An averaged model for switched systems with state jumps applicable for PWM descriptor systems

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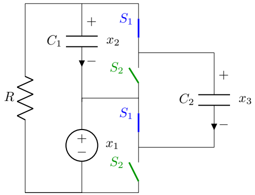
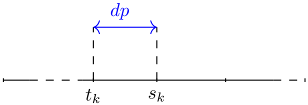
Why state jumps?



- $x_3 = x_2$



Switched descriptor systems can be modeled as switched differential equations with jumps at the switching time instants:

$$x^+ = \Pi_1 x^-, \quad t = t_k = kp \qquad\qquad x^+ = \Pi_2 x^-, \quad t = s_k = kp + dp$$

$$\dot{x} = F_1 x, \quad t \in (t_k, s_k) \qquad\qquad \dot{x} = F_2 x, \quad t \in (s_k, t_{k+1})$$

where t_k are the switching time instants at the multiple of the PWM period p, $s_k = kp + dp$ are the switching time instants internal to the period (d is the duty cycle) and $k \in \mathbb{N}$ is the index of the PWM period p.

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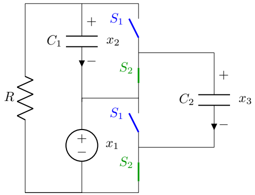

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- $x_3 = x_1$
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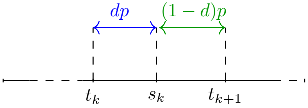

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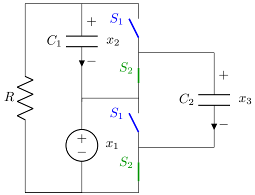

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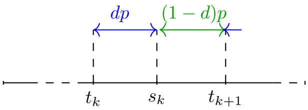

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Contents

Classical averaging for linear switched systems

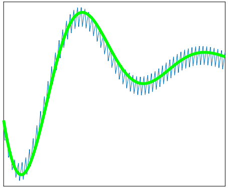
Switched system

$$\dot{x} = F_1 x,\ t \in [t_k, s_k)$$
$$\dot{x} = F_2 x,\ t \in [s_k, t_{k+1})$$

Smooth averaged system

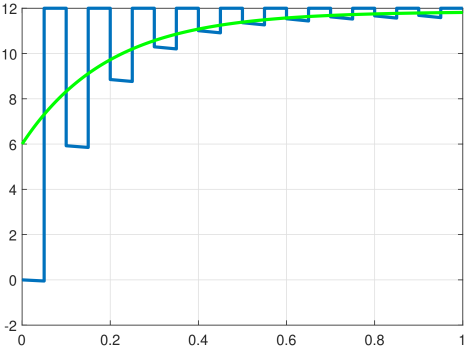
$$\dot{x}_{\text{av}} = F_{\text{av}} x$$
$$F_{\text{av}} = F_1 d + F_2 (1 - d),$$

In the presence of fast switchings the solution of the averaged model approximates that of the switched model with an error that goes to zero with the same order of the switching period.



$$x(t) = x_{av}(t) + \mathrm{O}(p)$$

Averaging for switched systems with state jumps



The research question:

Is it possible to apply averaging in the presence of state jumps?

Averaging for switched systems with state jumps

Switched system with state jump

$$\begin{aligned} x^+ &= \Pi_1 x^-, \quad t = t_k = kp \\ \dot{x} &= F_1 x, \quad t \in (t_k, s_k) \\ x^+ &= \Pi_2 x^-, \quad t = s_k = kp + dp \\ \dot{x} &= F_2 x, \quad t \in (s_k, t_{k+1}) \end{aligned}$$

Averaged model

$$\begin{aligned} \dot{x}_{\text{av}} &= \Pi_\cap F_{\text{av}} \Pi_\cap x_{\text{av}}, \\ x_{\text{av}}(0) &= \Pi_\cap x_0 \\ \Pi_\cap &= \Pi_2 \Pi_1 \\ F_{\text{av}} &= F_1 d + F_2 (1-d) \end{aligned}$$

Existing results require that Π_1 and Π_2 are idempotent matrices and one of the following conditions is satisfied

1. commutativity property $\Pi_2\Pi_1 = \Pi_1\Pi_2 \implies \Pi_\cap$ is idempotent
2. $\operatorname{im}\Pi_\cap \subseteq \operatorname{im}\Pi_i$ and $\ker\Pi_\cap \supseteq \ker\Pi_i$ with $i = 1, 2$.

