A weak Kalman decomposition approach for reduced realizations of switched linear systems

Md Sumon Hossain * **Stephan Trenn** *

* *Bernoulli Institute for Mathematics, Computer Science, and Artificial Intelligence, University of Groningen, The Netherlands.*

Abstract: We propose a novel reduction approach for switched linear systems with a fixed mode sequence based on subspaces related to the (time-varying) reachable and unobservable spaces. These subspaces are defined in such a way that they can be used to construct a weak Kalman decomposition, which is then in turn used to define a reduced switched linear system with an identical input-output behavior. The proposed method is illustrated with a low dimensional academic example.

Keywords: Kalman decomposition, reachability, observability, switched systems.

1. INTRODUCTION

Realization theory is an important and central topic of system theory. The realization problem deals with finding an equivalent internal description of a dynamical system from an external one. In general, it provides a theoretical foundation for model reduction, system identification and observer design. Indeed, transforming a system to a minimal order by preserving its input-output behavior could be seen as the first step towards model reduction.

Realization theory of switched systems has already been discussed in the literature, e.g. Petreczky (2006, 2007); Petreczky and van Schuppen (2010); Petreczky et al. (2013) and the references therein; however, in most of these references the switching signal is viewed as an "input", i.e. it is not possible to use these results when trying to find a (minimal) realization when a specific switching signal is given.

Without discussing realization theory, observability and reachability of switched systems have been studied in Sun et al. (2002); Tanwani et al. (2013); Petreczky et al. (2015); Küsters and Trenn (2018) and our approach is strongly inspired by these results.

We consider switched linear systems (SLS) of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)} x_k(t) + B_{\sigma(t)} u(t), \ t \in (s_k, s_{k+1}) \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)} x_{k-1}(s_k^-), \quad k \in \mathcal{Q} \\ y(t) = C_{\sigma(t)} x_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{1}$$

where $\sigma : \mathbb{R} \to \mathcal{Q} = \{1, 2, \ldots, \mathtt{m}\} \subseteq \mathbb{N}$ is the given switching signal with finitely many switching times $s_1 < s_2 < \ldots < s_\mathtt{m}$ in the bounded interval $[t_0, t_f)$ of interest and $x_k : (s_k, s_{k+1}) \to \mathbb{R}^{n_k}$ is the k-th piece of the state (whose dimension n_k may depend on the mode). For notational convenience, let $s_0 := t_0$, $s_{\mathtt{m}+1} := t_f$ of length $\tau_k := s_{k+1} - s_k$, $k \in \{0, 1, \ldots, \mathtt{m}\}$. We assume a zero initial condition i.e. we set $x_{-1}(t_0^-) := 0$. The input $u : \mathbb{R} \to \mathbb{R}^m$ and the output $y : \mathbb{R} \to \mathbb{R}^m$ in (1) are assumed to have the same dimension m. Here, $x(t^-)$ and $x(t^+)$ denote, respectively, the left- and right-sided limit at t, assuming these limits exists.

For each $p \in \{0, 1, 2, \ldots, \mathtt{m}\}$, the system matrices A_p, B_p, C_p of appropriate size describe the (continuous) dynamics corresponding to the linear system active in mode $p \in \{0, 1, 2, \ldots, \mathtt{m}\}$. Furthermore, $J_{p,q} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is the jump map from mode q to mode p. Note that due the different space dimensions the introduction of a jump map is necessary; on the other hand, in case all state dimensions are equal, the consideration of a jump map is "optional" and leads to so called impulsive systems.

Our goal is to find a reduced size switched system (with the same switching signal σ)

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)} \widehat{x}_k(t) + \widehat{B}_{\sigma(t)} u(t), \ t \in (s_k, s_{k+1}) \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} \widehat{x}_{k-1}(s_k^-), \quad k \in \mathcal{Q} \\ y(t) = \widehat{C}_{\sigma(t)} \widehat{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{2}$$

which has the same input-output behavior as the original system Σ_σ.

We assume in the following that the switching signal is fixed, hence, by relabeling the matrices, we can assume in most parts of the paper that $\sigma(t) = k$ on (s_k, s_{k+1}) and we denote the duration of k-th mode as $\tau_k = s_{k+1} - s_k$. In some slight abuse of notation we will simply speak in the following of the solution $x(\cdot)$ instead of the different solution pieces $x_k(\cdot)$. Furthermore, we will simply write $J_k := J_{\sigma(s_k^+),\sigma(s_k^-)} = J_{k,k-1}$ and $\widehat{J}_k := \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} = \widehat{J}_{k,k-1}$ in the following.

As already highlighted in our paper Hossain and Trenn (2021), which only considered the single switch case, a mode-wise reduction is not possible in general (see Example 4 in that reference); furthermore, Example 2 in that reference also shows that a switched system which is reachable and observable, is not necessarily minimal.

* Email Addresses: s.hossain@rug.nl; s.trenn@rug.nl;

Here we extend the preliminary results in Hossain and Trenn (2021) to the multiple switch case. In contrast to our earlier approach taken in Hossain and Trenn (2020), we are aiming to find a reduced model in the same system class (i.e. a piecewise-constant time-varying system instead of a continuously time-varying model).

The paper is organized as follows. In Section 2, the problem formulation and preliminaries are given with the characterization of reachability and observability of SLS. Section 3 discusses the main results with the proposed algorithm. Finally, some numerical results are shown in Section 4.

2. PRELIMINARIES

2.1 Weak Kalman decomposition

For non-switched linear systems with zero initial condition, the well known Kalman decomposition (KD) (Kalman, 1963) can be used to define a minimal realization. However, in the context of switched systems, the initial values for each (without first) mode are neither zero nor completely arbitrary, but are constraint to the reachable space of the previous mode. Furthermore, some unobservable state in one mode may become observable in a later mode. This motivates us to define a weak KD which takes into account an extended reachable space and restricted unobservable space.

