Funnel Control for Input Saturated System

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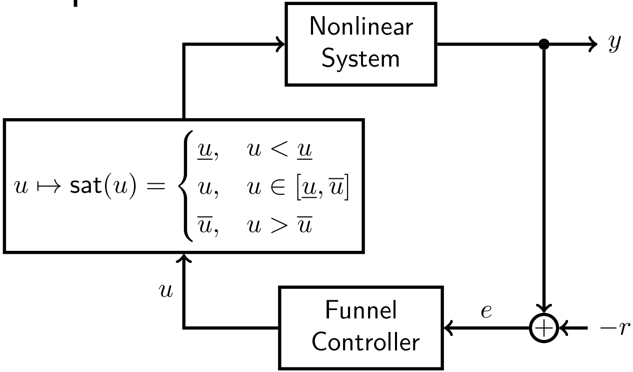
European Control Conference, 2022

Control setup



Goal: Output tracking with saturated input with prespecified performance (funnel)

Non-assumptions:

- No exact knowledge of controlled system.
- No future knowledge or model for reference signal.

System class

Consider SISO system

$$\dot{y} = f(p_f, y, z) + g(p_g, y, z) \cdot \mathrm{sat}_{[\underline{u}, \overline{u}]}(u), \qquad y(0) = y^0 \in \mathbb{R}, \qquad (1a)$$

$$\dot{z} = h(p_h, y, z), \qquad z(0) = z^0 \in \mathbb{R}^{n-1}, \qquad (1b)$$

A1 $g(p_g, y, z) > 0$ for all p_g, y and z.

A2 BIBO-stability of zero-dynamics: for all bounded and continuous p_h, y the solutions of (1b) satisfy

$$\|z(t)\| \leq b_z\big(\|p_{h[0,t)}\|_\infty, \|y_{[0,t)}\|_\infty, \|z(0)\|\big),$$

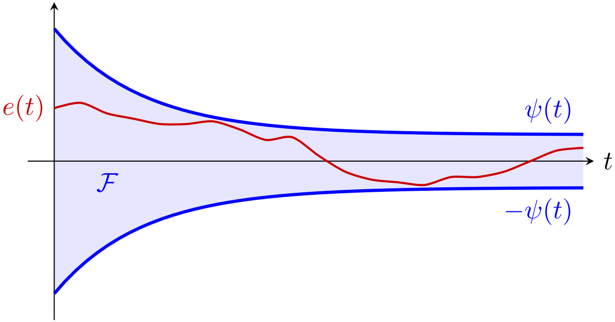
for some continuous function b_z. Furthermore, assume $z_0 \in Z_0$ for some bounded $Z_0 \subset \mathbb{R}^{n-1}$.

A3 The perturbations p_f, p_g and p_h are bounded by $p_f^{\max}$, $p_g^{\max}$, $p_h^{\max}$, respectively.

Funnel

Funnel = time-varying strict error bound $\psi : [0,\infty) \to (0,\infty)$

Control objective: $|e(t)| < \psi(t)$ for all $t \geq 0$, where $e(t) := y(t) - r(t)$



$$\mathcal{F} = \mathcal{F}(\psi) := \{(t,e) \mid |e| < \psi(t)\}$$

Classical funnel control

Further assumptions:

A4 $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is continuously differentiable, bounded and with bounded derivative; furthermore $\liminf_{t\to\infty} \psi(t) > 0$

A5 Reference signal $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable, bounded with bounded derivative.

A6 Initial error: $e(0) := y^0 - r(0) \in (\psi^-(0), \psi^+(0))$.

Theorem (cf. Ilchmann, Ryan, Sangwin 2002)

Under above assumptions, the funnel controller

$$u(t) = -k(t)e(t) \text{ with } k(t) = \frac{1}{\psi(t) - |e(t)|}$$

ensures that $e(t)$ remains within funnel $\mathcal{F}(\psi)$ while $k(t)$ remains bounded.

Short comings of classical funnel control

$$u(t) = -k(t)e(t) \text{ with } k(t) = \frac{1}{\psi(t) - |e(t)|}$$

Shortcomings from traditional funnel controller

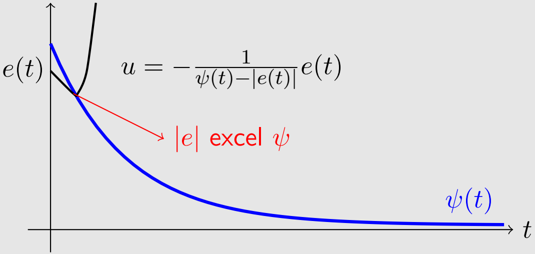
S1 Unbounded gain for asymptotic tracking
Asymptotic tracking $\Leftrightarrow \quad \psi(t) \underset{t\to\infty}{\to} 0 \quad \Rightarrow \quad \psi(t) - |e(t)| \underset{t\to\infty}{\to} 0$
$\Leftrightarrow \quad k(t) \underset{t\to\infty}{\to} \infty$

S2 The structure of such controller is quite fragile: once the needed input value exceeds the saturation bounds and the error leaves the funnel then the gain has the wrong sign and further destabilizes the closed loop.

