \tau \in [0,\tau_{q,\min}) \\ P_q & \text{otherwise,} \end{cases} \tag{27}$$

so that $R_q(0) = Q_q$ and $R_q(\tau_{q,\min}) = P_q$. For $\tau \geq \tau_{q,\min}$, (20) is guaranteed according to (25). Now, let us consider $R_q(\tau)$ in the interval $[0,\tau_{q,\min})$ where the time-derivative of $R_q(\tau)$ satisfies

$$\frac{\mathrm{d}R_q(\tau)}{\mathrm{d}\tau} = \frac{1}{\tau_{q,\min}}(P_q - Q_q). \tag{28}$$

Correspondingly, using (23), (20), (27) and (28) we can compute the time-derivative of V_q in (22) on $[0,\tau_{q,\min}]$ as follows

$$\begin{aligned} &\dot{V}_q(x,\tau) - \eta_q V_q(x,\tau) - \gamma_q\Big(k - V_q(x,\tau)\Big) = \\ &\quad \frac{\tau}{\tau_{q,\min}}\Big[\tilde{x}^\top \Xi_{q,1}\tilde{x} + \tilde{x}^\top P_q L_q + L_q^\top P_q \tilde{x} - \\ &\quad \gamma_q k\Big] + \Big(1 - \frac{\tau}{\tau_{q,\min}}\Big)\Big[\tilde{x}^\top \Xi_{q,2}\tilde{x} + \tilde{x}^\top Q_q L_q \\ &\quad + L_q^\top Q_q \tilde{x} - \gamma_q k\Big], \end{aligned} \tag{29}$$

where we have used the relation $\tilde{x} = x - x^\star$ in $V_q(x,\tau)$ above. The right-hand side of (29) can be written compactly as

$$\begin{aligned} &\frac{\tau}{\tau_{q,\min}}\begin{bmatrix}\tilde{x}^\top & 1\end{bmatrix}\begin{bmatrix} \Xi_{q,1} & P_q L_q \\ * & -\gamma_q k \end{bmatrix}\begin{bmatrix}\tilde{x} \\ 1\end{bmatrix} + \\ &\Big(1 - \frac{\tau}{\tau_{q,\min}}\Big)\begin{bmatrix}\tilde{x}^\top & 1\end{bmatrix}\begin{bmatrix} \Xi_{q,2} & Q_q L_q \\ * & -\gamma_q k \end{bmatrix}\begin{bmatrix}\tilde{x} \\ 1\end{bmatrix}. \end{aligned} \tag{30}$$

Correspondingly, using (24) and (30), it follows that

$$\dot{V}_q(x,\tau) - \eta_q V_q(x,\tau) - \gamma_q\Big(k - V_q(x,\tau)\Big) \le 0, \quad (31)$$

for all $\tau \in [0, \tau_{q,\min})$. According to (26), it implies that

$$\begin{aligned} V_q(x,\tau) &\le (\mu_q - \gamma'_q)V_q(x,\tau) \\ &\le \mu_q V_q(x,\tau) + \gamma'_q\Big(k - V_q(x,\tau)\Big). \end{aligned} \quad (32)$$

Similarly, for all $\tau \ge \tau_{q,\min}$, $\dot{V}_q(x,\tau) - \eta_q V_q(x,\tau) - \gamma_q(k - V_q(x,\tau))$ is negative definite according to (25). Consequently, in combination with (26) and (14), all hypotheses in Theorem 3.3 are satisfied and the claim of the lemma follows immediately. □

In general switched affine systems, L_q in (23) is not equal to zero and may not be identical among the different modes q when each mode has a different equilibrium point. The possibility of admitting a different equilibrium point for every mode makes it impossible to find a global quadratic common Lyapunov function given by (22).

Remark 4.3. *Since R_q is a convex combination of P_q and Q_q, then for any given $k > 0$, $\alpha(k)$ in (6) can be upper-bounded by*

$$\alpha(k) \le \overline{\alpha}(k) := \frac{\lambda_{max}}{\lambda_{\min}} k, \quad (33)$$

where $\lambda_{max} = \max\{\lambda(P_q), \lambda(Q_q)\}$, and $\lambda_{\min} = \min\{\lambda(P_q), \lambda(Q_q)\}$, $\forall q \in \mathcal{Q}$.