Lemma 1. Consider a system (A, B, C) and let $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ and $\underline{\mathcal{U}} \subseteq \ker C$ be two A-invariant subspaces (an extended reachable and a restricted unobservable space). For any coordinate transformation $\overline{T} = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4]$ with $\operatorname{im} \overline{V}^1 := \overline{\mathcal{R}} \cap \underline{\mathcal{U}}$, $\operatorname{im}[\overline{V}^1, \overline{V}^2] := \overline{\mathcal{R}}$, $\operatorname{im}[\overline{V}^1, \overline{V}^3] := \underline{\mathcal{U}}$, we have $(\overline{T}^{-1} A \overline{T}, \overline{T}^{-1} B, C\overline{T}) =$

$$\left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right). \tag{3}$$

In particular, $Ce^{At}B = C^2 e^{A^{22}t} B^2$, for all $t \in \mathbb{R}$.

Proof. Since $\overline{\mathcal{R}} \cap \underline{\mathcal{U}} = \operatorname{im} \overline{V}^1$ is A-invariant there is a matrix A^{11} of appropriate size such that $A\overline{V}^1 = \overline{V}^1 A^{11}$. The A-invariance of $\overline{\mathcal{R}}$ implies that $A\overline{V}^2 \subseteq \operatorname{im}[\overline{V}^1, \overline{V}^2]$, hence there exists A^{12}, A^{22} such that $A\overline{V}^2 = \overline{V}^1 A^{12} + \overline{V}^2 A^{22}$. Similarly, A-invariance of $\underline{\mathcal{U}}$ implies $A\overline{V}^3 \subseteq \operatorname{im}[\overline{V}^1, \overline{V}^3]$, hence there exists A^{13}, A^{33} such that $A\overline{V}^3 = \overline{V}^1 A^{13} + \overline{V}^3 A^{33}$. Finally, $\operatorname{im}[\overline{V}^1, \overline{V}^2, \overline{V}^3, \overline{V}^4] = \mathbb{R}^n$ implies existence of A^{14}, A^{24}, A^{34}, A^{44} such that $A\overline{V}^4 = \overline{V}^1 A^{14} + \overline{V}^2 A^{24} + \overline{V}^3 A^{34} + \overline{V}^4 A^{44}$. Combining all of the above, we obtain

$$A[\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] \begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix},$$

which shows that $\overline{T}^{-1} A \overline{T}$ has the desired block structure. Since $\operatorname{im} B \subseteq \overline{\mathcal{R}} = \operatorname{im}[\overline{V}^1, \overline{V}^2]$, there exists B^1, B^2 such that

$$B = \overline{V}^1 B^1 + \overline{V}^2 B^2 = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix},$$

from which the desired block structure of $\overline{T}^{-1} B$ follows. Finally, $\ker C \supseteq \underline{\mathcal{U}} = \operatorname{im}[\overline{V}^1\ \overline{V}^3]$ implies that $C[\overline{V}^1\ \overline{V}^3] = \{0\}$, and hence, for $C^2 := C\overline{V}^2$ and $C^4 := C\overline{V}^4$,

$$C\overline{T} = C[\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] = [0\ C^2\ 0\ C^4].$$

With these block structure, simple matrix multiplication leads to $Ce^{At}B = C^2 e^{A^{22}t} B^2$ for all $t \in \mathbb{R}$. □

For the formulation of the main results, we will need the following notations of invariant subspaces.

Definition 2. For $A \in \mathbb{R}^{n \times n}$ and a subspace $\mathcal{L} \subseteq \mathbb{R}^n$, let

$$\langle A \mid \mathcal{L} \rangle := \operatorname{im}[\mathcal{L}, A\mathcal{L}, \ldots, A^{n-1}\mathcal{L}]$$

be the smallest A-invariant subspace containing $\mathcal{L}$. Furthermore, let (here A^{-1} stands for the preimage, it is not assumed that A is invertible)

$$\langle \mathcal{L} \mid A \rangle := \mathcal{L} \cap A^{-1}\mathcal{L} \ldots \cap A^{-(n-1)}\mathcal{L}$$

be the largest A-invariant subspace contained in $\mathcal{L}$. △

Note that for any $C \in \mathbb{R}^{m \times n}$, we have

$$\langle \ker C \mid A \rangle = \ker[C^\top, (CA)^\top, \ldots, (CA^{n-1})^\top]^\top.$$

Furthermore, it is well known that for a linear system (A, B, C) the reachable space $\mathcal{R}$ is given by $\mathcal{R} = \langle A, \operatorname{im} B \rangle$ and the unobservable space $\mathcal{U}$ is given by $\langle \ker C, A \rangle$.

Remark 3. Clearly, the choice $\overline{\mathcal{R}} = \mathcal{R}$ and $\underline{\mathcal{U}} = \mathcal{U}$ in Lemma 1 leads to the well known KD. Furthermore, any A-invariant subspace $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ will be a superset of $\mathcal{R}$, because $\mathcal{R}$ is the smallest A-invariant subspace containing $\operatorname{im} B$; analogously, any A-invariant subspace $\underline{\mathcal{U}} \subseteq \ker C$ will be contained in $\mathcal{U}$. This motivation to call $\overline{\mathcal{R}} \supseteq \mathcal{R}$ an extended reachable space and $\underline{\mathcal{U}} \subseteq \mathcal{U}$ a restricted unobservable space in Lemma 1.

For a linear system (A, B, C) with given extended reachable space $\overline{\mathcal{R}}$ and restricted unobservable space $\underline{\mathcal{U}}$ the weak KD (3) immediately leads to the reduced system (A^{22}, B^2, C^2) which can be obtained from (A, B, C) by suitable left and right projection defined as follows.

