Impulse-controllability of system classes of switched differential algebraic equations

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Abstract

In this paper impulse controllability of system classes containing switched DAEs is studied. We introduce several notions of impulse-controllability of system classes and provide a characterization of strong impulse-controllability of system classes generated by arbitrary switching signals. In the case of a system class generated by switching signals with a fixed mode sequence it is shown that either all or almost all systems are impulse-controllable, or that all or almost all systems are impulse-uncontrollable. Sufficient conditions for all systems to be impulse-controllable or impulse-uncontrollable are presented. Furthermore, it is observed that although all systems are impulse-controllable, the input achieving impulse-free solutions might still depend on the switching times in the future, which causes some causality issues. Therefore, the concept of (quasi-) causal impulse-controllability is introduced and system classes which are (quasi-) causal are characterized. Finally necessary and sufficient conditions for a system class to be causal given some dwell-time are stated.

Keywords: Switched systems, Differential Algebraic Equations, Impulse-controllability, Geometric control

1 Introduction

We consider *switched differential algebraic equations* (switched DAEs) of the form

$$E_\sigma = A_\sigma x + B_\sigma u, \qquad x(0^-) = x_0, \tag{1}$$

where $\sigma : \mathbb{R} \to \mathbb{N}$ is the switching signal and $E_p, A_p \in \mathbb{R}^{n\times n}$, $B_p \in \mathbb{R}^{n\times m}$, for $p, n, m \in \mathbb{N}$. In general, trajectories of switched DAEs exhibit jumps (or even impulses), which may exclude classical solutions from existence. Therefore, we adopt the *piecewise-smooth distributional solution framework* introduced in [1]. An important property, called *impulse-controllability*, of these models is the ability to choose an input in such a way that no Dirac impulses are induced by the switches. In this contribution we will extend our recently established results [2] for the case of fixed switching signals to the case where the switching times are not known.

Differential algebraic equations (DAEs) arise naturally when modeling physical systems with certain algebraic constraints on the state variables; examples of applications of DAEs in electrical circuits (with distributional solutions) can be found, e.g., in [3]. These constraints are often eliminated such that the system is described by ordinary differential equations (ODEs). However, in the case of switched systems, the elimination process of the constraints is in general different for each individual mode and therefore there does not exist a description as a switched ODE with a common state variable for every mode in general. This problem can be overcome by studying switched DAEs directly.

Several structural properties of switched DAEs have been studied recently such as controllability by [4], stability/stabilizability by [5–8] and observability/detectability by [9–11]. Impulse-controllability has been studied in the non-swtiched case [12–15] and in the switched case in [2] for fixed switching signals.

In the case of component failure or cyber-physical attacks, the instance at which structural changes in the system occur is often unknown and they could happen at any time. This poses a problem when Dirac impulses in the state are to be avoided, since impulse controllability of switched DAEs is in general dependent on the switching times induced by the switching signal [2]. However, in some cases the existence of impulse-free solutions for all initial values does not depend on the switching signal. As an example consider any system generated by such a switching signal and the matrices

$$\begin{aligned} E_0 &= \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right] & A_0 &= \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right], & B_0 &= \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \\ E_1 &= \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right], & A_1 &= \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right], & B_1 &= \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]. \end{aligned}$$

i.e., each mode is given by

$$\begin{aligned} \text{mode 0:} \quad & \left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\dot{x} = \left[\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right]x + \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]u, \\ \text{mode 1:} \quad & \left[\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right]\dot{x} = \left[\begin{smallmatrix}1&0\\1&1\end{smallmatrix}\right]x. \end{aligned}$$

For every switching time $t_1 \in (0, \infty)$ and any order in which the modes appear the corresponding switched DAE is impulse controllable, since the input $u = x_1$ on $[t_0, t_1)$ and $u = 0$ on $[t_1, \infty)$ ensures impulse free solutions. Hence every system in the system class defined by these systems is impulse controllable.

Motivated by this example, the aim of this paper is to characterize the system classes for which any system contained in it is impulse controllable, regardless of the switching signal. Stated differently, we will present necessary and sufficient conditions under which there exist impulse free solutions of any switched system with modes governed by the matrices E_p, A_p and B_p and $p \in \{0, 1, ..., \mathtt{n}\}$. Furthermore, we will investigate system classes containing switched systems for which the order in which the modes appears is fixed, i.e., for a particular class of switching signals. For those system classes we will show that either all systems, almost all, none or almost none of the systems are impulse controllable. Then it is shown that although every system in such a system class is impulse-controllable, an input that guarantees impulse-free solution might depend on the switching times in the future, which causes a causality issue. Consequently, we introduce the concepts of (quasi-) causal impulse-controllability of system classes and provide characterizations. Finally, necessary and sufficient conditions for system classes to be causally impulse-controllable given some dwell-time are presented.

The remainder of the paper is structured as follows. The mathematical preliminaries are given in Section 2. The result regarding impulse controllability of system classes are contained by Section 3 and (quasi-) causal imulse-controllabillity is considered in Section 4. Conclusions and direction for further research are given in Section 5.

2 Mathematical Preliminaries

In this section we recall some notation and properties related to the non-switched DAE

$$E\dot{x} = Ax + Bu. \tag{2}$$

2.1 Properties and definitions for regular matrix pairs

In the following, we call a matrix pair (E, A) and the associated DAE (2) *regular* iff the polynomial $\det(sE - A)$ is not the zero polynomial. Recall the following result on the *quasi-Weierstrass form*[16].

Proposition 1. *A matrix pair $(E, A) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times n}$ is regular if, and only if, there exists invertible matrices $S, T \in \mathbb{R}^{n \times n}$ such that*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right), \tag{3}$$

where $J \in \mathbb{R}^{n_1 \times n_1}$, $0 \leq n_1 \leq n$, is some matrix and $N \in \mathbb{R}^{n_2 \times n_2}$, $n_2 := n - n_1$, is a nilpotent matrix.

The matrices S and T can be calculated by using the so-called *Wong sequences* [16, 17]:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, \quad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \quad i = 0, 1, ... \\ \mathcal{W}_0 &:= \{0\}, \quad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i), \quad i = 0, 1, ... \end{aligned}$$

The Wong sequences are nested and get stationary after finitely many iterations. The limiting subspaces are defined as follows:

$$\mathcal{V}^* := \bigcap_i \mathcal{V}_i, \qquad \mathcal{W}^* := \bigcup \mathcal{W}_i.$$

For any full rank matrices V, W with $\operatorname{im} V = \mathcal{V}^*$ and $\operatorname{im} W = \mathcal{W}^*$, the matrices $T := [V, W]$ and $S := [EV, AW]^{-1}$ are invertible and (3) holds.

Based on the Wong sequences we define the following projector and selectors.

Definition 2. Consider the regular matrix pair (E, A) with corresponding quasi-Weierstrass form (3). The *consistency projector* of (E, A) is given by

$$\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1},$$

the *differential* and *impulse selector* are given by

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S, \ \Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S.$$

In all three cases the block structure corresponds to the block structure of the quasi-Weierstrass form. Furthermore we define

$$\begin{aligned} A^{\text{diff}} &:= \Pi^{\text{diff}}_{(E,A)} A, \qquad E^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} E, \\ B^{\text{diff}} &:= \Pi^{\text{diff}}_{(E,A)} B, \qquad B^{\text{imp}} := \Pi^{\text{imp}}_{(E,A)} B. \end{aligned}$$

Note that all the above defined matrices do not depend on the specifically chosen transformation matrices S and T; they are uniquely determined by the original regular matrix pair (E, A). An important feature for DAEs is the so called consistency space, defined as follows:

Definition 3. Consider the DAE (2), then the *consistency space* is defined as

$$\mathcal{V}_{(E,A)} := \left\{ x_0 \in \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists \text{ smooth solution } x \text{ of} \\ E\dot{x} = Ax, \text{ with } x(0) = x_0 \end{array} \right\},$$

and the *augmented consistency space* is defined as

$$\mathcal{V}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{c} \exists \text{ smooth solutions } (x,u) \text{ of} \\ E\dot{x} = Ax + Bu \text{ and } x(0) = x_0 \end{array} \right\}.$$

In order to express (augmented) consistency spaces in terms of the Wong limits we need the following notation for matrices A,B of suitable sizes:

$$\langle A \mid B \rangle := \operatorname{im} \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}.$$

Proposition 4 ([18]). *Consider the regular DAE* (2), *then* $\mathcal{V}_{(E,A)} = \mathcal{V}^* = \operatorname{im} \Pi_{(E,A)} = \operatorname{im} \Pi^{\text{diff}}_{(E,A)}$ *and* $\mathcal{V}_{(E,A,B)} = \mathcal{V}^* \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

For studying impulsive solutions, we consider the space of *piecewise-smooth distributions* $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ from [1] as the solution space. For a piecewise-smooth distribution $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ the left-/right-evaluation $D(t^-)$ / $D(t^+)$ at any $t \in \mathbb{R}$ is well defined and it is also possible to define the impulse evaluation $D[t]$ for any $t \in \mathbb{R}$. Solving the DAE (2) with an inconsistent initial value is reinterpreted as the problem of finding a solution $(x,u) \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^{n+m}$ to the following initial-trajectory problem (ITP):

$$x_{(-\infty,0)} = x^0_{(-\infty,0)}, \tag{4a}$$

$$(E\dot{x})_{[0,\infty)} = (Ax + Bu)_{[0,\infty)}, \tag{4b}$$

where $x^0 \in (\mathbb{D}_{\text{pw}\mathcal{C}^\infty})^n$ is some initial trajectory, and $f_\mathcal{I}$ denotes the restriction of a piecewise-smooth distribution f to an interval $\mathcal{I}$. In [1] it is shown that the ITP (4) has a unique solution for any initial trajectory if, and only if, the matrix pair (E,A) is regular. As a direct consequence, the switched DAE (1) with regular matrix pairs is also uniquely solvable (with piecewise-smooth distributional solutions) for any switching signal with locally finitely many switches.

2.2 Properties of DAEs

Recall the following definitions and characterization of (impulse) controllability [18].

Proposition 5. *The reachable space of the regular DAE* (2) *defined as*

$$\mathcal{R} := \left\{ x_T \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0 \; \exists \text{ smooth solution } (x,u) \text{ of } (2) \\ \text{with } x(0) = 0 \text{ and } x(T) = x_T \end{array} \right\}$$

satisfies $\mathcal{R} = \langle A^{\text{diff}} \mid B^{\text{diff}} \rangle \oplus \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle$.

It is easily seen that the reachable space for (2) coincides with the controllable space, i.e.

$$\mathcal{R} = \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists T > 0\ \exists \text{ smooth solution } (x,u) \text{ of } (2) \\ \text{with } x(0) = x_0 \text{ and } x(T) = 0 \end{array} \right\}.$$

Corollary 6. *The augmented consistency space of* (2) *satisfies* $\mathcal{V}_{(E,A,B)} = \mathcal{V}_{(E,A)} + \mathcal{R} = \mathcal{V}_{(E,A)} \oplus \langle E^{\text{imp}}, B^{\text{imp}} \rangle$.