Funnel control

Shortcomings from traditional funnel controller

S2 Traditional controller may further destabilizes the closed loop



Ratio based funnel controller

For input saturated system, we propose a ratio based funnel controller

$$u(t) = \begin{cases} \alpha\big(\eta(t)\big) \cdot \underline{u}, & 0 \leq e(t) \leq \psi(t), \\ \alpha\big(\text{-}\eta(t)\big) \cdot \overline{u}, & \text{-}\psi(t) \leq e(t) < 0, \end{cases} \tag{2}$$

where $\eta(t) := e(t)/\psi(t)$, $\alpha : [0,1] \to [0,1]$ is continuous, $\alpha(0) = 0$, $\alpha(1) = 1$.

Theorem

Under the same assumptions as before, the ratio based funnel controller ensures $|e(t)| \leq \psi(t)$ *if*

$$\underline{u} < \frac{\min_{t\geq 0}\left(\dot{\psi}^{+}(t) + \dot{r}(t)\right) - F_{\max}}{G_{\min}}$$

$$\overline{u} > \frac{\max_{t\geq 0}(\dot{\psi}^{-}(t) + \dot{r}(t)) - F_{\min}}{G_{\min}}$$

Key proof steps

i) Standard ODE theory ensures closed loop has a unique maximal solution $(y,z):[0,\overline{\omega})\to\mathbb{R}\times\mathbb{R}^{n-1}$ for some $\overline{\omega}>0$.

ii) Seeking a contradiction, assume there is a minimal finite $\omega\in(0,\overline{\omega})$ such that $e(\omega)=\psi(\omega)$ (analogous arguments for $e(\omega)=-\psi(\omega)$).

iii) Boundedness of e, y, and z
$e(t)$ in funnel for $t\in[0,\omega]$ $\implies$ y bounded on $[0,\omega]$ $\implies$ z bounded on $[0,\omega]$.

iv) Boundedness of f and g (independent of ω)

$$F_{\min/\max} := \min/\max_{|p_f|\leq p_f^{\max},|y|\leq Y^{\max},|z|\leq Z^{\max}} f(p_f,y,z)$$

$$G_{\min} := \min_{|p_g|\leq p_g^{\max},|y|\leq Y^{\max},|z|\leq Z^{\max}} g(p_g,y,z) > 0$$

v) Contradicting conclusion for $\dot{e}(\omega)$

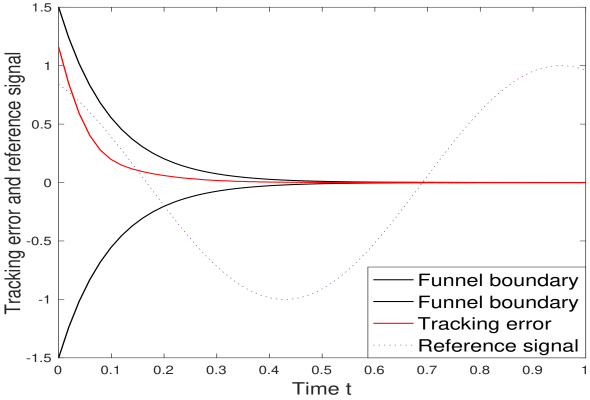
$$\begin{aligned}\dot{e}(\omega) &= f(p_f(\omega),y(\omega),z(\omega)) + g(p_g(\omega),y(\omega),z(\omega))\underline{u} - \dot{r}(\omega)\\ &\leq F_{\max} + G_{\min}\underline{u} - \dot{r}(\omega) < \min_{t\geq 0}(\dot{\psi}(t)+\dot{r}(t)) - \dot{r}(\omega) \leq \dot{\psi}(\omega)\end{aligned}$$

Example with sufficiently large input saturations

Consider system

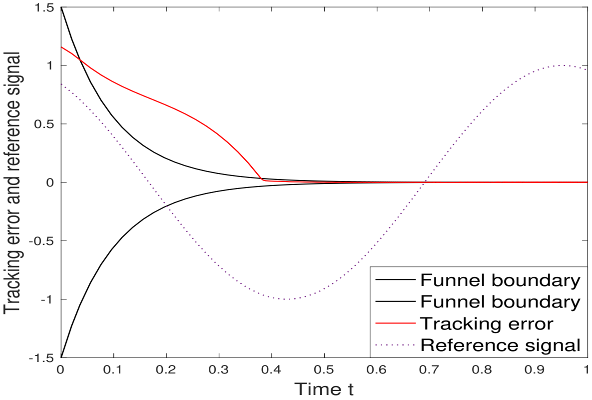
$$\dot{y}(t) = 2 + \sin(t) + u(t), \quad y(0) = 0.5,$$

with $r(t) = -\sin(6t-1)$, $\psi(t) = 1.5e^{-10t}$, $\alpha(\eta) = \eta$, $\overline{u} = 30$, $\underline{u} = -30$