Definition 4. For any coordinate transformation $\overline{T} = [\overline{V}^1, \overline{V}^2, \overline{V}^3, \overline{V}^4]$ as in Lemma 1, let

$$[(\overline{W}^1)^\top, (\overline{W}^2)^\top, (\overline{W}^3)^\top, (\overline{W}^4)^\top]^\top := \overline{T}^{-1}.$$

Then, $\overline{W}^2$ and $\overline{V}^2$ are called the *weak KD left-projector* and *weak KD right-projector*, respectively. △

By definition, of the left- and right-projectors, $\overline{W}^2 \overline{V}^2 = I$ and $(A^{22}, B^2, C^2) = (\overline{W}^2 A \overline{V}^2, \overline{W}^2 B, C\overline{V}^2)$. Our approach relies on defining suitable extended reachable and restricted unobservable spaces for each of the modes of the switched system (1). Towards this goal, we first provide expression for the exact (time-varying) reachable and unobservable spaces for (1) in the following.

2.2 Exact (time-varying) reachability space

It is easily seen that the solution of (1) is given recursively by, for $t \in [s_k, s_{k+1})$ and $k = 1, \ldots, \mathtt{m}$,

$$x(t) := e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^{t} e^{A_k(t-s)} B_k u(s) ds. \tag{4}$$

The output equation is given by

$$y(t) = C_k x(t),\ t \in [s_k, s_{k+1}), k = 0, 1, \ldots, \mathtt{m}. \tag{5}$$

Let us now introduce the following formal definition of the reachable space of (1) on the intervals $[t_0, s_k)$, $k = 1, 2, \ldots, \mathtt{m}$.

Definition 5. The reachable space of the switched system (1) on time interval $[t_0, t)$ is defined by

$$\mathcal{R}_{[t_0,t)} := \left\{ x(t^-) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \text{ of } (1) \\ \text{with } x(t_0^-) = 0 \end{array} \right\}.$$

We call the switched system (1) *reachable* (on (t_0, t_f)) if, and only if,

$$\mathcal{R}_{[t_0,t_f)} = \mathbb{R}^{n_\mathtt{m}}. \qquad \triangle$$

To calculate the reachability spaces of (1), the known information of each switching time interval needs to carry over from a switching time interval to the next time interval. Let $\mathcal{R}_k = \langle A_k \mid \operatorname{im} B_k \rangle$ be the local reachable subspace for mode k. We will show then that the reachable space at the end of the k-th mode is defined by the following recursive equation, $k = 1, 2, \ldots, \mathtt{m}$:

$$\begin{aligned} \mathcal{M}_1 &:= \mathcal{R}_0, \\ \mathcal{M}_{k+1} &:= \mathcal{R}_k + e^{A_k \tau_k} J_k \mathcal{M}_k, \end{aligned} \tag{6}$$

where $\tau_k := s_{k+1} - s_k$ is the duration of mode k. The intuition behind the sequence (6) is as follows. By starting the zero initial values of first mode, clearly $\mathcal{R}_{[t_0,s_1)} = \mathcal{R}_0$, continuing recursively, the reachable space at switch number $k+1$, is obtained by propagating forward the reachable space just before switch k in time, i.e. first jump via J_k and then propagate according to the matrix exponential (the time-evolution for a zero input). Finally, to take into account the effect of the input, the local reachable space of mode k is added. This intuition is formalized as follows.

Lemma 6. (Cf. Küsters and Trenn (2016)). For all $1 \leq k \leq \mathtt{m} + 1$,

$$\mathcal{M}_k = \mathcal{R}_{[t_0,s_k)}.$$

In particular, (1) is reachable if, and only if $\mathcal{M}_{\mathtt{m}+1} = \mathbb{R}^{n_\mathtt{m}}$.

Proof. The proof is similar to the proof of (Küsters and Trenn, 2016, Thm. 27 & Lem. 26) and therefore omitted. □

From (6) it is clear that the reachable spaces depend on the switching times (in fact, on the mode durations τ_k) and this dependency cannot be avoided in general as the following example shows. In particular, the overall reachability of the switched system (1) on (t_0, t_f) depends on the switching times and how long each mode is active.

Example 7. (Dependency on the switching times). Consider the switched system (1) given by

$$\begin{aligned} A_0 = A_2 &= \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], A_1 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \\ B_0 = B_2 &= \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], B_1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]. \end{aligned}$$

with $J_1 = J_2 = I$. It is noted that none of the pairs (A_i, B_i) is reachable. Consider the switching signal σ with the mode sequence $0 \to 1 \to 2$ and switching times s_1, s_2. Let $\{e_1, e_2\}$ denote the natural basis vectors for $\mathbb{R}^2$. Clearly, $\mathcal{R}_0 = \mathcal{R}_2 := \operatorname{span}\{e_1\}$, $\mathcal{R}_1 := \{0\}$, $e^{A_1\tau} = \left[\begin{smallmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{smallmatrix}\right]$ and $e^{A_2\tau} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Hence,

$$\begin{aligned} \mathcal{M}_1 &= \mathcal{R}_0 = \operatorname{span}\{e_1\}, \\ \mathcal{M}_2 &= \mathcal{R}_1 + e^{A_1\tau_1} J_1 \mathcal{M}_1 = \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ \sin\tau_1 \end{smallmatrix}\right] \right\}, \\ \mathcal{M}_3 &= \mathcal{R}_2 + e^{A_2\tau_2} J_2 \mathcal{M}_2 = \operatorname{span}\{e_1\} + \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ \sin\tau_1 \end{smallmatrix}\right] \right\}. \end{aligned}$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$ then $\mathcal{M}_3 = \operatorname{span}\{e_1\}$, otherwise $\mathcal{M}_3 = \mathbb{R}^2$. This clearly shows that the overall reachability of a switched system depends on the switching times. △

Note that although $\mathcal{M}_{k+1} \supseteq \mathcal{R}_k \supseteq \operatorname{im} B_k$ the space $\mathcal{M}_{k+1}$ is not a suitable extended reachable space for the mode (A_k, B_k, C_k) in the sense of Lemma 1, because it is *not* A_k-invariant in general. Before addressing this problem in Section 2.4, we recall first the "dual" space of the reachability spaces: the unobservable spaces.