Definition 7. The DAE (2) is impulse controllable if for all initial conditions $x_0 \in \mathbb{R}^n$ there exists a solution (x,u) of the ITP (4) such that $x(0^-) = x_0$ and $(x,u)[0] = 0$, i.e. the state and the input are impulse free at $t = 0$. The space of impulse controllable states of the DAE (2) is given by

$$\mathcal{C}^{\text{imp}}_{(E,A,B)} := \left\{ x_0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \text{ of } (4) \\ \text{s.t. } x(0^-) = x_0 \text{ and } (x,u)[0] = 0. \end{array} \right\}.$$

In particular, the DAE (2) is impulse controllable if and only if $\mathcal{C}^{\text{imp}}_{(E,A,B)} = \mathbb{R}^n$.

Impulse controllability can be characterized geometrically as follows (cf. [15, 19]).

Lemma 8. *The regular DAE* (2) *is impulse controllable if and only if*

$$\operatorname{im} E + A \ker E + \operatorname{im} B = \mathbb{R}^n.$$

Furthermore,

$$\begin{aligned} \mathcal{C}^{\text{imp}}_{(E,A,B)} &= \mathcal{V}_{(E,A,B)} + \ker E = \mathcal{V}_{(E,A)} + \mathcal{R} + \ker E \\ &= \mathcal{V}_{(E,A)} \oplus \mathcal{D}^{\text{imp}} = \operatorname{im} \Pi_{(E,A)} \oplus \mathcal{D}^{\text{imp}}. \end{aligned}$$

where $\mathcal{D}^{\text{imp}} := \langle E^{\text{imp}} \mid B^{\text{imp}} \rangle + \ker E$.

According to [20] if the input $u(\cdot)$ is sufficiently smooth, trajectories of (2) are continuous and given by

$$\begin{aligned} x(t) = x_u(t, t_0;\ x_0) &= e^{A^{\text{diff}}(t-t_0)} \Pi_{(E,A)} x_0 \\ &+ \int_{t_0}^{t} e^{A^{\text{diff}}(t-s)} B^{\text{diff}} u(s)\, ds - \sum_{i=0}^{n-1} (E^{\text{imp}})^i B^{\text{imp}} u^{(i)}(t). \end{aligned} \tag{5}$$

In the case of a family of matrix triples $\{(E_p, A_p, B_p)\}_{p=0}^{\mathtt{n}}$ for some $\mathtt{n} \in \mathbb{N}$, we will adopt the shorthand notation $\Pi_p := \Pi_{(E_p,A_p)}$, $\Pi_p^{\text{diff}} := \Pi^{\text{diff}}_{(E_p,A_p)}$ and $\Pi_p^{\text{imp}} := \Pi^{\text{imp}}_{(E_p,A_p)}$ for the consistency projector and the consistency selectors.

The matrices A_p^{diff}, B_p^{diff}, E_p^{imp}, B_p^{imp} are defined accordingly. The impulse controllable space for mode p is denoted by $\mathcal{C}_p^{\text{imp}} := \mathcal{C}_{(E_p,A_p,B_p)}^{\text{imp}}$.

3 Impulse controllability of system classes

3.1 System classes

The concepts introduced above will be used in the following to study impulse controllability of system classes containing switched DAEs. We will focus our attention on finite time intervals with finitely many mode changes within this interval. Since we do not want to fix the length of the interval of interest a priori, we simply assume that the last mode remains active until $t = \infty$. In other words, we restrict our attention to classes of switching signals which are defined on the interval $[t_0, \infty)$ and have finitely many mode changes. The corresponding class of switching signals with at most $\mathtt{n} \in \mathbb{N}$ mode changes is formally defined as follows.

Definition 9 (Arbitrary switching signals)**.** The class of (arbitrary) switching signals $\mathcal{S}_{\mathtt{n}}$ is defined as the set of all $\sigma : \mathbb{R} \to \{0, 1, ..., \mathtt{n}\}$ of the form

$$\sigma(t) = q_p \quad t \in [t_p, t_{p+1}) \tag{6}$$

where $\mathbf{q} := (q_0, q_1, \dots, q_{\mathtt{n}}) \in \{0, 1, \dots, \mathtt{n}\}^{\mathtt{n}+1}$ is the *mode sequence* of σ and $t_1 < t_2 < ... < t_{\mathtt{n}}$ are the $\mathtt{n} \in \mathbb{N}$ switching times in $(0, \infty)$ with $t_0 := 0$ and $t_{\mathtt{n}+1} := \infty$ for notational convenience. Furthermore, for a given sequence of switching times, let $\tau_i := t_{i+1} - t_i$, $i = 0, 1, \dots, \mathtt{n} - 1$ and

$$\boldsymbol{\tau} := (\tau_0, \tau_1, \dots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}, \tag{7}$$

the sequence of (finite) mode-durations.

Note that in the above definition, we do not exclude the situation that $q_p = q_{p+1}$ for some p, effectively leading to a switching signal with less then $\mathtt{n}$ switches. Consequently, for such a switching signal the mode duration $\boldsymbol{\tau}$ is not uniquely defined, as the switching time t_{p+1} can be altered without changing the actual switching signal. Nevertheless, this does not lead to any technical problems in the following and we will use $\sigma \in \mathcal{S}_{\mathtt{n}}$ and the corresponding pair $(\mathbf{q}, \boldsymbol{\tau}) \in \mathbb{N}^{\mathtt{n}+1} \times \mathbb{R}^{\mathtt{n}}_{>0}$ interchangeably.

Definition 10 (Fixed mode sequence switching signals)**.** The class of switching signals with fixed mode sequence $\mathbf{q} \in \mathbb{N}^{\mathtt{n}+1}$ is denoted by $\overline{\mathcal{S}}_{\mathbf{q}}$, i.e. $\overline{\mathcal{S}}_{\mathbf{q}}$ contains all switching signals associated to $(\mathbf{q}, \boldsymbol{\tau})$ for some $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$. For the canonical mode sequence $\mathbf{q} = (0, 1, 2, \dots, \mathtt{n})$ we simply write $\overline{\mathcal{S}}_{\mathtt{n}} := \overline{\mathcal{S}}_{(0,\dots,\mathtt{n})}$.

Definition 11 (System classes)**.** For a family of matrix triplets $\{(E_p, A_p, B_p)\}_{p=0}^{\mathtt{n}}$ with regular pairs (E_p, A_p), the *system class* $\Sigma_{\mathtt{n}}$ of associated switched (regular) DAEs (1) under arbitrary switching is given by

$$\Sigma_{\mathtt{n}} := \{(E_\sigma, A_\sigma, B_\sigma) \mid \sigma \in \mathcal{S}_{\mathtt{n}}\},$$

where $(E_\sigma, A_\sigma, B_\sigma)$ is understood as a triple of (piecewise-constant) time-varying matrices for each specific switching signal $\sigma : (t_0, \infty) \to \{0, 1, \ldots, \mathtt{n}\}$.

The corresponding system class $\overline{\Sigma}_{\mathtt{n}}$ of switched DAEs with fixed mode sequence $\mathbf{q} = (0, 1, \ldots, \mathtt{n})$ is given by

$$\overline{\Sigma}_{\mathtt{n}} := \left\{(E_\sigma, A_\sigma, B_\sigma) \,\middle|\, \sigma \in \overline{\mathcal{S}}_{\mathtt{n}}\right\}.$$

3.2 Strong impulse controllability of $\Sigma_{\mathtt{n}}$

For an individual switched DAE (1) given by the (time-varying) matrix triple $(E_\sigma, A_\sigma, B_\sigma)$, impulse controllability is defined as the property that Dirac impulses can be avoided regardless of the initial condition. This is formalized as follows.

Definition 12 (Impulse controllability)**.** The switched DAE $(E_\sigma, A_\sigma, B_\sigma)$ for a fixed switching signal $\sigma \in \mathcal{S}_n$ is called *impulse controllable* iff for all $x_0 \in \mathcal{V}_{(E_{q_0}, A_{q_0}, B_{q_0})}$ there exists a solution $(x, u) \in \mathbb{D}^{n+m}_{\text{pw}\mathcal{C}^\infty}$ with $x(t_0^+) = x_0$ which is impulse free.

The whole system class $\Sigma_{\mathtt{n}}$ associated to the family $\{(E_p, A_p, B_p)\}_{p=0}^{\mathtt{n}}$ is called strongly impulse controllable, if $(E_\sigma, A_\sigma, B_\sigma)$ is impulse controllable for all $\sigma \in \mathcal{S}_{\mathtt{n}}$.

Remark 13. This definition of impulse controllability of an individual switched DAEs is very similar to the definition in [2], which is restricted to a bounded interval and is in fact equivalent when considering the finite interval $[t_0, t_f)$ for some $t_f > t_{\mathtt{n}}$. Furthermore, note that an individual switched system with constant switching signal is by definition always impulse controllable, because only consistent initial values are considered (cf. the discussion after [2, Def. 9]).

Some system classes are trivially strongly impulse controllable (e.g. when each individual mode is impulse controllable or the switched DAEs is in fact non-switching because $(E_p, A_p, B_p) = (E_q, A_q, B_q)$ for all p, q, cf. the discussion after [2, Def 9]).

However, the following example shows that there exists non-trivial example of strongly impulse controllable system classes.

Example 14. Consider a switched DAE (1) with mode triplets

$$\begin{aligned}(E_0, A_0, B_0) &= \left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)\\(E_1, A_1, B_1) &= \left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}0\\0\end{bmatrix}\right).\end{aligned} \tag{8}$$

It is easily seen that the corresponding augemented consistency and impulse controllable spaces satisfy $\mathcal{V}_0 = \mathcal{C}_0^{\text{imp}} = \mathbb{R}^2$ and $\mathcal{V}_1 = \mathcal{C}_1^{\text{imp}} = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$.

The corresponding system class Σ_1 is strongly impulse controllable, which can be seen by considering all possible cases for the switching signals: Switching signals with $\mathbf{q} = (0, 0)$ or $\mathbf{q} = (1, 1)$ are trivially impulse controllable as a non-switched DAE (with consistent initial values); for mode sequence $\mathbf{q} = (0, 1)$ it is possible to choose a smooth input on (t_0, t_1) such that $x_2(t_1^-) = 0$ and hence no impulse occurs at the switching time t_1; for the mode sequence $(1, 0)$ the input $u(t) = 0$ will result in an impulse free solution for all initial values in $\mathcal{V}_1 = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ ◇

In the case of switched DAEs with a single switch the following characterization of impulse controllability is a simple consequence from the results in [2].

Lemma 15 (cf. [2, Thm. 14 & Lem. 17])**.** *A switched DAE* $(E_\sigma, A_\sigma, B_\sigma) \in \Sigma_1$ *with mode sequence* $\mathbf{q} = (0, 1)$ *is impulse controllable if, and only if,*

$$\operatorname{im}\Pi_0 \subseteq \mathcal{C}_1^{\text{imp}} + \mathcal{R}_0. \tag{9}$$

The single-switch result can directly be used to arrive at a characterization of strong impulse controllability as follows.

Theorem 16. *Consider the system class* $\Sigma_{\mathtt{n}}$ *associated to* $\{E_p, A_p, B_p\}_{p=0}^{\mathtt{n}}$ *with corresponding (individual) consistency projectors* Π_p*, impulse controllable spaces* $\mathcal{C}_p^{\text{imp}}$ *and reachability spaces* $\mathcal{R}_p$*. Then* $\Sigma_{\mathtt{n}}$ *is strongly impulse controllable if, and only if,*

$$\operatorname{im}\Pi_i \subseteq \mathcal{C}_j^{\text{imp}} + \mathcal{R}_i \tag{10}$$

for all $i, j \in \{0, 1, ..., \mathtt{n}\}$*.*

Proof Necessity of (10) is clear by considering switching signals with mode sequences of the form $\mathbf{q} = (i, j, q_2, \ldots, q_{\mathtt{n}})$ together with Lemma 15 and the obvious fact that an impulse-free solution needs to be impulse free on the initial interval $[t_0, t_2)$ as well.