Equipped with the LMI conditions in Lemma 4.2, the following theorem provides sufficient conditions for practical stability of the switched affine system (19).

Theorem 4.4. *(Practical stability) Let the sets $\Omega_1 = N^{\alpha}(k)$ and $\Omega_2 = L(k)$ ($\Omega_1 \subset \Omega_2$). Suppose that the hypotheses in Lemma 4.2 hold. Then for all initial states in Ω_1, i.e. $x(t_0) \in N^{\alpha}(k)$, the trajectories of switched system (19) remain in the set Ω_2, i.e. $x(t) \in L(k)$, for every switching signals $\sigma(t)$ satisfying (14).*

Similar to Corollary 3.5, if all subsystems of (19) are unstable, the results in Lemma 4.2 can be used to establish the following remark.

Remark 4.5. *Suppose that the hypotheses in Lemma 4.2 hold with $\mathcal{Q} = \mathcal{U}$. Then the trajectories of switched affine system (23) will remain in $L(k)$ after time $T > 0$ for any switching signals satisfying (17). In addition, if $P_q > Q_q$ then Ω_1 and Ω_2 can be estimated by $\bigcap_{q\in\mathcal{Q}}\{\tilde{x}_q \mid \tilde{x}_q^{\top} Q_q \tilde{x}_q \le \alpha(k)\}$ and $\bigcup_{q\in\mathcal{Q}}\{\tilde{x}_q \mid \tilde{x}_q^{\top} Q_q \tilde{x}_q \le \beta_q(k)\}$, respectively.*

Let us illustrate the applicability of the LMI conditions in Lemma 4.2. By a direct application of Lemma 4.2, we establish the stability of a switched system with stable and unstable subsystems in Example 2 below, and it is followed by the stability of a switched system with all unstable subsystems in Example 3.

Example 2: Let us consider the switched system (19) composed of both unstable ($q = 1$) and stable ($q = 2$) subsystems as follows

$$\begin{aligned} q = 1: \quad \dot{x} &= \begin{bmatrix} -2 & 0.5 \\ 0.5 & 0 \end{bmatrix} x + \begin{bmatrix} 1.4 \\ -0.4 \end{bmatrix}, \\ q = 2: \quad \dot{x} &= \begin{bmatrix} 0 & 1 \\ -0.5 & -2 \end{bmatrix} x + \begin{bmatrix} -0.9 \\ 2.4 \end{bmatrix}, \end{aligned} \quad (34)$$

and we set the parameter $k = 2$. Then by applying Lemma 4.2 to this switched affine system, where we fix $x^\star = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\top}$, $\gamma_1 = \gamma_2 = 0.05$, $\eta_1 = 0.34$, $\eta_2 = -0.24$, $\mu_1 = 0.5$, $\mu_2 = 2$, $\gamma'_1 = \gamma'_2 = 0$, it can be checked that using the following symmetric constant matrices

$$\begin{aligned} P_i &: \begin{bmatrix} 0.9160 & -0.0841 \\ -0.0841 & 0.3847 \end{bmatrix}, \begin{bmatrix} 0.0788 & 0.0296 \\ 0.0296 & 0.1767 \end{bmatrix}, \\ Q_i &: \begin{bmatrix} 0.1350 & 0.0624 \\ 0.0624 & 0.3511 \end{bmatrix}, \begin{bmatrix} 0.1596 & 0.0186 \\ 0.0186 & 0.1789 \end{bmatrix}, \end{aligned} \quad (35)$$

the LMI problem given by (24)-(26) is feasible. Correspondingly, we have $\tau_{1,\min} = 2$, $\tau_{1,max} = 2.04$, $\tau_{2,\min} = 2.89$. An upper bound of $\alpha(2)$ is given by (33) as $\overline{\alpha}(2) = 26.3546$ and $\frac{\lambda_{max}}{\lambda_{\min}} = 13.1773$. According to (11), $\beta_1 = 52.7092$, $\beta_2 = 26.3546$. Then, we have,

$N(2) = \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 2\Big\} \cup \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 2\Big\}$,

$N^{\alpha}(2) \subseteq N^{\overline{\alpha}}(2) = \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 26.3546\Big\} \cap \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 26.3546\Big\}$;

$L(2) = \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} Q_1 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 52.7092\Big\} \cup \Big\{x_1, x_2 : \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix}^{\top} P_2 \begin{bmatrix} x_1 - 1 \\ x_2 - 1 \end{bmatrix} \le 26.3546\Big\}$.