2.3 Exact (time-varying) unobservability space

Definition 8. The unobservable space of the switched system (1) on time interval $[t, t_f)$ is defined by

$$\mathcal{U}_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u = 0) \text{ such that} \\ y = 0 \text{ of } (1) \text{ on } [t, t_f) \end{array} \right\}.$$

We call the switched system (1) *observable* (on $[t_0, t_f)$) if, and only if,

$$\mathcal{U}_{[t_0,t_f)} = \{0\}. \qquad \triangle$$

Similar as for the reachable spaces, we aim to express the unobservable spaces recursively. Starting from the last mode it is clear that the unobservable space is the same as the classical unobservable space $\mathcal{U}_\mathtt{m} = \langle \ker C_\mathtt{m} \mid A_\mathtt{m} \rangle$. Recursively, the unobservable space at switch number k can now be propagated backwards in time by first taking the preimage under the jump J_k and then further propagating it back with the continuous flow of mode $k-1$, i.e. by $e^{-A_{k-1}\tau_{k-1}}$. Finally, this propagated space needs to be combined with the local unobservable space of mode $k-1$ given by $\mathcal{U}_{k-1} = \langle \ker C_{k-1} \mid A_{k-1} \rangle$. This motivates the definition of the following sequence of subspaces, $k = \mathtt{m}, \mathtt{m}-1, \ldots, 1$:

$$\begin{aligned} \mathcal{N}_\mathtt{m} &:= \mathcal{U}_\mathtt{m}, \\ \mathcal{N}_{k-1} &:= \mathcal{U}_{k-1} \cap \left(e^{-A_{k-1}\tau_{k-1}} J_k^{-1} \mathcal{N}_k \right). \end{aligned} \tag{7}$$

Lemma 9. (Cf. Tanwani et al. (2011)). For all $0 \leq k \leq \mathtt{m}$,

$$\mathcal{N}_k = \mathcal{U}_{[s_k,t_f)}.$$

In particular, (1) is observable if, and only if $\mathcal{N}_0 = \{0\}$.

Proof. The proof is similar to the proof of (Küsters and Trenn, 2016, Thm. 27 & Lem. 26) and therefore omitted. □

Example 10. (Dependency on the switching times). Recall Example 7 with output submatrices

$$C_0 = C_2 = \left[\begin{smallmatrix} 0 & 1 \end{smallmatrix}\right], C_1 = \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right],$$

It is noted that none of the pairs (A_i, C_i) is observable. Clearly, $\mathcal{U}_0 = \mathcal{U}_2 = \operatorname{span}\{e_1\}$, $\mathcal{U}_1 = \mathbb{R}^2$, $e^{-A_1\tau} = \left[\begin{smallmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{smallmatrix}\right]$ and $e^{-A_2\tau} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Hence,

$$\begin{aligned} \mathcal{N}_2 &= \mathcal{U}_2 = \operatorname{span}\{e_1\}, \\ \mathcal{N}_1 &= \mathcal{U}_1 \cap e^{-A_1\tau_1} J_2^{-1} \mathcal{N}_2 = \mathbb{R}^2 \cap \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{smallmatrix}\right] \right\}, \\ \mathcal{N}_0 &= \mathcal{U}_0 \cap e^{-A_0\tau_0} J_1^{-1} \mathcal{N}_1 = \operatorname{span}\{e_1\} \cap \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{smallmatrix}\right] \right\} \end{aligned}$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$, then $\mathcal{N}_0 := \operatorname{span}\{e_1\}$, otherwise $\mathcal{N}_0 = \{0\}$. Therefore, the overall observability of (1) depends on the switching time. △

Note that similar to the reachability spaces, although the unobservable spaces $\mathcal{N}_k$ satisfy $\mathcal{N}_k \subseteq \mathcal{U}_k \subseteq \ker C$, they are not A_k-invariant and hence, are not restricted unobservable spaces in the sense of Lemma 1.

2.4 Extended reachable / restricted unobservable spaces for switched system

So far, we have seen that the reachability spaces and observability spaces of (1) depend on the switching time. Even worse, when looking at the reachable / unobservable space at a particular time $t \in (s_k, s_{k+1})$ between two switches, then it is easily seen that these spaces in general also depend on the considered time t and a reduction method based on the exact reachability / observability spaces will necessarily result in general time-varying coordinate transformations / projections (cf. our previously proposed reduction method (Hossain and Trenn, 2020)) and would not lead to a reduced system of the desired form (2).

To circumvent this problem, we introduce suitable extended reachable and restricted unobservable spaces for the switched system (1). The key idea is based on the fact that for any subspace $\mathcal{H} \subseteq \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n\times n}$ the following subspace relationship holds:

$$\langle \mathcal{H} \mid A\rangle \subseteq e^{At}\mathcal{H} \subseteq \langle A \mid \mathcal{H}\rangle. \tag{8}$$