Sufficiency of (10) is also clear by considering each switched system $(E_\sigma, A_\sigma, B_\sigma)$ as a concatenation of single switch switched DAEs and the ability to choose the input independently around the switching times to ensure impulse freeness at each individual switch (as a consequence of Lemma 15). □

Remark 17. The characterization of strong impulse controllability of $\Sigma_{\mathtt{n}}$ via (10) is much simpler than the characterization of impulse-controllability of an individual switched system as given in [2, Thm. 21] which is based on a rather complicated recursive subspace sequence (discussed in detail in the next subsection, see (12)) and depends on the specific mode durations $\boldsymbol{\tau}$. The underlying reason is that strong impulse controllability is by definition independent from the mode durations and, furthermore, can be reduced to the single switch case (as utilized in the proof of Theorem 16).

3.3 Impulse controllability of $\overline{\Sigma}_n$

As can be seen from Theorem 16, verifying whether a system class $\Sigma_{\mathtt{n}}$ is strongly impulse controllable can be done by verifying impulse controllability of all possible single switch switched DAEs. However, if a mode sequence is fixed, these conditions are only sufficient and not necessary in general. In fact, defining strong impulse controllability for $\overline{\Sigma}_n$ analogously as in Definition 12 (see also the forthcoming Defintion 20), we have the following consequence from Lemma 15.

Corollary 18. *The system class $\overline{\Sigma}_n$ of switched systems with fixed mode sequence $\mathbf{q} = (0, 1, 2, \ldots, \mathtt{n})$ is strongly impulse controllable if*

$$\operatorname{im}\Pi_k \subseteq \mathcal{C}^{\text{imp}}_{k+1} + \mathcal{R}_k \quad \forall k \in \{0, 1, \ldots, \mathtt{n}-1\}. \tag{11}$$

The following examples shows that (11) is indeed only sufficient and not necessary in general.

Example 19. Consider the system class $\overline{\Sigma}_{\mathtt{n}}$ with $\mathtt{n} = 2$ and modes $(E_0, A_0, B_0) = (I, 0, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ $(E_1, A_1, B_1) = (I, 0, 0)$ $(E_2, A_2, B_2) = (\left[\begin{smallmatrix}0&0\\1&0\end{smallmatrix}\right], I, 0)$. It is easily seen that $\overline{\Sigma}_{\mathtt{n}}$ is strongly impulse controllable; in fact, for any switching time t_1 and any initial value it is possible to choose the input u on $[0, t_1)$ such $x_1(t_1^-) = 0$, in the second mode the state then remains constant and hence $x_1(t_2^-) = x_1(t_1^-) = 0$ which then implies that at the the last switch x_1 does not jump and hence no Dirac impulse is induced. However, condition (11) is not satisfied for the mode pair $(1, 2)$; indeed $\operatorname{im}\Pi_1 = \mathbb{R}^2$ is not contained in $\mathcal{C}^{\text{imp}}_2 + \mathcal{R}_1 = \operatorname{im}\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right] + \{0\}$.

The above example shows that characterization of impulse controllability of $\overline{\Sigma}_{\mathtt{n}}$ cannot simply be reduced to the single switch case anymore. In particular, it will turn out that it is possible that a switched system with fixed mode sequence has some isolated mode duration for which impulse controllability is lost, but for all remaining mode duration it is impulse controllable. Furthermore, for arbitrary switching signals it is not possible that *none* of the systems in $\Sigma_{\mathtt{n}}$ are impulse uncontrollable (see Remark 13), however, for a fixed mode sequence it is indeed possible, that *all* of the systems in $\overline{\Sigma}_{\mathtt{n}}$ are *not* impulse controllable. Finally, it is also possible that for some specific mode durations a system in $\overline{\Sigma}_{\mathtt{n}}$ is impulse controllable, while for all remaining mode durations the systems are not impulse controllable. This motivates us to introduce the following different notions of impulse controllability for the system class $\overline{\Sigma}_{\mathtt{n}}$.

Definition 20 (Strong and essential impulse (un-)controllability for $\overline{\Sigma}_{\mathtt{n}}$)**.** Consider the class $\overline{\Sigma}_{\mathtt{n}}$ of switched systems (1) with fixed mode sequence $\mathbf{q} = (0, 1, 2, \ldots, \mathtt{n})$ and arbitrary mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$.

- $\overline{\Sigma}_{\mathtt{n}}$ is called *strongly impulse controllable* if all $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ are impulse controllable.
- $\overline{\Sigma}_{\mathtt{n}}$ is called *essentially impulse controllable* if the set of all mode durations $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ of $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ which are not impulse controllable has measure zero in $\mathbb{R}^{\mathtt{n}}_{>0}$.
- $\overline{\Sigma}_{\mathtt{n}}$ is called *strongly impulse uncontrollable* if all $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ are not impulse controllable.
- $\overline{\Sigma}_{\mathtt{n}}$ is called *essentially impulse uncontrollable* if the set of all mode durations $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ of $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ which are impulse controllable has measure zero in $\mathbb{R}^{\mathtt{n}}_{>0}$.

First note that clearly every strongly impulse (un-)controllable system class is also essentially impulse (un-)controllable.

Example 19 already provides a nontrivial example for a strongly impulse controllable $\overline{\Sigma}_{\mathtt{n}}$, and every $\overline{\Sigma}_{\mathtt{n}}$ with two modes which do not satisfy the single-switch impulse controllability condition (9) is an example for a strongly impulse uncontrollable $\overline{\Sigma}_{\mathtt{n}}$. In order to justify the introduction of the notion of essential impulse (un-)controllability we will provide in the following examples which are essentially impulse (un-)controllable but not strongly impulse (un-)controllable.

Example 21 (Essentially, but not strongly, impulse controllable class)**.** Consider the switched system class $\overline{\Sigma}_2$ with modes

$$\begin{aligned}
(E_0, A_0, B_0) &= (I, 0, \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])\\
(E_1, A_1, B_1) &= (I, \left[\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right], 0)\\
(E_2, A_2, B_2) &= (\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right], I, 0).
\end{aligned}$$

For any mode duration $\boldsymbol{\tau} = (\tau_0, \tau_1)$ we see that the solution of the corresponding switched DAE (2) with (arbitrary) initial value $x(0^+) = \left(\begin{smallmatrix}x_{01}\\x_{02}\end{smallmatrix}\right)$ is given by

$$\begin{aligned}
x(t) &= \begin{pmatrix} x_{01} + \int_0^t u \\ x_{02} \end{pmatrix}, \quad t \in (0, t_1),\\
x[t_1] &= 0,\\
x(t) &= \left[\begin{smallmatrix}\cos(t-t_1) & \sin(t-t_1)\\ -\sin(t-t_1) & \cos(t-t_1)\end{smallmatrix}\right] x(t_1^-), \quad t \in (t_1, t_2),\\
x[t_2] &= -\left[\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right] x(t_2^-)\delta_{t_2}\\
x(t) &= 0, \quad t > t_2.
\end{aligned}$$

For the specific mode duration $\tau_2 = 2\pi$ we see that $x(t_2^-) = x(t_1^-)$, hence the second component of $x(t_2^-)$ is x_{02}, independently of the choice of the input u. However, for $x_{02} \neq 0$ this leads to an unavoidable Dirac impulse at $t = t_2$, i.e. $\overline{\Sigma}_{\mathtt{n}}$ is not strongly impulse controllable. On the other hand, for all $\tau_2 \neq k\pi$, it is easily seen that there exists an input u on $(0, t_1)$ resulting in a suitable first entry of $x(t_1^-)$ such that the rotation in mode 1 leads to $x_2(t_2^-)$ having a zero second component and hence resulting in an impulse-free switch at $t = t_2$. This shows that $\overline{\Sigma}_{\mathtt{n}}$ is indeed essentially impulse controllable.

Example 22 (Essentially, but not strongly, impulse uncontrollable class)**.** Consider the switched system class $\overline{\Sigma}_2$ with modes

$$
\begin{aligned}
(E_0, A_0, B_0) &= \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], 0\right) \\
(E_1, A_1, B_1) &= \left(I, \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right], 0\right) \\
(E_2, A_2, B_2) &= \left(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right], I, 0\right).
\end{aligned}
$$

Note that for this example the input is not effecting the dynamics at all, so impulse controllability reduces to impulse freeness. Clearly, the solution in the initial mode is given by $x(t) = \left[\begin{smallmatrix} x_{01} \\ 0 \end{smallmatrix}\right]$ and afterwards the solutions are given as in Example 21 (because modes 1 and 2 are identical to the ones there). Consequently, for $\tau_1 = 2\pi$ we have $x(t_2^-) = x(t_1^-) = \left[\begin{smallmatrix} x_{01} \\ 0 \end{smallmatrix}\right]$, which results in an impulse-free solution of the switched DAE, i.e. $\overline{\Sigma}_2$ is not stronly impulse uncontrollable. Nevertheless, for any $\tau_1 \neq k\pi$ we see that the second component of $x(t_2^-)$ is non-zero (if $x_{01} \neq 0$) and hence a Dirac impulse occurs at $t = t_2$. This means that $\overline{\Sigma}_2$ is essentially impulse uncontrollable.

We are now ready to formulate our first main result concerning impulse controllability of the class of switched DAEs with fixed mode sequence.

Theorem 23. *Consider a class $\overline{\Sigma}_{\mathtt{n}}$ of switched systems* (1) *with fixed mode sequence $\mathbf{q} = (0, 1, 2, \ldots, \mathtt{n})$. Then $\overline{\Sigma}_{\mathtt{n}}$ is either essentially impulse controllable or essentially impulse uncontrollable.*

Proof The proof utilizes properties of analytic matrices which are recalled in the Appendix.

Case 1: All systems in $\overline{\Sigma}_{\mathtt{n}}$ are impulse controllable.
By definition $\overline{\Sigma}_{\mathtt{n}}$ is then strongly impulse controllable and in particular essentially impulse controllable.

Case 2: There exists at least one impulse uncontrollable system in $\overline{\Sigma}_{\mathtt{n}}$.
In view of Lemma 46 in the Appendix we can choose an analytic matrix $N_0 : \mathbb{R}^{\mathtt{n}} \to \mathbb{R}^{n \times k_0}$ with generically full rank such that $\operatorname{im} N_0(\boldsymbol{\tau}) = \mathcal{K}_0^{\boldsymbol{\tau}}$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}$.

Case 2a: For all impulse uncontrollable mode durations $\overline{\boldsymbol{\tau}} \in \mathbb{R}^{\mathtt{n}}_{>0}$ we have that $\operatorname{im} N_0(\overline{\boldsymbol{\tau}}) \neq \mathcal{K}_0^{\overline{\boldsymbol{\tau}}}$ or $N_0(\overline{\boldsymbol{\tau}})$ does not have full rank.
In this case the set of impulse uncontrollable mode durations is contained in a set of measure zero, hence $\overline{\Sigma}_{\mathtt{n}}$ is essentially impulse controllable.