Figure 3 shows the trajectories of switched system (34) with a periodic switching signal σ satisfying $\tau_1 = 2$, $\tau_2 = 3$ and with four different initial conditions $\begin{bmatrix} 2.584 \\ -7.86 \end{bmatrix}$, $\begin{bmatrix} -0.056 \\ 9.848 \end{bmatrix}$, $\begin{bmatrix} -13.432 \\ 4.92 \end{bmatrix}$, $\begin{bmatrix} 15.608 \\ -2.36 \end{bmatrix}$.

Example 3: Let us consider the switched system (19) composed of two unstable subsystems as follows

$$\begin{aligned} q = 1: \quad \dot{x} &= \begin{bmatrix} -1.9 & 0.6 \\ 0.6 & -0.1 \end{bmatrix} x + \begin{bmatrix} 1.4 \\ -0.6 \end{bmatrix}, \\ q = 2: \quad \dot{x} &= \begin{bmatrix} 0.1 & -0.9 \\ 0.1 & -1.4 \end{bmatrix} x + \begin{bmatrix} 0.7 \\ 1.4 \end{bmatrix}, \end{aligned} \quad (36)$$

and let us set $k = 5$. Then by applying Lemma 4.2 to this switched affine system, where we fix $x^\star = \begin{bmatrix} 1 & 1 \end{bmatrix}^{\top}$,

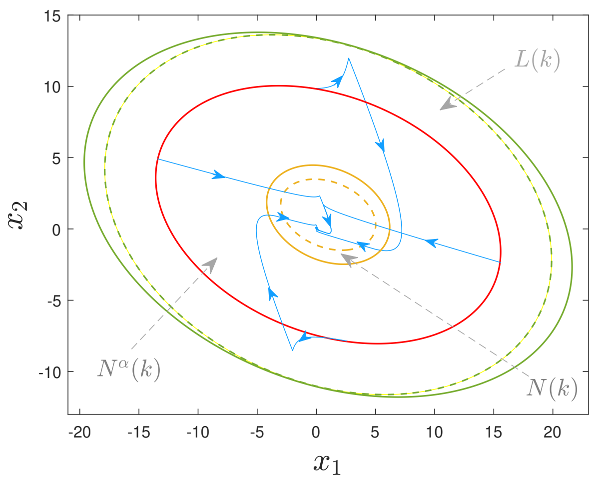

Figure 3: The plot of trajectories of switched system in Example 3 initialized at $\begin{bmatrix}2.584\\-7.86\end{bmatrix}$, $\begin{bmatrix}-0.056\\9.848\end{bmatrix}$, $\begin{bmatrix}-13.432\\4.92\end{bmatrix}$, $\begin{bmatrix}15.608\\-2.36\end{bmatrix}$, and using a periodic switching signal with $\tau_1 = 2$, $\tau_2 = 3$, the green solid line is $L(k)$, the red solid line is $N^{\alpha}(k)$, the orange line is $N(k)$.