By replacing the matrix-exponentials in the constructions of the reachable / unobservable spaces by the corresponding A-invariant subspace, we arrive at the following sequences (cf. Tanwani et al. (2011) for the unobservable spaces):

$$\begin{aligned}\overline{\mathcal{R}}_0 &:= \mathcal{R}_0,\\ \overline{\mathcal{R}}_k &:= \mathcal{R}_k + \langle A_k \mid J_k\overline{\mathcal{R}}_{k-1}\rangle, k = 1,\dots,\mathtt{m};\end{aligned} \tag{9}$$

$$\begin{aligned}\underline{\mathcal{U}}_{\mathtt{m}} &:= \mathcal{U}_{\mathtt{m}},\\ \underline{\mathcal{U}}_{k-1} &:= \mathcal{U}_{k-1} \cap \langle J_k^{-1}\underline{\mathcal{U}}_k \mid A_{k-1}\rangle, k = \mathtt{m},\dots,2,1.\end{aligned} \tag{10}$$

In view of (8), it is easy to see that

$$\overline{\mathcal{R}}_k \supseteq \mathcal{M}_{k+1} \supseteq \mathcal{R}_k \quad\text{and}\quad \underline{\mathcal{U}}_k \subseteq \mathcal{N}_k \subseteq \mathcal{U}_k.$$

In particular, $\overline{\mathcal{R}}_{\mathtt{m}} = \mathbb{R}^{n_{\mathtt{m}}}$ and $\underline{\mathcal{U}}_0 = \{0\}$ are necessary conditions for reachability and observability, respectively, of the overall switched system (3).

Finally, observe that by construction both $\overline{\mathcal{R}}_k$ and $\underline{\mathcal{U}}_k$ are A_k-invariant, i.e. they are extended reachable / restricted unobservable spaces in the sense of Lemma 1 and we are now ready to propose our main result about the reduction of switched systems of the form (1).

3. MAIN RESULT: PROPOSED REDUCTION METHOD

We now propose a method to compute a reduced realization (2) of (1) for a given switching signal.

Step 1. Compute the sequence of extended reachable subspaces $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \cdots, \overline{\mathcal{R}}_{\mathtt{m}}$ and restricted unobservable subspaces $\underline{\mathcal{U}}_0, \underline{\mathcal{U}}_1, \cdots, \underline{\mathcal{U}}_{\mathtt{m}}$ as in (9) and (10).
Step 2. Apply Lemma 1 to (A_k, B_k, C_k) with $(\overline{\mathcal{R}}_k, \underline{\mathcal{U}}_k)$ to compute the left- and right-projectors $\overline{W}_k^2, \overline{V}_k^2$, and let

$$\left(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k\right) = \left(\overline{W}_k^2 A_k \overline{V}_k^2, \overline{W}_k^2 B_k, C_k \overline{V}_k^2\right).$$

Step 3. Calculate the reduced jump map

$$\widehat{J}_k := \overline{W}_k^2 J_k \overline{V}_{k-1}^2.$$

Before showing that the resulting reduced system (2) is indeed a realization of (1), we first highlight an important connection between the solutions of both systems.

Lemma 11. Consider the switched system Σ_σ as in (1) and the reduced system $\widehat{\Sigma}_\sigma$ as in (2) obtained by the left- and right-projectors $\overline{W}_{\sigma(\cdot)}^2, \overline{V}_{\sigma(\cdot)}^2$. If $x(\cdot)$ is a solution of Σ_σ then $\widehat{x}(\cdot) := \overline{W}_{\sigma(\cdot)}^2 x(\cdot)$ is a solution of $\widehat{\Sigma}_\sigma$.

Proof. Consider any time interval (s_k, s_{k+1}) between two switches, then, for $t \in (s_k, s_{k+1})$,

$$\begin{aligned}\dot{\widehat{x}}(t) &= \overline{W}_k^2 \dot{x} = \overline{W}_k^2 A_k x(t) + \overline{W}_k^2 B u(t)\\ &= [0\ \widehat{A}_k\ 0\ *]\ \overline{T}_k^{-1} x(t) + B_2^k u(t),\end{aligned}$$

where $\overline{T}_k = [\overline{V}_k^1\ \overline{V}_k^2\ \overline{V}_k^3\ \overline{V}_k^4]$ is the coordinate transformation according to Lemma 1 for mode k. Since $x(t) \in \mathcal{R}_{[t_0,t)} \subseteq \overline{\mathcal{R}}_k = \operatorname{im}[\overline{V}_k^1, \overline{V}_k^2]$ it follows that $\overline{T}_k^{-1} x(t) = [*\ \widehat{x}(t)^\top\ 0\ 0]^\top$ and hence, as claimed, for all $t \in (s_k, s_{k+1})$

$$\dot{\widehat{x}}(t) = \widehat{A}_k \widehat{x}(t) + \widehat{B}_k u(t).$$

In particular, due to unique solvability of linear ODEs, for any solutions x of Σ_σ and $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ the following implication holds:

$$\overline{W}_k^2 x(s_k^+) = \widehat{x}(s_k^+) \implies \forall t \in (s_k, s_{k+1}) : \overline{W}_k^2 x(t) = \widehat{x}(t).$$

To show that $\widehat{x} = \overline{W}_\sigma^2 x$ is indeed a global solution of $\widehat{\Sigma}_\sigma$ it therefore remains to be shown that

$$\overline{W}_k^2 x(s_k^+) = \widehat{J}_k \overline{W}_{k-1}^2 x(s_k^-). \tag{11}$$

In fact,

$$\begin{aligned}\overline{W}_k^2 x(s_k^+) &= \overline{W}_k^2 J_k x(s_k^-) = \overline{W}_k^2 J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-)\\ &= \overline{W}_k^2 J_k [\overline{V}_{k-1}^1\ \overline{V}_{k-1}^2\ \overline{V}_{k-1}^3\ \overline{V}_{k-1}^4] \begin{pmatrix} * \\ \overline{W}_{k-1}^2 x(s_k^-) \\ 0 \\ 0\end{pmatrix}.\end{aligned}$$