Case 2b: There exists an impulse uncontrollable mode duration $\overline{\boldsymbol{\tau}} \in \mathbb{R}^{\mathtt{n}}_{>0}$ such that $\operatorname{im} N_0(\overline{\boldsymbol{\tau}}) = \mathcal{K}_0^{\overline{\boldsymbol{\tau}}}$ and $N_0(\overline{\boldsymbol{\tau}})$ has full rank.
Since impulse-controllability for a specific switching signal is equivalent to (13) we have

$$\mathcal{V}_{[E_0,A_0,B_0]} \not\subseteq \mathcal{K}_0^{\overline{\boldsymbol{\tau}}} = \operatorname{im} N_0(\overline{\boldsymbol{\tau}}).$$

Hence there exists a vector $v \in \mathcal{V}_{[E_0,A_0,B_0]}$ such that $M(\boldsymbol{\tau}) := \operatorname{rank}[N(\boldsymbol{\tau}), v]$ has full rank for $\boldsymbol{\tau} = \overline{\boldsymbol{\tau}}$. In particular, M is an analytic matrix for which $\boldsymbol{\tau} \mapsto \det M(\boldsymbol{\tau})^\top M(\boldsymbol{\tau})$ is not identically zero, i.e. M is generically full rank. Consequently, $v \notin \operatorname{im} N(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ and hence

$$\mathcal{V}_{[E_0,A_0,B_0]} \not\subseteq \operatorname{im} N_0(\boldsymbol{\tau}) = \mathcal{K}_0^{\boldsymbol{\tau}} \quad \text{for a.a. } \boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}.$$

This implies that almost all systems in $\overline{\Sigma}_{\mathtt{n}}$ are impulse uncontrollable, i.e. $\overline{\Sigma}_{\mathtt{n}}$ is essentially impulse uncontrollable. This concludes the proof as no other cases are possible. □

Remark 24. Theorem 23 states that the classes of switched DAEs with fixed mode sequences fall into four disjoint categories: 1) strongly impulse controllable, 2) essentially (but not strongly) impulse controllable, 3) essentially (but not strongly) impulse uncontrollable, 4) strongly impulse uncontrollable. Interestingly, there are only *three* categories for the notions of observability and controllability for switched systems with a fixed mode sequences (cf. [21] for observability, which by the duality arguments of [22] also carry over to controllability). The underlying reason is that the characterization of impulse controllability is expressed in terms of sums *and* intersections of certain subspaces (see the forthcoming discussion) which can result in a singular dimension drop as well as a singular dimension increase in the involved duration-dependent subspaces; this in contrast to the observability (reachability) subspaces, which only involve intersections (sums).

In order to further investigate the different notions of impulse controllability for the system class $\overline{\Sigma}_{\mathtt{n}}$, we need to introduce certain sequences of subspaces, which are inspired by the backward approach from [2]. For each switched DAE $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ with corrresponding mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$ define

$$\begin{aligned} \mathcal{K}_{\mathtt{n}}^{\boldsymbol{\tau}} &:= \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}, \\ \mathcal{K}_{i-1}^{\boldsymbol{\tau}} &:= \Big(\operatorname{im} \Pi_{i-1} \cap (e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1})\Big) \oplus \mathcal{D}_{i-1}^{\mathrm{imp}}, \\ & \quad i = \mathtt{n}, \mathtt{n}-1, \ldots, 1. \end{aligned} \tag{12}$$

In view of invertibility of each exponential term $e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}}$ in (12) and A_{i-1}^{diff}-inveriance of the subspaces $\operatorname{im} \Pi_{i-1}$ and $\mathcal{R}_{i-1}$, it follows that the recursive definition (12) can equivalently be written as

$$\mathcal{K}_{i-1}^{\boldsymbol{\tau}} = e^{-A_{i-1}^{\mathrm{diff}}\tau_{i-1}} \left(\operatorname{im} \Pi_{i-1} \cap (\mathcal{K}_i^{\boldsymbol{\tau}} + \mathcal{R}_{i-1})\right) \oplus \mathcal{D}_{i-1}^{\mathrm{imp}}.$$

The relevance of the subspaces $\mathcal{K}_{i-1}^{\boldsymbol{\tau}}$ is highlighted by the following result.

Lemma 25 (Cf. [2, Lem. 19]). *Consider a switched DAE $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ with mode duration $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots, \tau_{\mathtt{n}-1}) \in \mathbb{R}^{\mathtt{n}}_{>0}$ and $\mathcal{K}_i^{\boldsymbol{\tau}}$ given by (12). Then*

$$\mathcal{K}_i^{\boldsymbol{\tau}} = \left\{ x_i \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists \textit{ impulse-free sol. } (x,u) \textit{ of } (1) \\ \textit{on } [t_i, t_f) \textit{ with } x(t_i^-) = x_i \end{array} \right\}$$

Proof The proof follows inductively with the same arguments as used in the proof of [2, Lem. 19] and is therefore omitted. □

Corollary 26 ([2, Thm. 21]). *The switched DAE $(E_\sigma, A_\sigma, B_\sigma) \in \overline{\Sigma}_{\mathtt{n}}$ with fixed mode sequence and with mode duration $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$ is impulse controllable if, and only if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \mathcal{K}_0^{\boldsymbol{\tau}}. \tag{13}$$

An obvious characterization of strong impulse (un-)controllability of the system class $\overline{\Sigma}_{\mathtt{n}}$ is therefore the condition that (13) does (not) hold for all $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}}_{>0}$. However, this characterization is not very insightful and impracticable because uncountably many subspace sequence need to be calculated. We can obtain more practible (sufficient) conditions for strong impulse (un-)controllability, by using the fact that for any subspace $\mathcal{S}$, any matrix A and any $t \in \mathbb{R}$ we have

$$\langle \mathcal{S} \mid A \rangle \subseteq e^{At}\mathcal{S} \subseteq \langle A \mid \mathcal{S} \rangle, \tag{14}$$

where $\langle \mathcal{S} \mid A \rangle$ denotes the largest A-invariant subspace contained in $\mathcal{S}$ and $\langle A \mid \mathcal{S} \rangle$ denotes the smallest A-invariant subspace containing $\mathcal{S}$. In fact, we can construct an over- and underestimation of $\mathcal{K}_i^{\boldsymbol{\tau}}$ as follows:

$$\overline{\mathcal{K}}_{i-1} := \left\langle A_{i-1}^{\mathrm{diff}} \mid \operatorname{im} \Pi_{i-1} \cap (\overline{\mathcal{K}}_i + \mathcal{R}_{i-1}) \right\rangle \oplus \mathcal{D}_{i-1}^{\mathrm{imp}}, \tag{15}$$

$$\underline{\mathcal{K}}_{i-1} := \left\langle \operatorname{im} \Pi_{i-1} \cap (\underline{\mathcal{K}}_i + \mathcal{R}_{i-1}) \mid A_{i-1}^{\mathrm{diff}} \right\rangle \oplus \mathcal{D}_{i-1}^{\mathrm{imp}}, \tag{16}$$

each for $i = \mathtt{n}, \mathtt{n}-1, \ldots, 1$ and with $\overline{\mathcal{K}}_{\mathtt{n}} = \underline{\mathcal{K}}_{\mathtt{n}} = \mathcal{C}_{\mathtt{n}}^{\mathrm{imp}}$. By construction we have $\underline{\mathcal{K}}_i \subseteq \mathcal{K}_i^{\boldsymbol{\tau}} \subseteq \overline{\mathcal{K}}_i$, which immediately leads to the following sufficient condition for strong impulse (un-)controllability.

Corollary 27. *The system class $\overline{\Sigma}_{\mathtt{n}}$ is strongly impulse controllable if*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \underline{\mathcal{K}}_0$$

and it is strongly impulse uncontrollable if

$$\mathcal{V}_{(E_0,A_0,B_0)} \not\subseteq \overline{\mathcal{K}}_0.$$

Remark 28. It is also possible to obtain under- and overestimation of $\mathcal{K}_i^{\boldsymbol{\tau}}$ by using (14) directly in (12), however it turns out that this leads to smaller underestimations and bigger overestimations and hence leads to more conservative sufficient conditions.

Remark 29 (Sufficient condition for essential impulse (un-) controllability)**.** It seems there is not simple weaker sufficient condition compared to the ones provided in Corollary 27 to guarantee *essential* impulse (un-) controllability. However, if condition (13) is satisfied for some *random* set of duration times, then $\overline{\Sigma}_{\mathtt{n}}$ is essentially impulse controllable with probability one and if (13) is not satisfied, then $\overline{\Sigma}_{\mathtt{n}}$ is essentially impulse uncontrollable with probability one. In practise this seems to be a reliable way to check for essential impulse (un-)controllability.

4 (Quasi)-causal impulse controllability

So far we have presented several sufficient conditions for strong impulse controllability, which is concerned with the existence of an input (depending on the initial value) which results in an impulse free solution. Clearly, this "impulse-avoiding" input in general depends on the switching signal and in particular for the system class $\overline{\Sigma}_{\mathtt{n}}$ with known mode sequence it is not clear whether an impulse-avoiding input can be constructed *independently* of the (unknown) mode durations. The following example shows, that indeed the impulse-avoiding input may depend on future mode durations.

Example 30 (Non-causal impulse controllability)**.** Consider the class $\overline{\Sigma}_2$ of switched systems with fixed mode sequence $\mathbf{q} = (0, 1, 2)$ and with modes given by

$$\begin{aligned}
(E_0, A_0, B_0) &= (I, 0, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]),\\
(E_1, A_1, B_1) &= (I, \left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right], 0),\\
(E_2, A_2, B_2) &= (\left[\begin{smallmatrix}0&0\\1&1\end{smallmatrix}\right], \left[\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right], 0).
\end{aligned}$$

For a given switching signal with mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1) \in \mathbb{R}^2_{>0}$ the sequence (12) is given by

$$\begin{aligned}
\mathcal{K}_2^{\boldsymbol{\tau}} &= \mathcal{C}_2^{\text{imp}} = \operatorname{im}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\\
\mathcal{K}_1^{\boldsymbol{\tau}} &= \operatorname{span}\left\{e^{A_1\tau_1}\left[\begin{smallmatrix}1\\-1\end{smallmatrix}\right]\right\} = \operatorname{span}\left\{\left[\begin{smallmatrix}1\\e^{-\tau_1}\end{smallmatrix}\right]\right\},\\
\mathcal{K}_0^{\boldsymbol{\tau}} &= \mathcal{K}_1^{\boldsymbol{\tau}} + \mathcal{R}_0 = \operatorname{span}\left\{\left[\begin{smallmatrix}1\\e^{-\tau_1}\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\right\} = \mathbb{R}^2.
\end{aligned}$$

Hence the system class is strongly impulse controllable. However, for two mode durations $\boldsymbol{\tau} = (\tau_0, \tau_1)$ and $\overline{\boldsymbol{\tau}} = (\overline{\tau}_0, \overline{\tau}_1)$ with $\tau_1 \neq \overline{\tau}_1$ we have that

$$\mathcal{K}_1^{\boldsymbol{\tau}} \cap \mathcal{K}_1^{\overline{\boldsymbol{\tau}}} = \{0\}.$$