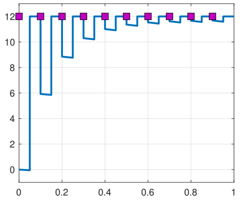
None of these conditions are satisfied for the switched capacitor circuit considered as a motivating example.

Contents

A discrete-time model

Switched system with state jump

$$x^+ = \Pi_1 x^-, \quad t = t_k = kp$$
$$\dot{x} = F_1 x, \quad t \in (t_k, s_k)$$
$$x^+ = \Pi_2 x^-, \quad t = s_k = kp + dp$$
$$\dot{x} = F_2 x, \quad t \in (s_k, t_{k+1})$$



Discrete-time model

$$z_{k+1} = \Phi_p z_k$$
$$\Phi_p = \Pi_\cap + \Lambda p, \quad \Pi_\cap = \Pi_2 \Pi_1$$
$$\Lambda = \Pi_2 F_1 d + F_2 \Pi_1 (1-d)$$
$$k = 0, 1, 2, \ldots$$

We are interested to compare the exact solution of the switched system at the multiple of the switching period before the jump, say x_k^-, with the solution z_k.

A discrete-time model

The solutions of the two systems at the multiple of the switching period p can be written as:

Switched system with state jump

$$x_k = \Theta_p^k x_0$$
$$\Theta_p = e^{F_2(1-d)p}\Pi_2 e^{F_1 dp}\Pi_1$$

Discrete-time model

$$z_k = \Phi_p^k z_0$$
$$\Phi_p = \Pi_\cap + \Lambda p, \quad \Pi_\cap = \Pi_2\Pi_1$$
$$\Lambda = \Pi_2 F_1 d + F_2 \Pi_1 (1-d)$$

Lemma

Assume there exists $\gamma \geq 0$ *such that* $\|\Phi_p\| \leq 1 + \gamma p$. *Then*

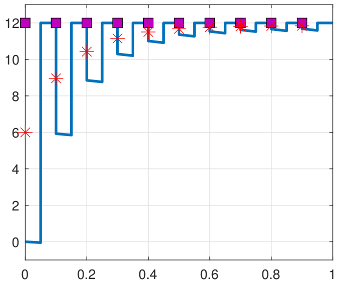
$$x_k^- = z_k + \mathrm{O}(p)$$

holds for all $k \in \mathbb{N}$ *and any* $z_0 = x_0^- + \mathrm{O}(p)$.

A discrete-time model with output

Switched system with state jump

$$m_k = \frac{1}{p}\int_{kp}^{(k+1)p} x(t)\mathrm{d}t$$



Discrete-time model with output

$$\begin{aligned} z_{k+1} &= \Phi_p z_k \\ \mu_k &= \Gamma z_k \\ \Phi_p &= \Pi_\cap + \Lambda p, \quad \Pi_\cap = \Pi_2\Pi_1 \\ \Lambda &= \Pi_2 F_1 d + F_2 \Pi_1 (1-d) \\ \Gamma &= \Pi_1 d + \Pi_\cap (1-d) \end{aligned}$$

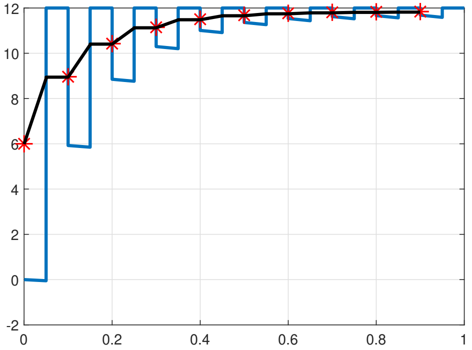
Lemma

Assume there exists $\gamma \geq 0$ such that $\|\Phi_p\| \leq 1 + \gamma p$. Then

$$m_k = \mu_k + \mathrm{O}(p)$$

holds for all $k \in \mathbb{N}$ and any $z_0 = x_0^- + \mathrm{O}(p)$.

The main result



The research question:

Is the solution of the discrete model able to approximate the moving average of the exact solution of the switching system?