From (10), it is easily seen that $J_k \underline{\mathcal{U}}_{k-1} \subseteq \underline{\mathcal{U}}_k$, hence, $\operatorname{im} J_k \overline{V}_{k-1}^1 \subseteq \operatorname{im} J_k [\overline{V}_{k-1}^1\ \overline{V}_{k-1}^3] = J_k \underline{\mathcal{U}}_{k-1} \subseteq J_k \underline{\mathcal{U}}_{k-1} \subseteq \underline{\mathcal{U}}_k = \operatorname{im}[\overline{V}_k^1\ \overline{V}_k^3] \subseteq \ker \overline{W}_k^2$, i.e. $\overline{W}_k^2 J_k \overline{V}_{k-1}^1 = 0$, from which it follows that

$$\overline{W}_k^2 x(s_k^+) = \overline{W}_k^2 J_k \overline{V}_{k-1}^2 \overline{W}_{k-1}^2 x(s_k^-)$$

as desired. □

As a consequence of the above and of the uniqueness of solutions it follows that every solution $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ with zero initial value and given input u satisfies $\widehat{x} = \overline{W}_\sigma^2 x$ where x is the solution of Σ_σ with zero initial value and the same input u. We will now prove that the corresponding outputs are indeed equal.

Theorem 12. Consider the switched system Σ_σ as in (1) and the reduced system $\widehat{\Sigma}_\sigma$ as in (2) obtained by the above reduction method. Then Σ_σ and $\widehat{\Sigma}_\sigma$ are input-output equivalent in the sense that for all inputs u the output y of (1) with initial condition $x(t_0^-) = 0$ equals the output $\widehat{y}$ of (2) with initial condition $\widehat{x}(t_0^-) = 0$.

Proof. The output of Σ_σ on $[s_k, s_{k+1})$ is given by

$$\begin{aligned}y(t) &= C_k e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^t C_k e^{A_k(t-s)} B_k u(s) ds\\ &=: y_{\mathtt{J}}(t) + y_{\mathtt{u}}(t).\end{aligned}$$

Inserting suitable identity matrices,

$$y_{\mathtt{J}} = C_k \overline{T}_k e^{\overline{T}_k^{-1} A_k \overline{T}_k (t-s_k)} \overline{T}_k^{-1} J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-),$$

$$y_{\mathtt{u}}(t) = \int_{s_k}^{t} C_k \overline{T}_k e^{\overline{T}_k^{-1} A_k \overline{T}_k (t-s)} \overline{T}_k^{-1} B_k u(s) ds,$$

where $\overline{T}_k = [\overline{V}_k^1\ \overline{V}_k^2\ \overline{V}_k^3\ \overline{V}_k^4]$ is the coordinate transformation according to Lemma 1 for mode k. The special block structure of the matrices $\overline{T}_k^{-1} A_k \overline{T}_k$, $\overline{T}_k^{-1} B_k$, $C_k \overline{T}_k$ implied by Lemma 1 immediately leads to

$$y_{\mathtt{u}}(t) = \int_{s_k}^{t} \widehat{C}_k e^{\widehat{A}_k (t-s)} \widehat{B}_k u(s) ds.$$

Hence, for showing $\widehat{y}(t) = y(t) = y_{\mathtt{J}}(t) + y_{\mathtt{u}}(t)$ it remains to be shown that

$$y_{\mathtt{J}}(t) = \widehat{C}_k e^{\widehat{A}_k (t-s_k)} \widehat{J}_k \widehat{x}(s_k^-). \tag{12}$$

With similar arguments as used to establish (11) in Lemma 11, we can show that

$$\overline{T}_k^{-1} J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-) = \begin{pmatrix} * \\ \widehat{J}_k \overline{W}_k^2 x(s_k^-) \\ 0 \\ 0 \end{pmatrix}.$$

Using the already established fact in Lemma 11, that $\overline{W}_k^2 x(s_k^-) = \widehat{x}(s_k^-)$ together with the special block structures of $\overline{T}_k^{-1} A_k \overline{T}_k$, $\overline{T}_k^{-1} B_k$, $C_k \overline{T}_k$, we can conclude that (12) holds. □

Remark 13. Our proposed reduction method is independent from the actual mode durations τ_k and only depends on the mode-sequence. We conjecture that the reduced systems is a *minimal* realization for *almost all* mode durations. In general, it is not possible to obtain a duration-independent realization which is minimal for *all* mode durations, see the following example.

Example 14. Consider a switched system with modes

$$A_0 = A_2 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, A_1 = \begin{bmatrix} 0&0&0 \\ 0&0&-1 \\ 0&1&0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_0 = C_1 = [\,1\ 0\ 0\,], C_2 = [\,1\ 1\ 0\,].$$

with $J_1 = J_2 = I$. Assume the mode sequence $0 \to 1 \to 2$. Fix the switching time duration $\tau_1 = \pi/2$ for mode 1. Then the original solution x and output y of each time interval can be characterized as follows:

$$t \in (t_0, s_1): \ x(t) = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \quad y(t) = C_0 x(t) = [\,1\ 0\ 0\,] \begin{bmatrix} * \\ * \\ 0 \end{bmatrix},$$
$$t \in (s_1, s_1 + \tfrac{\pi}{2}): \ x(t) = \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \quad y(t) = C_1 x(t) = [\,1\ 0\ 0\,] \begin{bmatrix} * \\ * \\ * \end{bmatrix},$$
$$x(s_2) = x(s_1 + \tfrac{\pi}{2}) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}$$
$$t \in (s_2, t_f): \ x(t) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \quad y(t) = C_2 x(t) = [\,1\ 1\ 0\,] \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}.$$

Clearly, the second and third state do not affect the output for this specific switching signal. In particular, it is easily seen that the overall input-output behaviour is described by the (nonswitched) system $\dot{\widehat{x}} = u$, $y = \widehat{x}$. However, if we apply our proposed method, then the sequence of reachable and unobservable spaces are given by

$$\mathcal{M}_1 = \operatorname{im} B_0, \quad \mathcal{N}_0 = \{0\},$$
$$\mathcal{M}_2 = \mathbb{R}^3, \quad \mathcal{N}_1 = \{0\},$$
$$\mathcal{M}_3 = \mathbb{R}^3, \quad \mathcal{N}_2 = \operatorname{span}\{e_3\}.$$

Indeed, the sequences produce a switched system with modes in dimensions 1, 3 and 2, respectively, instead of a one dimensional minimal systems. Nevertheless, one should note that for $\tau_1 \neq k\pi/2$, our method actually produces a minimal realization. △

We conclude the theoretical part of this contribution by showing that our proposed method results at least in a minimal realization in the sense that its not possible to reduced it further with the same reduction method.