Since the first mode is not null-controllable, this means that the value of the state $x(t_1^-)$ explicitly depends on the future mode-duration in order to guarantee impulse freeness. For example, for the (consistent) initial condition $x(0^+) = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, it follows for the first state component that $x_1(t_2^-) = 1$ as $\dot{x}_1 = 0$ in the zeroth and first mode. Hence in order to ensure an impulse-free solution it is required that the second state component satisfies $x_2(t_2^-) = -1$. This is achieved if and only if $x_2(t_1^-) = e^{-\tau_1}$. Consequently, the control on the interval $(0, t_1)$ needs to ensure that $x_2(t_1^-) = e^{-\tau_1}$ and therefore necessarily depends on the future mode duration τ_1. ◇

4.1 Quasi-causality of $\overline{\Sigma}_{\mathtt{n}}$

In some application it may be the case that the current mode duration is known once the mode is activated, but the mode durations of the future modes are not known yet; for example, if a switch is induced by shutting down or decoupling components for scheduled maintenance whose duration is known upfront. In this case causality of the input means that it should be independent from the future mode durations, but it can utilize the knowledge when the next switch happens. This somewhat weaker notion of causal impulse controllability is called *quasi-causal impulse controllability* and is defined in terms of the existence of a family of input-defining maps

$$\mathcal{U}_t : (\sigma_{(t_0,t)}, x_0) \mapsto u_{(t_0,t)}$$

such that for all $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$ and all initial values $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ the corresponding solution $(x, u)_{(t_0,t)}$ of $(E_\sigma, A_\sigma, B_\sigma)$ on (t_0, t) satisfying $x(t_0^+) = x_0$ is impulse free. Additionally, we have to require that the map $\mathcal{U}_t$ is itself quasi-causal, *i.e.*, for all switching times t_i and $s > t_i$ the following holds

$$\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0) = \mathcal{U}_s(\sigma_{(t_0,s)}, x_0)_{(t_0,t_i)}. \tag{17}$$

Observe that for two switching signals $\sigma, \bar{\sigma} \in \mathcal{S}_{\mathtt{n}}$ satisfying $\sigma_{(t_0,s)} = \bar{\sigma}_{(t_0,s)}$ for some $s \in (t_i, t_{i+1})$ it may occur that $\mathcal{U}_s(\sigma_{(t_0,s)}, x_0) \neq \mathcal{U}_s(\bar{\sigma}_{(t_0,s)}, x_0)$.

Before presenting conditions for quasi impulse-controllability we will present the following lemma, which is required in the proofs to come.

Lemma 31. *For all* $p \in \{0, 1, ..., , \mathtt{n} - 1\}$ *and* $\underline{\mathcal{K}}_p$ *as in* (16) *we have*

$$\underline{\mathcal{K}}_p = \left\{ x_p \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \forall \tau > 0 \;\exists \text{ impulse-free solution } (x, u) \\ \text{on } [t_p, t_p + \tau) \text{ of } E_p\dot{x} = A_p x + B_p u, \\ \text{with } x(t_p^-) = x_p \text{ and } x(\tau^-) \in \underline{\mathcal{K}}_{p+1} \end{array} \right\},$$

i.e. the subspace $\underline{\mathcal{K}}_p$ *consists of all initial states for mode* p *which can be controlled impulse-freely into the subspace* $\underline{\mathcal{K}}_{p+1}$ *within a given time duration* $\tau > 0$.

Before providing the proof we want to highlight that in the statement above the impulse avoiding input in general depends on τ, i.e. on the mode duration of the current mode, whereas the subspaces given by (16) are independent from the mode duration (but depend on the mode sequence).

Proof Let $x_p \in \underline{\mathcal{K}}_p$. Then $x_p = w + v$ for some $w \in \langle \operatorname{im}\Pi_p \cap \big(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\big) \mid A_p^{\text{diff}} \rangle$ and $v \in \mathcal{D}_p^{\text{imp}}$. Recall that any $v \in \mathcal{D}_p^{\text{imp}}$ can be impulse-freely controlled to zero with a smooth input for any given time duration $\tau > 0$. Hence, in view of linearity, it suffices to consider the case $x_p \in \langle \operatorname{im}\Pi_p \cap \big(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\big) \mid A_p^{\text{diff}} \rangle$. It follows then from A_p^{diff}-invariance that for $\tau \in \mathbb{R}$

$$e^{A_p^{\text{diff}}\tau}\Pi_p x_p = k_{p+1}^\tau + \eta^\tau.$$

for some $k_{p+1}^\tau \in \underline{\mathcal{K}}_{p+1}$ and $\eta^\tau \in \mathcal{R}_p$. In particular, there exists a smooth input u defined on $[t_p, t_p + \tau)$ which stears the state x from zero to $-\eta^\tau$. Applying the same input for the initial value $x(t_p^-) = x_p$ results in

$$\begin{aligned} x_u((t_p+\tau)^-, x_p) &= e^{A_p^{\text{diff}}\tau}\Pi_p x_p - \eta^\tau \\ &= k_{p+1}^\tau + \eta^\tau - \eta^\tau \\ &= k_{p+1}^\tau \end{aligned}$$

as desired.

Conversely, let x_p be such that for all τ there exists an impulse free solution (x, u) of $E_p\dot{x} = A_p x + B_p u$ with $x(t_p^-) = x_p$ and $x((t_p+\tau)^-) \in \underline{\mathcal{K}}_{p+1}$. Using the same inductive arguments as in Lemma 25 and utilizing A_p^{diff} invariance of $\operatorname{im}\Pi_p$, $\mathcal{R}_p$, $\mathcal{D}_p^{\text{imp}}$, it then follows for all $\tau \in \mathbb{R}$ that

$$\begin{aligned} x_p &\in \operatorname{im}\Pi_p \cap \big(e^{-A_p^{\text{diff}}\tau}\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\big) \oplus \mathcal{D}_p^{\text{imp}} \\ &= e^{-A_p^{\text{diff}}\tau}\Big(\operatorname{im}\Pi_p \cap \big(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\big) \oplus \mathcal{D}_p^{\text{imp}}\Big) \end{aligned}$$

As this holds for all $\tau > 0$ we obtain

$$\begin{aligned} x_p &\in \bigcap_{\tau>0} e^{-A_p^{\text{diff}}\tau}\Big(\operatorname{im}\Pi_p \cap \big(\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p\big) \oplus \mathcal{D}_p^{\text{imp}}\Big) \\ &= \langle \operatorname{im}\Pi_p \cap (\underline{\mathcal{K}}_{p+1} + \mathcal{R}_p) \oplus \mathcal{D}_p^{\text{imp}} \mid A_p^{\text{diff}} \rangle = \underline{\mathcal{K}}_p, \end{aligned}$$

which follows from the general facts, that $\bigcap_{\tau>0} e^{-A\tau}\mathcal{V} = \langle \mathcal{V} \mid A \rangle$ and $\langle \mathcal{V} + \mathcal{W} \mid A \rangle = \langle \mathcal{V} \mid A \rangle + \mathcal{W}$ for any matrix $A \in \mathbb{R}^{n\times n}$ and any subspaces $\mathcal{V}, \mathcal{W} \subseteq \mathbb{R}^n$ of which $\mathcal{W}$ is A-invariant. This concludes the proof. □

Given this result, we can present the following simple characterization of quasi-causally impulse controllable system classes.

Theorem 32. *The system class $\overline{\Sigma}_{\mathrm{n}}$ is quasi-causally impulse-controllable if and only if*

$$\mathcal{V}_{[E_0,A_0,B_0]} \subseteq \underline{\mathcal{K}}_0$$

Proof ($\Rightarrow$) Suppose the system class is quasi-causally impulse controllable. Consider the solution (x, u) of (1) with $x(t_0^+) = x_0$ and $u_{(t_0,t_f)}$ given by $\mathcal{U}_{t_f}(\sigma_{(t_0,t_f)}, x_0)$. Then by definition, the solution (x, u) is impulse-free on (t_0, t_f), in particular, $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}} = \underline{\mathcal{K}}_{\mathtt{n}}$ for all possible switching signals.

In the following, we want to show by induction that $x(t_i^-) \in \underline{\mathcal{K}}_i$ for $i \in \{\mathtt{n} - 1, \dots, 1, 0\}$. Hence, inductively, we may assume that if (x, u) satisfies $x(t_0^+) = x_0$ and u is defined by $\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0)$, then $x(t_i^-) \in \underline{\mathcal{K}}_i$ for all switching signals. We want to show that $x(t_{i-1}^-) \in \underline{\mathcal{K}}_{i-1}$ for any solution (x, u) of (1) with $x(t_0^+) = x_0$ and u given by $\mathcal{U}_{t_{i-1}}(\sigma_{(t_0,t_{i-1})}, x_0)$. For any $\tau > 0$ consider the switching signal $\bar{\sigma}$ with $\bar{\sigma}_{(t_0,t_{i-1})} = \sigma_{(t_0,t_{i-1})}$ and $\bar{t}_i = \bar{t}_{i-1} + \tau = t_{i-1} + \tau$. Let $\bar{u}$ be given by $\mathcal{U}_{\bar{\sqcup}_\rangle}(\bar{\sigma}_{(\sqcup_/,\bar{\sqcup}_\rangle)}, \S_/)$, then the corresponding solution $(\bar{x}, \bar{u})$ is impulse-free and by induction assumption satisfies $\bar{x}(\bar{t}_i) \in \underline{\mathcal{K}}_i$. Since $\tau > 0$ was arbitary, Lemma 31 yields that $\bar{x}(t_{i-1}^-) \in \underline{\mathcal{K}}_{i-1}$ By causality, $u_{(t_0,t_{i-1})} = \bar{u}_{(t_0,t_{i-1})}$ and hence $x(t_{i-1}^-) = \bar{x}(t_{i-1}^-)$ which concludes the inductive proof. Since for all $x_0 \in \mathcal{V}_{[E_0,A_0,B_0]}$ there exists an impulse-free solution (x, u) satisfying $x(t_0^+) = x(t_0^-) = x_0$ we can conclude that $x_0 \in \underline{\mathcal{K}}_0$ and hence

$$\mathcal{V}_{[E_0,A_0,B_0]} \subseteq \underline{\mathcal{K}}_0.$$

($\Leftarrow$) Let $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$. Recall that by definition for all $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$, for each mode $p \in \{0, 1, ..., \mathtt{n} - 1\}$ and each $x_p \in \underline{\mathcal{K}}_p$ there exists an input $u^p(\cdot, x_p)$ on $[t_p, t_{p+1})$ such that the solution x of mode p satisfies $x(t_p^-) = x_p$ and $x(t_{p+1}^-) \in \underline{\mathcal{K}}_{p+1}$. Now, concatenate these inputs inductively as follows: $u(t) := u^0(t, x_0)$ for $t \in [t_0, t_1)$ and $u(t) := u^p(t, x(t_p^-))$ for $t \in [t_p, t_{p+1})$ where $x(t_p^-)$ is the value of the solution x corresponding to the already defined input u on $[t_0, t_p)$. Finally, by assumption $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}}$, hence the input u can be extended on $[t_{\mathtt{n}}, \infty)$ in such a way that the solution remains impulse-free. Altogether we can define $\mathcal{U}_{t_i}(\sigma_{(t_0,t_i)}, x_0) := u_{(t_0,t_i)}$ which satisfies the quasi-causality properties for all switching signals and all x_0. Hence the system class is quasi-causally impulse-controllable. ☐