The main result

Moving average of switched system

$$
\begin{aligned}
x^+ &= \Pi_1 x^-, \quad t = t_k = kp \\
\dot{x} &= F_1 x, \quad t \in (t_k, s_k) \\
x^+ &= \Pi_2 x^-, \quad t = s_k = kp + dp \\
\dot{x} &= F_2 x, \quad t \in (s_k, t_{k+1})
\end{aligned}
$$

$$
m(t) = \frac{1}{p}\int_t^{t+p} x(\tau)\mathrm{d}\tau
$$

Discrete-time model with output

$$
\begin{aligned}
z_{k+1} &= \Phi_p z_k \\
\mu_k &= \Gamma z_k \\
\Phi_p &= \Pi_\cap + \Lambda p, \quad \Pi_\cap = \Pi_2 \Pi_1 \\
\Lambda &= \Pi_2 F_1 d + F_2 \Pi_1 (1-d) \\
\Gamma &= \Pi_1 d + \Pi_\cap (1-d)
\end{aligned}
$$

Theorem

Assume exists $\gamma \geq 0$ *such that* $\|\Phi_p\| \leq 1 + \gamma p$ *and* $(\Pi_\cap - I)\Phi_p^k = \mathrm{O}(p)$. *Then*

$$
m(t) = \mu_k + \mathrm{O}(p)
$$

holds for any $t \in (0, T-p]$ *and for all* $z_0 = x_0^- + \mathrm{O}(p)$ *where* $k = \frac{t}{p}$.

The main result: an intepretation

The algebraic condition

$$(\Pi_\cap - I)\Phi_p^k = \mathrm{O}(p) \tag{1}$$

with

$$\begin{aligned}\Phi_p &= \Pi_\cap + \Lambda p \\ \Pi_\cap &= \Pi_2\Pi_1 \\ \Lambda &= \Pi_2 F_1 d + F_2 \Pi_1 (1-d)\end{aligned}$$

ensures that the solution of the proposed discrete-time model approximates the moving average of the real solution with an error of order of the switching period p.

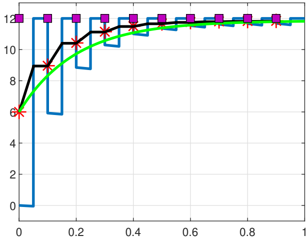
If $\Pi_\cap$ is idempotent and commutes with Λ then $(\Pi_\cap - I)\Phi_p^k = \mathrm{O}(p)$ is trivial.

Hence, the interpretation of (1) is that sufficiently high powers of $\Pi_\cap$ should behave similarly to an idempotent matrix and "approximately should commute" with Λ.

The conjectured continuous-time averaged model

Switched system with state jump

$$\begin{aligned} x^+ &= \Pi_1 x^-, \quad t = t_k = kp \\ \dot{x} &= F_1 x, \quad t \in (t_k, s_k) \\ x^+ &= \Pi_2 x^-, \quad t = s_k = kp + dp \\ \dot{x} &= F_2 x, \quad t \in (s_k, t_{k+1}) \end{aligned}$$



Continuous averaged model

$$\begin{aligned} \dot{\tilde{z}} &= A_p \tilde{z}, \quad t \in \mathbb{R}_+ \\ \mu &= \Gamma \tilde{z} \\ A_p &= \frac{1}{p}(\Phi_p - I) \\ \Phi_p &= \Pi_\cap + \Lambda p, \quad \Pi_\cap = \Pi_2 \Pi_1 \\ \Lambda &= \Pi_2 F_1 d + F_2 \Pi_1 (1-d) \\ \Gamma &= \Pi_1 d + \Pi_\cap (1-d) \end{aligned}$$

If $\Pi_i = I$ for $i = 1, 2$ then $\Phi_p = I + \Lambda p$ and $A = F_1 d + F_2(1-d)$ which corresponds to the classical continuous-time averaged model for switched ODEs.

Contents

Conclusion and perspective

- A new averaged model for PWM descriptor systems has been proposed.
- The solution of the discrete-time averaged model approximates the moving average of the solution of the original system with an error of order of the switching period.
- The averaging result requires weaker conditions with respect to the averaged models presented in the previous literature.
- Numerical simulations of a switched capacitor circuit have validated the theoretical results.
- Future work will be the formal analysis of the conjectured continuous-time averaged model and the extension of the averaging results to the case of more than two modes and time-varying duty cycles.

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