Theorem 15. Consider the switched system Σ_σ and the reduced switched system $\widehat{\Sigma}_\sigma$ resulting from our proposed method. Let $\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)}$ and $\underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)}$ be the sequences of reachable and unobservable spaces, respectively, of $\widehat{\Sigma}_\sigma$. Then,

$$\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)} = \mathbb{R}^{\widehat{n}_{\sigma(\cdot)}}, \quad \underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)} = \{0\}.$$

In particular, the left- and right-projectors for a potential further reduction are given by identity matrices, i.e. no further reduction occurs.

Proof. Our proposed methods yields for each mode k a coordinate transformation $\overline{T}_k$ such that (A_k, B_k, C_k) is transformed to

$$\left(\begin{bmatrix} A_k^{11} & A_k^{12} & A_k^{13} & A_k^{14} \\ 0 & \widehat{A}_k & 0 & A_k^{24} \\ 0 & 0 & A_k^{33} & A_k^{34} \\ 0 & 0 & 0 & A_k^{44} \end{bmatrix}, \begin{bmatrix} B_k^1 \\ \widehat{B}_k \\ 0 \\ 0 \end{bmatrix}, [\,0\ \widehat{C}_k\ 0\ C_k^4\,] \right), \tag{13}$$

where $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$ is the input-output equivalent reduced system for mode k. By construction, the reachable and unobservable spaces are given by

$$\overline{\mathcal{R}}_k = \overline{T}_k \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\mathcal{U}}_k = \overline{T}_k \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix},$$

respectively. Let $\overline{\widehat{\mathcal{R}}}_k$ and $\underline{\widehat{\mathcal{U}}}_k$ respectively, be the extended reachable and restricted unobservable space of $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$ according to (9) and (10).

Seeking a contradiction assume $\overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}$ (Case I), or $\underline{\widehat{\mathcal{U}}}_k \neq \{0\}$ (Case II) for some k.

Case I: For $k = 0$ we see that from $\overline{\mathcal{R}}_0 = \mathcal{R}_0$, it follows that the pair $(\widehat{A}_0, \widehat{B}_0)$ must be reachable and hence $\overline{\widehat{\mathcal{R}}}_0 = \widehat{\mathcal{R}}_0 = \mathbb{R}^{\widehat{n}_0}$. Assume now inductively that for some k we have $\overline{\widehat{\mathcal{R}}}_{k-1} = \mathbb{R}^{\widehat{n}_{k-1}}$ and $\overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}$. Since $\overline{\widehat{\mathcal{R}}}_k$ is $\widehat{A}_k$-invariant and contains $\operatorname{im} \widehat{B}_k$, we can choose a coordinate transformation $\overline{\widehat{T}}_k$ such that $(\widehat{A}_k, \widehat{B}_k)$ is transformed to

$$\left(\begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix}, \begin{bmatrix} \widehat{B}_k^1 \\ 0 \end{bmatrix} \right), \tag{14}$$

and $\operatorname{im} \overline{\widehat{T}}_k \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \overline{\widehat{\mathcal{R}}}_k$. By adjusting the original coordinate transformation $\overline{T}_k$, we can assume in the following that $(\widehat{A}_k, \widehat{B}_k)$ is actually equal to (14). In particular, we then have

$$\operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \overline{\widehat{\mathcal{R}}}_k = \widehat{\mathcal{R}}_k + \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle.$$

Since $\widehat{\mathcal{R}}_k = \langle \widehat{A}_k \mid \widehat{B}_k \rangle \subseteq \operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, we can conclude that $\operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \supseteq \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle = \langle \widehat{A}_k \mid \operatorname{im} \widehat{J}_k \rangle \supseteq \operatorname{im} \widehat{J}_k$. Therefore, (A_k, B_k, J_k) is actually transformed to

$$\left(\begin{bmatrix} * & * & * & * \\ 0 & \begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix} & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * \\ \widehat{B}_k^1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ J_k^1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right).$$

From this we arrive at the following contradiction:

$$\operatorname{im}\begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \overline{\mathcal{R}}_k = \mathcal{R}_k + \langle A_k \mid J_k \overline{R}_{k-1} \rangle \subseteq \operatorname{im}\begin{bmatrix} I & 0 \\ 0 & \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we have inductively shown that $\overline{\widehat{\mathcal{R}}}_k = \mathbb{R}^{\widehat{n}_k}$, for all mode k.

Case II: Assume $\widehat{\underline{\mathcal{U}}}_k \neq \{0\}$. Analogously as in Case I, the contradiction

$$\underline{\mathcal{U}}_k \neq \operatorname{im}\begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix},$$

arises, the details are omitted. □

4. NUMERICAL EXAMPLE

We demonstrate the operation of the proposed reduction method for the switched system.

Example 16. Consider a switched system with modes

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 0.2 & 1 \\ 0 & 1 & 0.1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 1 \end{bmatrix} \right),$$

$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \right), J_1 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(A_2, B_2, C_2) = \left(\begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0.2 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right), J_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

Assume the mode sequence $0 \to 1 \to 2$. We apply the proposed reduction method and the reduced systems can be obtained as follows.