4.2 Causal impulse-controllability of $\overline{\Sigma}_{\mathtt{n}}$

Knowledge of the current mode duration can not always be assumed, hence we want to provide in this subsection a characterization of a more strict causality notion. In particular, we make the above definition of quasi-causal impulse controllability stronger by requiring the causality property (17) of U_t to hold *for all* $t \in (t_0, \infty)$ and not only for the switching times $t = t_i$ of the corresponding switching signal. A key idea to characterize this stronger notion of causality are so called *controlled invariant subspaces* which are subspaces associated to a DAE $E\dot{x} = Ax + Bu$ which have the property that for any initial value in such a subspace there exists an input u such that the trajectory x remains in that subspace, cf. [23, 24]. It is well known that any controlled invariant subspace $\mathcal{V} \subseteq \mathcal{V}_{E,A,B}$ is (A, E, B)-invariant, i.e. $A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$; in particular, the augmented consistency space $\mathcal{V}_{E,A,B}$ is the largest controlled invariant subspace. For the class $\overline{\Sigma}_{\mathtt{n}}$ of switched DAEs with known mode sequence to be causally impulse it is now intuitively clear that at the the switch the state trajectory has to jump immediately into a controlled invariant subspace which is contained in a suitable subspace for the following mode. This intuition is formalized by the following sequence of subspaces

$$\underline{\mathcal{C}}_{i-1} := \langle \underline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle + \ker E_{i-1},$$

for $i \in \{\mathtt{n}, \mathtt{n}-1, ..., 1\}$ and with $\underline{\mathcal{C}}_{\mathtt{n}} := \mathcal{C}_{\mathtt{n}}^{\text{imp}}$; furthermore, $\langle \underline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle$ denotes the largest $(A_{i-1}, E_{i-1}, B_{i-1})$ invariant subspace contained in $\overline{\mathcal{C}}_i$. Note that such a subspace can be calculated with a subspace sequence similar to the Wong sequences, see [14, Thm. 10].

Theorem 33. *The system class $\overline{\Sigma}_{\mathtt{n}}$ is causally impulse controllable if, and only if,*

$$\mathcal{V}_{(E_0,A_0,B_0)} \subseteq \underline{\mathcal{C}}_0. \tag{18}$$

Proof (⇒) Suppose the system class $\overline{\Sigma}_{\mathtt{n}}$ is causally impulse controllable. Then for any given switching signal $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$ there exists an impulse-free solution (x, u) where $u_{[t_0,t)} = \mathcal{U}_t(\sigma_{[t_0,t)}, x_0)$.

We will proof by induction that $x(t_i^-) \in \underline{\mathcal{C}}_i$ for all $i \in \{\mathtt{n}, \mathtt{n}-1, ..., 1\}$. Since (x, u) is impulse-free, it follows that $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}} = \underline{\mathcal{C}}_{\mathtt{n}}$. Hence we assume that the statement holds for i and continue to proof the statement for $i-1$. Consider now another switching signal $\tilde{\sigma} \in \overline{\mathcal{S}}_{\mathtt{n}}$ such that $\sigma_{(t_0,t_i)} = \tilde{\sigma}_{(t_0,t_i)}$ (in particular, $\tilde{t}_i \geq t_i$) and with corresponding impulse free solution $(\tilde{x}, \tilde{u})$, where $\tilde{u}_{[t_0,t)} = \mathcal{U}_t(\tilde{\sigma}_{[t_0,t)}, x_0)$ By the inductive assumption we have $\tilde{x}(\tilde{t}_i^-) \in \underline{\mathcal{C}}_i$. Consequently, we can always find an input $\tilde{u}$ on $[t_i, \tilde{t}_i)$ which ensures that the trajectory $\tilde{x}$ which starts at $x(t_i^-) \in \underline{\mathcal{C}}_i$ stays in the same subspace for arbitrary $\tilde{t}_i > t_i$ under the dynamics of $E_{i-1}\dot{x} = A_{i-1}x + B_{i-1}u$. Consequently, $x(t_i^-)$ must be contained in the largest controlled invariant subspace within $\overline{\mathcal{C}}_i$, i.e. $x(t_i^-) \in \langle \underline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle$. Since this is true for any mode duration $t_i - t_{i-1}$ it follows that $x(t_{i-1}^+) \in \langle \underline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle$. Since x is impulse free, it follows that $x(t_{i-1}^-) - x(t_{i-1}^+) \in \ker E_{i-1}$ (otherwise the Dirac impulse occuring in $\dot{x}$ at t_{i-1} must also occur on the right hand side of the DAE, which is not possible because x and u are impulse free), this shows that $x(t_{i-1}^-) \in \underline{\mathcal{C}}_{i-1}$. Now we can conclude that $x_0 \in \underline{\mathcal{C}}_0$ and since this holds for all $x_0 \in \mathcal{V}_{(E_0,A_0,B_0)}$ we have shown the necessity part of the statement. (⇐) Let $x_i \in \underline{\mathcal{C}}_i$, then there exists $x_i^+ \in \langle \underline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle \subseteq \mathcal{V}_{(E_i,A_i,B_i)}$ and $\xi_i \in \ker E_i$ such that $x_i = x_i^+ + \xi_i$. Choose an input u on $[t_i, \infty)$ such that the solution x of $E_i\dot{x} = Ax_i + B_iu$ with consistent initial condition $x(t_i^+) = x_i^+$ satisfies $x(t^+) \in \langle \underline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle \subseteq \underline{\mathcal{C}}_{i+1}$ for all $t \in [t_i, \infty)$. Furthermore, observe that the zero distribution on $[t_i, \infty)$ is an (impulse free) solution of the (inconsistent) initial value problem $E_i\dot{x} = A_ix$, $x(t_i^-) = \xi_i$. Consequently, the previously chosen (x, u) is also a solution of $E_i\dot{x} = Ax_i + B_iu$ with inconsistent initial value $x(t_i^-) = x_i$. Hence for a given switching signal and (consistent) initial condition $x_0 \in \underline{\mathcal{C}}_0$, we can successively construct an input (independent of the mode durations), such that the resulting solution x is impulse free and satisfies $x(t_i^-) \in \underline{\mathcal{C}}_{i-1}$. In particular, $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}}$ which implies that u can be defined on $[t_{\mathtt{n}}, \infty)$ such that the resulting solution remains impulse free, which concludes the proof. □

The above condition on causal impulse controllability is in most situation too restrictive because the controller must be designed in such a way that at a switch the correct input must be chosen to avoid a Dirac impulse and at the same time the state right after the switch must be an element of a controlled invariant subspace contained in the impulse controllable subspace of the next mode. This is required because if some non-instanteneous control action is needed to drive the state into a suitable subspace, then this control input (which needs a duration $d > 0$ to arrive at that subspace) would not work for a switching duration smaller than this d (and hence causality would be violated). However, in most practical situation, a dwell time for the switching signal can be assumed, i.e. there exists $d > 0$ such that $t_{i+1} - t_i \geq d$ for all switching times. Under such a dwell-time condition, we are able to prove a less restrictive characterization of causal impulse-controllability. Towards this goal, we define an enlarged version of the subspace sequence (18) for the system class $\overline{\Sigma}_{\mathtt{n}}$ as follows:

$$\overline{\mathcal{C}}_{i-1} := \left\langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \right\rangle + \mathcal{R}_{i-1} + \ker E_{i-1} \tag{19}$$

for $i \in \{\mathtt{n}, \mathtt{n}-1, ..., 1\}$.

Theorem 34. *The system class $\overline{\Sigma}_{\mathtt{n}}$ with some dwell time $d > 0$ is causally impulse controllable if and only if*

$$\mathcal{V}_{(E_0, A_0, B_0)} \subseteq \overline{\mathcal{C}}_0.$$

Note that the above characterization of causal impulse controllability is independent of the dwell time $d > 0$, however the input map $\mathcal{U}_t$ will depend on it. The proof of the theorem utilizes the following property of (A, E, B)-invariant subspaces.

Lemma 35. *Let (E, A) be a regular matrix pair with corresponding consistency projector Π and flow matrix A^{diff}. Then for any (A, E, B) invariant subspace $\mathcal{V}$ we have*

a) $\Pi\mathcal{V} \subseteq \langle \mathcal{V} + \mathcal{R} \mid A^{\mathrm{diff}} \rangle \subseteq \mathcal{V} + \mathcal{R}$,

b) $A^{\mathrm{diff}}\mathcal{V} \subseteq \mathcal{V} + \mathcal{R}$.

Proof a) Let $x \in \mathcal{V}$. Then there exists an input such that $x(t) \in \mathcal{V}$ for all $t \geq 0$. Consequently,

$$e^{A^{\mathrm{diff}} t}\Pi x_0 \in \mathcal{V} + \mathcal{R}$$

for all $t \geq 0$, i.e. $\Pi x_0 \in \bigcap_{t>0} e^{-A^{\mathrm{diff}} t}(\mathcal{V} + \mathcal{R})$. Hence $\Pi x_0 \in \langle \mathcal{V} + \mathcal{R} \mid A^{\mathrm{diff}} \rangle$.

b) Since $A^{\mathrm{diff}}\Pi = A^{\mathrm{diff}}$ it follows from a) that for each $x \in \mathcal{V}$,

$$\begin{aligned} A^{\mathrm{diff}} x = A^{\mathrm{diff}}\Pi x &\in A^{\mathrm{diff}}\langle \mathcal{V} + \mathcal{R} \mid A^{\mathrm{diff}} \rangle \\ &\subseteq \langle \mathcal{V} + \mathcal{R} \mid A^{\mathrm{diff}} \rangle \subseteq \mathcal{V} + \mathcal{R}. \end{aligned}$$

□

Proof of Theorem 34 ($\Rightarrow$) Suppose the system class $\overline{\Sigma}_{\mathtt{n}}$ with dwell time $d > 0$ is causally impulse controllable. Then for any given switching signal $\sigma \in \overline{\mathcal{S}}_{\mathtt{n}}$ (with dwell time $d > 0$) there exists an impulse-free solution (x, u) where $u_{[t_0,t)} = \mathcal{U}_t(\sigma_{[t_0,t)}, x_0)$.

We will proof by induction that $x(t_i^-) \in \underline{\mathcal{C}}_i$ for all $i \in \{\mathtt{n}, \mathtt{n}{-}1, ..., 1\}$. Since (x, u) is impulse-free, it follows that $x(t_{\mathtt{n}}^-) \in \mathcal{C}_{\mathtt{n}}^{\text{imp}} = \underline{\mathcal{C}}_{\mathtt{n}}$. Hence we assume that the statement holds for i and continue to proof the statement for $i - 1$. Using the same arguments as in the proof of Theorem 33, we can show that $x(t_i^-) \in \langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle$.

Consequently, it follows from the solution formula for differential algebraic equations that

$$e^{A_{i-1}^{\text{diff}} \tau_{i-1}} \Pi_{i-1} x(t_{i-1}^-) \in \langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1}$$

and hence

$$\begin{aligned} &\Pi_{i-1} x(t_{i-1}^-) \\ &\in e^{-A_{i-1}^{\text{diff}} \tau_{i-1}} \langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle + e^{-A_{i-1}^{\text{diff}} \tau_{i-1}} \mathcal{R}_{i-1} \\ &\subseteq \langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1}, \end{aligned}$$

where we utilized that $\mathcal{R}_{i-1}$ is A_{i-1}^{diff}-invariant together with Lemma 35.b).