$\gamma_1 = \gamma_2 = 0.06$, $\eta_1 = \eta_2 = 0.34$, $\mu_1 = \mu_2 = 0.5$, $\gamma_1' = \gamma_2' = 0$, it can be checked that using the following symmetric constant matrices

$$\begin{aligned} P_i &: \begin{bmatrix}6.6543 & -1.0418\\-1.0418 & 3.7555\end{bmatrix}, \begin{bmatrix}2.0998 & -0.6941\\-0.6941 & 6.8937\end{bmatrix}, \\ Q_i &: \begin{bmatrix}1.0475 & -0.3351\\-0.3351 & 3.3797\end{bmatrix}, \begin{bmatrix}2.0716 & -0.9015\\-0.9015 & 1.7611\end{bmatrix}, \end{aligned} \tag{37}$$

the LMI problem given by (24)-(26) is feasible. Correspondingly, we have $\tau_{1,\min} = \tau_{2,\min} = 2$, $\tau_{1,max} = \tau_{2,max} = 2.04$. An upper bound for $\alpha(5)$ is given by (33) as $\overline{\alpha}(5) = 34.95$ and $\frac{\lambda_{max}}{\lambda_{\min}} = 6.99$. According to (11), $\beta_1 = \beta_2 = 69.9$. According to Remark 4.5,
$N(5) = \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_1 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 5\right\} \cup \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_2 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 5\right\}$;
$N^{\alpha}(5) \subseteq N^{\overline{\alpha}}(5) = \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_1 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 34.95\right\} \cap \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_2 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 34.95\right\}$;
$L(5) = \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_1 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 69.9\right\} \cup \left\{x_1, x_2 : \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix}^{\top} Q_2 \begin{bmatrix}x_1-1\\x_2-1\end{bmatrix} \le 69.9\right\}$. Figure 4 shows the plot of trajectories of the switched system with periodic switching signal σ satisfying $\tau_1 = \tau_2 = 2$ and with four different initial conditions $\begin{bmatrix}-3.3\\0.49\end{bmatrix}$, $\begin{bmatrix}5.35\\1.58\end{bmatrix}$, $\begin{bmatrix}-0.37\\-2.26\end{bmatrix}$, $\begin{bmatrix}1.65\\4.26\end{bmatrix}$. The figure shows that when the trajectory starts on the boundary of $N(\alpha)$, the trajectory stays in $L(k)$ for all time. We note that the first switching moment in the trajectory, which starts from $\begin{bmatrix}1.65\\4.26\end{bmatrix}$, illustrates the second case of step one in the proof of Theorem 3.3, i.e. for an unstable system, the trajectory can go into $N(k)$ and later escape from $N(k)$ on the next switching moment.

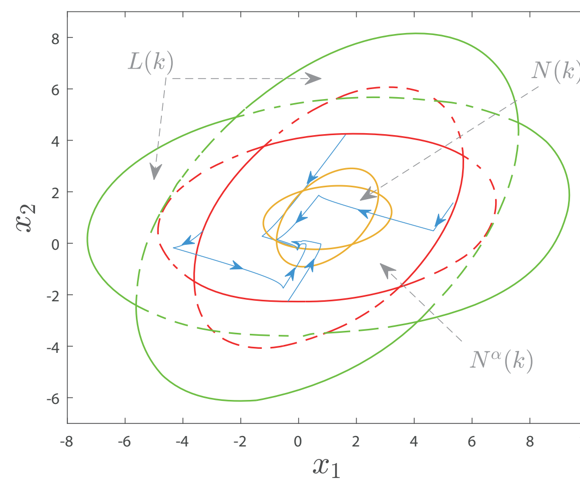

Figure 4: The plot of trajectories of switched system in Example 3 initialized at $\begin{bmatrix}-3.3\\0.49\end{bmatrix}$, $\begin{bmatrix}5.35\\1.58\end{bmatrix}$, $\begin{bmatrix}-0.37\\-2.26\end{bmatrix}$, $\begin{bmatrix}1.65\\4.26\end{bmatrix}$, and using a periodic switching signal with $\tau_1 = \tau_2 = 2$, the green solid line is $L(k)$, the red solid line is $N^{\alpha}(k)$, the orange line is $N(k)$.

5. CONCLUSION

In this paper, the set convergence properties of switched systems with mixed stable-unstable modes have been studied. Based on the dwell-time and leave-time property of the switching signals, multiple Lyapunov functions are defined and used to characterise the set of initial conditions that admits an attractive set, which all trajectories will converge to. Based on these sufficient conditions, LMI conditions are presented that allow for numerical validation on the practical stability of switched affine systems with computable dwell-time.

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