Same example with insufficient input

Consider same system with input saturations $\overline{u} = 10$, $\underline{u} = -\overline{u}$



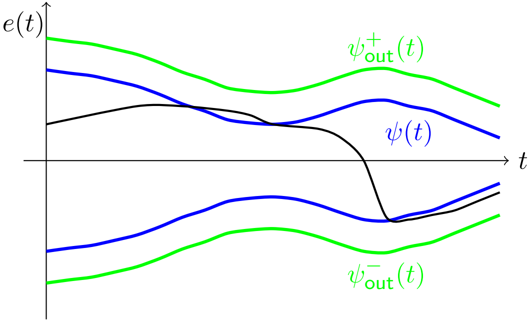
Here $\alpha(\eta) = 1$ for $\eta > 0$ trivially extends the domain of the funnel controller globally

Constrained funnel

Example shows: insufficient saturations can be handled robustly

Question

Can we provide guarantees for the behaviour outside the original funnel?



Goal: Define constraint funnel ψ_{out} for which $|e(t)| < \psi_{\text{out}}$

Feasibility requirement

We assume saturation values satisfy

$$\overline{d} = \overline{d}(\overline{u}) := \overline{F}_{\min} + \overline{G}_{\min}\overline{u} - \sup_{t\geq 0} \dot{r} > 0,$$

$$\underline{d} = \underline{d}(\underline{u}) := \overline{F}_{\max} + \overline{G}_{\min}\underline{u} - \inf_{t\geq 0} \dot{r} < 0,$$

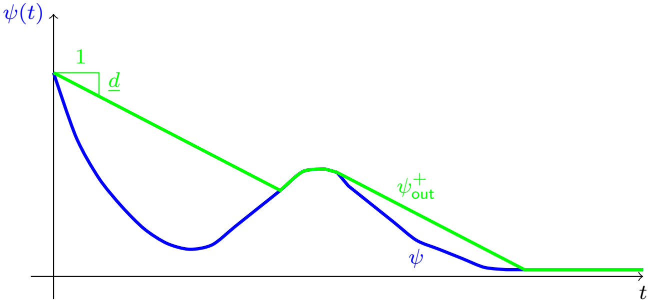
Intuition

$u(t) = \overline{u} \quad \Longrightarrow \quad \dot{e}(t) \geq \overline{d} > 0$ $\qquad$ $u(t) = \underline{u} \quad \Longrightarrow \quad \dot{e}(t) \geq \underline{d} < 0$

Constrained funnel

Given the actual prescribed funnel ψ and the feasibility constants $\underline{d} < 0 < \overline{d}$ we can now construct the constrained funnel as follows:

$$\begin{aligned} \psi^+_{\text{out}}(t) &:= \min\left\{s \geq \psi(\tau) + \underline{d}\cdot(t-\tau) \;\middle|\; \tau \in [0,t]\right\} \\ \psi^-_{\text{out}}(t) &:= \max\left\{s \leq -\psi(\tau) + \overline{d}\cdot(t-\tau) \;\middle|\; \tau \in [0,t]\right\} \end{aligned} \tag{3}$$



Constrained funnel

1. Extending the domain of the controller definition (2) also for $e \notin [-\psi, \psi]$ by simply setting $\alpha(\eta) = 1$ for all $\eta > 1$.
2. Defining a *constraint funnel*

$$\mathcal{F}_{\text{out}} := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \psi^-_{\text{out}}(t) \leq e \leq \psi^+_{\text{out}}(t) \right\}$$

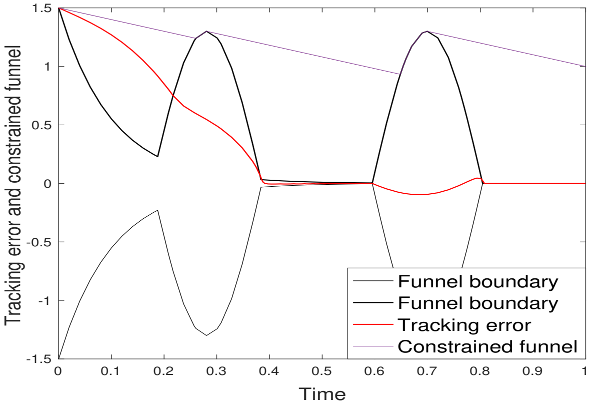
Theorem

Under the same assumptions as before together with $\overline{d}(\overline{u}) > 0$ and $\underline{d}(\underline{u}) < 0$ the ratio-based funnel controller ensures

$$e(t) \in [\psi^-_{out}(t), \psi^+_{out}(t)], \quad \forall t \geq 0$$

Example

Consider system with insufficient control input $\overline{u} = 10$, $\underline{u} = -\overline{u}$. $y(0) = 1.5$, $\psi(t) = \max\{1.5e^{-10t}, 1.3\sin(15t + 0.5)\}$.



Conclusion

Problem statement

- System class
- Funnel

Ratio based funnel controller

- Controller design
- Examples

Constrained funnel boundary

- Constrained funnel design
- Example