Since (x, u) is impulse-free it follows that $x(t_{i-1}^-) \in \mathcal{C}_{i-1}^{\text{imp}}$ and hence $(I - \Pi_{i-1}) x(t_{i-1}^-) \in \mathcal{R}_{i-1} + \ker E_{i-1}$. Altogether, we conclude the inductive proof by observing that

$$\begin{aligned} x(t_{i-1}^-) &= \Pi_{i-1} x(t_{i-1}^-) + (I - \Pi_{i-1}) x(t_{i-1}^-) \\ &\in \langle \overline{\mathcal{C}}_i \mid A_{i-1}, E_{i-1}, B_{i-1} \rangle + \mathcal{R}_{i-1} + \ker E_{i-1} \\ &= \overline{\mathcal{C}}_{i-1}. \end{aligned}$$

Now we can conclude that $x_0 \in \overline{\mathcal{C}}_0$ and since this holds for all $x_0 \in \mathcal{V}_{[E_0, A_0, B_0]}$ we have shown the necessity part of the statement.

($\Leftarrow$) Let $x_i \in \overline{\mathcal{C}}_i \in \mathcal{V}_{(E_i, A_i, B_i)} + \ker E_i = \mathcal{C}_i^{\text{imp}}$, hence there exists an input $\widehat{u}$ on $[t_i, t_{i+1} + d)$ such that the corresponding solution $\widehat{x}$ of mode i with (inconsistent) initial condition $\widehat{x}(t_i^-) = x_i$ is impulse free. Furthermore, $\widehat{x}((t_i{+}d)^-) = e^{A_i^{\text{diff}} d} \Pi_i x_i + \widehat{\eta}_i$ for some $\widehat{\eta}_i \in \mathcal{R}_i$. From Lemma 35 it follows that

$$\begin{aligned} e^{A_i^{\text{diff}} d} \Pi_i x_i &\in e^{A_i^{\text{diff}} d} \Pi_i \left(\langle \overline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle + \mathcal{R}_i + \ker E_i \right). \\ &\subseteq \langle \overline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle + \mathcal{R}_i. \end{aligned}$$

Consequently, $e^{A_i^{\text{diff}} d} \Pi_i x_i = c_{i+1} + \eta_i$ for some $c_{i+1} \in \langle \overline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle$ and $\eta_i \in \mathcal{R}_i$.

Now choose a smooth input $\widetilde{u}$ on $[t_i, t_i + d)$ such that corresponding solution $\widetilde{x}$ of mode i with initial condition $\widetilde{x}(t_i^-) = 0$, satisfies $\widetilde{x}((t_i + d)^-) = -\eta_i - \widehat{\eta}_i$. Now let $u := \widehat{u} + \widetilde{u}$ then, by linearity, the corresponding solution x of mode i with (inconsistent) initial condition $x(t_i^-) = x_i$ is impulse free and satisfies

$$\begin{aligned} x((t_i + d)^-) &= \widehat{x}((t_i + d)^-) + \widetilde{x}((t_i + d)^-) \\ &= e^{A_i^{\text{diff}} d} \Pi_i x_i + \widehat{\eta}_i - \eta_i - \widehat{\eta}_i = c_{i+1}. \end{aligned}$$

Due to the controlled invariance of $\langle \overline{\mathcal{C}}_{i+1} \mid A_i, E_i, B_i \rangle$ it is possible to extend u onto $[t_i, t_{i+1})$ such that the corresponding solution satisfies $x(t^-) \in \overline{\mathcal{C}}_{i+1}^{\text{imp}}$ for all $t \in [t_i + d, t_{i+1})$. Now, concatenate these inputs inductively with the corresponding initial conditions $x(t_i^-)$ obtained from the previous input it follows that the overall input is causal (in particular, independent of the mode duration) and achieves and impulse free solution on $[t_0, t_i)$ with $x(t_i^-) \in \overline{\mathcal{C}}_i$, $i = 1, 2, \dots \mathtt{n}$. Finally, by assumption $x(t_\mathtt{n}^-) \in \mathcal{C}_\mathtt{n}^{\text{imp}}$, hence the input u can by extended also on $[t_\mathtt{n}, \infty)$ in such a way that the solution remains impulse free. Altogether, we can define $\mathcal{U}(\sigma_{[t_0,t)}, x_0) := u_{[t_0,t)}$ which satisfies the causality properties with a dwell time for all switching signals and all x_0. □

Remark 36. Since $\langle \operatorname{im} \Pi_0 \cap (\underline{\mathcal{K}}_1 + \mathcal{R}_0) \mid A_0^{\text{diff}} \rangle \subseteq \operatorname{im} \Pi_0 \subseteq \mathcal{V}_{(E_0,A_0,B_0)}$, $\langle \overline{\mathcal{C}}_1 \mid A_0, E_0, B_0 \rangle \subseteq \mathcal{V}_{(E_0,A_0,B_0)}$ and $\langle \underline{\mathcal{C}}_1 \mid A_0, E_0, B_0 \rangle \subseteq \mathcal{V}_{(E_0,A_0,B_0)}$ and, by Lemma 8,

$$\mathcal{C}_0^{\text{imp}} = \operatorname{im} \Pi_0 + \mathcal{R}_0 + \ker E_0 = \mathcal{V}_{(E_0,A_0,B_0)} + \ker E_0,$$

it follows that

$$\ker E_0 \subseteq \underline{\mathcal{C}}_0 \subseteq \overline{\mathcal{C}}_0 \subseteq \underline{\mathcal{K}}_0 \subseteq \mathcal{C}_0^{\text{imp}}.$$

Consequently, we have the following equivalent characterizations for quasi-causal impulse controllability, causal impulse-controllability and causal impulse-controllability with a dwell-time of $\overline{\Sigma}_\mathtt{n}$, respectively:

$$\begin{aligned} \mathcal{C}_0^{\text{imp}} &= \underline{\mathcal{K}}_0, \\ \mathcal{C}_0^{\text{imp}} &= \underline{\mathcal{C}}_0, \\ \mathcal{C}_0^{\text{imp}} &= \overline{\mathcal{C}}_0. \end{aligned}$$

4.3 Causal impulse controllability for $\Sigma_\mathtt{n}$

We conclude this section by considering causality also for the case of unknown mode sequence, i.e. for the system class $\Sigma_\mathtt{n}$. The definition of (quasi)-causality given above carries over to the system class $\Sigma_\mathtt{n}$ without change (apart from considering switching signals in $\mathcal{S}_\mathtt{n}$ instead of $\overline{\mathcal{S}}_\mathtt{n}$). Since $\Sigma_\mathtt{n}$ contains all switched systems with a single switch, we can immediately necessary conditions for (quasi-) causal impulse controllability (with dwell time). In fact, similar as in Theorem 16 these necessary conditions turn out to be sufficient as well.

Corollary 37. *Consider the system class of switched systems $\Sigma_\mathtt{n}$ of switched DAEs with arbitrary mode sequence and arbitrary mode durations.*

a) $\Sigma_\mathtt{n}$ is quasi-causally impulse controllable if, and only if, for all $i, j \in \{0, 1, \dots, \mathtt{n}\}$

$$\mathcal{C}_i^{\text{imp}} = \langle \operatorname{im} \Pi_i \cap (\mathcal{C}_j^{\text{imp}} + \mathcal{R}_i) \mid A_i^{\text{diff}} \rangle \oplus \mathcal{D}_i^{\text{imp}}.$$

b) $\Sigma_\mathtt{n}$ is causally impulse controllable if, and only if, for all $i, j \in \{0, 1, \dots, \mathtt{n}\}$

$$\mathcal{C}_i^{\text{imp}} = \langle \mathcal{C}_j^{\text{imp}} \mid E_i, A_i, B_i \rangle + \ker E_i.$$

c) $\Sigma_{\mathtt{n}}$ with dwell time $d > 0$ is causally impulse controllable if, and only if, for all $i, j \in \{0, 1, \ldots, \mathtt{n}\}$

$$\mathcal{C}_i^{\text{imp}} = \langle \mathcal{C}_j^{\text{imp}} \mid E_i, A_i, B_i \rangle + \mathcal{R}_i + \ker E_i.$$

5 Conclusion

In this paper impulse-controllability of system classes of switched DAEs have been considered. It was shown that strong impulse-controllability of system classes generated by arbitrary switching signals is equivalent to impulse-controllability of every switched system with a single switch. In the case the system class contains systems with a fixed mode sequence, either all or almost all systems are impulse-(un)controllable and sufficient conditions for strong impulse-(un)controllability are given. Finally, we considered the notions of (quasi-) causal impulse-controllability and controllability and characterized system classes with these properties.

A natural direction of research is to design controllers that achieve impulse-free solutions. In the case of causal impulse-controllable systems, it seems that there should exist a switched feedback controller that guarantees impulse-free solutions. However, for systems in a class that is causally impulse-controllable given some dwell-time or quasi-causally impulse-controllable, the controller design seems not so straight forward. Furthermore, it remains an open question whether simple necessary conditions for essential impulse-(un)controllability of system classes can be stated.

Acknowledgments. This work was supported by the NWO Vidi-grant 639.032.733.

A Proofs

A.1 Proof of Theorem 23

The proof of Theorem 23 relies on utilizing properties of analytic functions, which are recalled first.

Definition 38. A function $f : \mathbb{R}^p \to \mathbb{R}$ is called *analytic* if for each $x \in \mathbb{R}^p$ the function f may be presented by a convergent power series in some neighborhood of x.

A useful property of analytic functions is the following well known result.

Lemma 39 (Cf. [25, Cor I.A.10])**.** *The zero-set of a non-trivial analytic function $f : \mathbb{R}^p \to \mathbb{R}$ has (Lebesgue) measure zero.*

The notion of analycity can be extended to matrix-valued function as follows.

Definition 40. The matrix valued function $M : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ is called an *analytic matrix* if each entry $m_{ij} : \mathbb{R}^p \to \mathbb{R}$ of M is an analytic function.

Definition 41. A analytic matrix $M : \mathbb{R}^p \to \mathbb{R}^{m\times n}$ is called *generically full rank* if either $\det\big(M(\boldsymbol{\tau})^\top M(\boldsymbol{\tau})\big) \neq 0$ for almost all[1] $\boldsymbol{\tau} \in \mathbb{R}^p$ or $\det\big(M(\boldsymbol{\tau})M(\boldsymbol{\tau})^\top\big) \neq 0$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.