Step 1. Computed sequence of reachable and unobservable spaces are given by

$$(\overline{\mathcal{R}}_0, \underline{\mathcal{U}}_0) = (\operatorname{span}\{e_1\}, \{0\}),$$
$$(\overline{\mathcal{R}}_1, \underline{\mathcal{U}}_1) = (\operatorname{span}\{e_1, e_2\}, \operatorname{span}\{e_3\}),$$
$$(\overline{\mathcal{R}}_2, \underline{\mathcal{U}}_2) = (\mathbb{R}^3, \operatorname{span}\{e_1, e_3\}).$$

Step 2. Via the proposed method, we obtain the sequence of left- and right-projectors as

$$(\overline{W}_0^2, \overline{V}_0^2) = \left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}^\top, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right), (\overline{W}_1^2, \overline{V}_1^2) = \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}^\top, \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \right),$$

$$(\overline{W}_2^2, \overline{V}_2^2) = \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}^\top, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right).$$

The reduced switched systems are given by

$$(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) = (\overline{W}_0^2 A_0 \overline{V}_0^2, \overline{W}_0^2 B_0, C_0 \overline{V}_0^2) = (1, 1, 2),$$
$$(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) = \left(\overline{W}_1^2 A_1 \overline{V}_1^2, \overline{W}_1^2 B_1, C_1 \overline{V}_1^2\right)$$
$$= \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \end{bmatrix} \right),$$
$$(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) = \left(\overline{W}_2^2 A_2 \overline{V}_2^2, \overline{W}_2^2 B_2, C_2 \overline{V}_2^2\right) = (0.2, 2, 1).$$

Step 3. Computed jump maps are $\widehat{J}_1 = \begin{bmatrix} 0 \\ -3 \end{bmatrix}$, $\widehat{J}_2 = \begin{bmatrix} 1 & 0 \end{bmatrix}$.

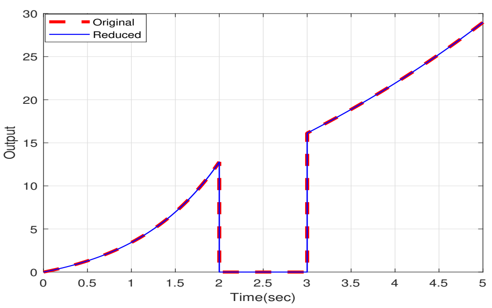

Fig. 1. Outputs of original and the reduced systems.

Figure (1) shows the output of the original and its reduced switched system for input $u(t) = 1$ with switching times $s_1 = 2$ and $s_2 = 3$ and clearly both outputs coincide. △

5. CONCLUSION

We have presented a method concerning reduced realization of switched linear systems for general switching signals with known switching sequence. Our method is based on a weak Kalman decomposition of each mode defined in terms of suitable extended reachable and restricted unobservable spaces. An important feature of our method is the independence from the precise switching times and we conjecture that the resulting reduced system has minimal size for almost all switching times. We provide an example, that in general the dimension of the minimal realization depends on the specific switching times.

By extending our approach to suitable subspaces of easily reachable states and difficult to observe states, we see high potential to design novel model reduction methods for switched systems. Furthermore, our ideas should carry over also to switched descriptor systems, however, the presence of Dirac impulses needs further careful investigation.

REFERENCES

Hossain, M.S. and Trenn, S. (2020). A time-varying gramian based model reduction approach for linear switched systems. *IFAC-PapersOnLine*, 53(2), 5629–5634.

Hossain, M.S. and Trenn, S. (2021). Minimal realization for linear switched systems with a single switch. In *2021 European Control Conference (ECC)*, 1168–1173.

Kalman, R.E. (1963). Mathematical description of linear dynamical systems. *SIAM J. Control Optim.*, 1, 152–192. doi: 10.1137/0301010.

Küsters, F. and Trenn, S. (2016). Duality of switched DAEs. *Math. Control Signals Syst.*, 28(3), 25. doi:10.1007/s00498-016-0177-2.

Küsters, F. and Trenn, S. (2018). Switch observability for switched linear systems. *Automatica*, 87, 121–127. doi: 10.1016/j.automatica.2017.09.024.

Petreczky, M. (2006). *Realization Theory of Hybrid Systems*. Ph.D. thesis, Vrije Universiteit, Amsterdam.

Petreczky, M. (2007). Realization theory for linear switched systems: Formal power series approach. *Systems & Control Letters*, 56(9-10), 588–595.

Petreczky, M., Bako, L., and Van Schuppen, J.H. (2013). Realization theory of discrete-time linear switched systems. *Automatica*, 49(11), 3337–3344.

Petreczky, M., Tanwani, A., and Trenn, S. (2015). Observability of switched linear systems. In M. Djemai and M. Defoort (eds.), *Hybrid Dynamical Systems*, volume 457 of *Lecture Notes in Control and Information Sciences*, 205–240. Springer-Verlag.

Petreczky, M. and van Schuppen, J.H. (2010). Realization theory for linear hybrid systems. *IEEE Trans. Autom. Control*, 55(10), 2282–2297.

Sun, Z., Ge, S.S., and Lee, T.H. (2002). Controllability and reachability criteria for switched linear systems. *Automatica*, 38, 775–786.

Tanwani, A., Shim, H., and Liberzon, D. (2011). Observability implies observer design for switched linear systems. In *Proc. ACM Conf. Hybrid Systems: Computation and Control*, 3 – 12.

Tanwani, A., Shim, H., and Liberzon, D. (2013). Observability for switched linear systems: Characterization and observer design. *IEEE Trans. Autom. Control*, 58(4), 891–904. doi: 10.1109/TAC.2012.2224257.