Lemma 42. *Let $A \in \mathbb{R}^{n\times n}$, $W : \mathbb{R}^p \to \mathbb{R}^{n\times k}$ a generically full rank analytic matrix and $\mathcal{R} \subseteq \mathbb{R}^n$ some subspace. Then there exists an analytic matrix $N : \mathbb{R}^{p+1} \to \mathbb{R}^{n\times q}$ which is generically full rank and such that for a.a. $(\tau_0, \boldsymbol{\tau}) \in \mathbb{R}^{p+1}$*

$$\operatorname{im} N(\tau_0, \boldsymbol{\tau}) = e^{A\tau_0} \operatorname{im} W(\boldsymbol{\tau}) + \mathcal{R}. \tag{20}$$

Proof We use $\mathcal{N}_{\tau_0,\boldsymbol{\tau}} \subseteq \mathbb{R}^n$ as short hand notation for the right-hand side of (20) in the following. Pick any $(\overline{\tau}_0, \overline{\boldsymbol{\tau}}) \in \mathbb{R}^{p+1}$ such that $\dim \mathcal{N}_{\overline{\tau}_0,\overline{\boldsymbol{\tau}}} = \max_{(\tau_0,\boldsymbol{\tau})} \dim \mathcal{N}_{\tau_0,\boldsymbol{\tau}} =: q$ and let $r_1, ..., r_l \in \mathbb{R}^n$ be a basis of $\mathcal{R}$. Choose $B_W \in \mathbb{R}^{k\times(q-l)}$ such that $[\overline{w}_1, \dots, \overline{w}_{q-l}] = W(\overline{\boldsymbol{\tau}})B_W$ yields a basis

$$r_1, \dots, r_l, e^{A\overline{\tau}_0}\overline{w}_1, \dots, e^{A\overline{\tau}_0}\overline{w}_{q-l}$$

of $\mathcal{N}_{\overline{\tau}_0,\overline{\boldsymbol{\tau}}}$. Consider now the matrix valued function $N : \mathbb{R}^{p+1} \to \mathbb{R}^{n\times q}$ defined by

$$N(\tau_0, \boldsymbol{\tau}) := \Big[r_1, \dots, r_l, e^{A\tau_0} W(\boldsymbol{\tau})B_W\Big].$$

This matrix is analytic because the matrix exponential is analytic and the product of two analytic matrices is again analytic. By construction

$$\det\Big(N(\overline{\tau}_0, \overline{\tau})^\top N(\overline{\tau}_0, \overline{\tau})\Big) \neq 0,$$

and hence the analytic function $(\tau_0, \boldsymbol{\tau}) \mapsto \det\Big(N(\tau_0, \boldsymbol{\tau})^\top N(\tau_0, \boldsymbol{\tau})\Big)$ is not identically zero. In view of Lemma 39 it therefore follows that N is generically full rank.

It remains to be shown that (20) holds. By construction, $\operatorname{im} N(\tau_0, \boldsymbol{\tau}) \subseteq \mathcal{N}_{\tau_0,\boldsymbol{\tau}}$ for all $(\tau_0, \boldsymbol{\tau}) \in \mathbb{R}^{p+1}$. Furthermore, since $\dim \mathcal{N}_{\tau_0,\boldsymbol{\tau}} \leq q$ and $\dim \operatorname{im} N(\tau_0, \boldsymbol{\tau}) = q$ for a.a. $(\tau_0, \boldsymbol{\tau}) \in \mathbb{R}^{p+1}$ the claim follows. □

Remark 43. It is indeed possible that for some specific $(\tau_0, \boldsymbol{\tau})$ we have $\operatorname{im} N(\tau_0, \boldsymbol{\tau}) \subsetneq N_{\tau_0,\boldsymbol{\tau}}$. As an example consider for $\alpha > 0$

$$\begin{aligned} W(\tau_1) &= \operatorname{span}\left\{\left[\begin{smallmatrix} e_1^\tau \\ e_1^\tau - e^\alpha \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ e_1^\tau \end{smallmatrix}\right]\right\} := \operatorname{span}\{w_1(\tau_1), w_2(\tau_2)\}, \\ \mathcal{R} &= \operatorname{span}\left\{\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]\right\} := \operatorname{span}\{r_1\}, \quad A = 0. \end{aligned}$$

Then clearly, $e^{A\tau_0}W(\tau_1) + \mathcal{R} = \mathbb{R}^2$ for all $(\tau_0, \tau_1) \in \mathbb{R}^2$. However, while the choice $N(\tau_0, \tau_1) := [r_1, w_1(\tau_1)]$ satisfies

$$\operatorname{im} N(\tau_0, \tau_1) = e^{A\tau_0}W(\tau_1) + \mathcal{R} = \mathbb{R}^2 \quad \text{for a.a. } (\tau_0, \tau_1),$$

[1] A property $P(\boldsymbol{\tau})$ is said to hold for almost all (a.a.) $\boldsymbol{\tau} \in \mathbb{R}^p$, if there exists $S \subseteq \mathbb{R}^p$ of Lebesgue measure zero, such that $P(\boldsymbol{\tau})$ holds for all $\boldsymbol{\tau} \in \mathbb{R}^p \setminus S$

for $\tau_1 = \alpha$ we have

$$\operatorname{im} N(\tau_0, \alpha) = \operatorname{span}\{r_1\} \neq \mathbb{R}^2.$$

Lemma 44. *Let $W : \mathbb{R}^p \to \mathbb{R}^{q \times n}$, $n > q$, be an analytic matrix with generically full rank. Then there exists an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n \times (n-q)}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \ker W(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.*

Proof By considering the field of meromorphic functions (i.e. fractions of scalar-valued analytic functions), we can apply Gauss-Jordan eliminations on $W(\boldsymbol{\tau})$ to obtain a reduced row echolon form (RREF), which contains meromorphic entries and whose kernel for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$ equals $\ker W(\boldsymbol{\tau})$. Identically as for constant matrices, a full rank matrix $\overline{N}(\boldsymbol{\tau}) \in \mathbb{R}^{n \times (n-q)}$, can be easily constructed from the (meromorphic) entries of the obtained RREF such that $W(\boldsymbol{\tau})\overline{N}(\boldsymbol{\tau}) = 0$ for all $\boldsymbol{\tau}$ for which $\overline{N}(\boldsymbol{\tau})$ is well-defined. As a final step, let $N(\boldsymbol{\tau}) = \overline{N}(\boldsymbol{\tau}) \begin{bmatrix} \alpha_1(\boldsymbol{\tau}) & & \\ & \ddots & \\ & & \alpha_{n-q}(\boldsymbol{\tau}) \end{bmatrix}$, where $\alpha_i(\boldsymbol{\tau})$ is the product of all denominators of the entries in the i-th column of $\overline{N}(\boldsymbol{\tau})$. Then $M(\boldsymbol{\tau})N(\boldsymbol{\tau}) = 0$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$ and $\boldsymbol{\tau} \mapsto N(\boldsymbol{\tau})$ is an analytic matrix and has generically the same rank as $\overline{N}$, i.e. N is generically full rank. □

Lemma 45. *Let $W : \mathbb{R}^p \to \mathbb{R}^{n \times k}$, $k \leq n$, be an analytic matrix with generically full rank. Then for any $\Pi \in \mathbb{R}^{n \times n}$ there exists an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n \times m}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$.*

Proof By Lemma 44 there exists an analytic matrix $\overline{N} : \mathbb{R}^p \to \mathbb{R}^{n \times q}$ with generically full rank and $\operatorname{im} \overline{N}(\boldsymbol{\tau}) = \ker W(\boldsymbol{\tau})^\top$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^p$. Consequently,

$$\begin{aligned} (\operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau}))^\perp &= \ker \Pi^\top + \ker W(\boldsymbol{\tau})^\top, \\ &= \ker \Pi^\top + \operatorname{im} \overline{N}(\boldsymbol{\tau}). \end{aligned}$$

Applying Lemma 42 for $\mathcal{R} = \ker \Pi^\top$ and $A = 0$, we find an analytic matrix $\widetilde{N} : \mathbb{R}^p \to \mathbb{R}^{n \times \widetilde{q}}$ with generically full rank such that $\operatorname{im} \widetilde{N}(\boldsymbol{\tau}) = \ker \Pi^\top + \operatorname{im} \overline{M}(\boldsymbol{\tau})$ for a.a. $\boldsymbol{\tau}$. Finally, using Lemma 44 again we can find an analytic matrix $N : \mathbb{R}^p \to \mathbb{R}^{n \times q}$, $q = n - \widetilde{q}$ with generically full rank such that $\operatorname{im} N(\boldsymbol{\tau}) = \ker \widetilde{N}(\boldsymbol{\tau})^\top$ for a.a. $\boldsymbol{\tau}$. Altogether, we have for a.a. $\boldsymbol{\tau}$

$$\operatorname{im} \Pi \cap \operatorname{im} W(\boldsymbol{\tau}) = \left(\operatorname{im} \widetilde{N}(\boldsymbol{\tau})\right)^\perp = \ker \widetilde{N}(\boldsymbol{\tau})^\top = \operatorname{im} N(\boldsymbol{\tau}).$$

□

Lemma 46. *Consider the sequence (12). Then for all $i \in \{0, 1, ..., \mathtt{n}\}$ there exists an analytic matrix $N_i : \mathbb{R}^{\mathtt{n}-i} \to \mathbb{R}^{n \times k_i}$ with generically full rank such that $\operatorname{im} N_i(\boldsymbol{\tau}) = \mathcal{K}_i^{\boldsymbol{\tau}}$ for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}-i}$.*

Proof For $i = \mathtt{n}$ we use the convention that a constant full rank matrix is interpreted as an analytic matrix depending on an empty tuple $\boldsymbol{\tau} = () \in \mathbb{R}^0$, then the claim is correct by simply choosing the columns of $N_{\mathtt{n}}(\boldsymbol{\tau})$ as a (constant) basis of $\mathcal{C}_{\mathtt{n}}^{\text{imp}}$. We now proceed inductively and assume the claim is correct for some $i \in \{1, 2, \ldots, \mathtt{n}\}$. Let $\mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}} := e^{-A_{i-1}^{\text{diff}}\tau_{i-1}} \operatorname{im} N_i(\boldsymbol{\tau}) + \mathcal{R}_{i-1}$ and $\mathcal{R}_{i-1}^{\text{imp}} := \langle E_{i-1}^{\text{imp}} \mid B_{i-1}^{\text{imp}} \rangle + \ker E_{i-1}$, then

$$\mathcal{K}_{i-1}^{(\tau_{i-1},\boldsymbol{\tau})} = \left(\operatorname{im} \Pi_{i-1} \cap \mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}}\right) + \mathcal{R}_{i-1}^{\text{imp}}$$

for a.a. $\boldsymbol{\tau} \in \mathbb{R}^{\mathtt{n}-i}$ and all $\tau_{i-1} \in \mathbb{R}$. Utilizing Lemmas 42 and 45 we find analytic and generically full rank matrices $\widetilde{N}_{i-1} : \mathbb{R}^{\mathtt{n}-(i-1)} \to \mathbb{R}^{n \times \widetilde{k}_i}$, $\overline{N}_{i-1} : \mathbb{R}^{\mathtt{n}-i+1} \to \mathbb{R}^{n \times \overline{k}_i}$, $N_{i-1} : \mathbb{R}^{\mathtt{n}-(i-1)} \to \mathbb{R}^{n \times k_i}$ such that a.a. $(\tau_{i-1}, \boldsymbol{\tau}) \in \mathbb{R}^{\mathtt{n}-(i-1)}$

$$\begin{aligned}
\operatorname{im} \widetilde{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \mathcal{N}_{\tau_{i-1},\boldsymbol{\tau}}, \\
\operatorname{im} \overline{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \operatorname{im} \Pi_{i-1} \cap \operatorname{im} \widetilde{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}), \\
\operatorname{im} N_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) &= \operatorname{im} \overline{N}_{i-1}(\tau_{i-1}, \boldsymbol{\tau}) + \mathcal{R}_{i-1}^{\text{imp}},
\end{aligned}$$

i.e. $\mathcal{K}_{i-1}^{(\tau_{i-1},\boldsymbol{\tau})} = \operatorname{im} N_{i-1}(\tau_{i-1}, \boldsymbol{\tau})$ as desired. □

With these preliminary results related to analytic matrices we are now in the position to proof Theorem 23.

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