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Switched differential algebraic equations: Jumps and impulses

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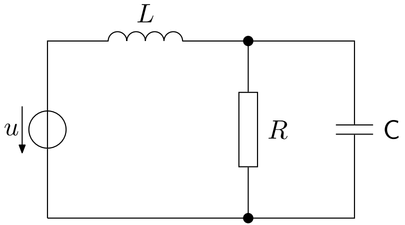
Why DAEs?

Electric circuit modelling



Physical variables

voltage and current for each circuit element

Defining equations

- element behaviors (voltage-current relation)
- Kirchhoff laws (voltage-loops, current-nodes)

Basic circuit elements:

- Resistors: $v_R(t) = Ri_R(t)$
- Capacitor: $C\frac{\mathrm{d}}{\mathrm{d}t}v_C(t) = i_C(t)$
- Inductor: $L\frac{\mathrm{d}}{\mathrm{d}t}i_L(t) = v_L(t)$
- Voltage source: $v_S(t) = u(t)$ (current i_S free)

Kirchhoff laws:

- $i_s = i_L$
- $i_L = i_R + i_C$
- $v_s = v_L + v_R$
- $v_R = v_C$

We already have arrived at a DAE model!

With $x = (v_R, i_R, v_C, i_C, v_L, i_L, v_S, i_S)$ we have $E\dot{x} = Ax + Bu$

Different circuit modeling frameworks

DAE-model:

$$\begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&C&0&0&0&0&0\\ 0&0&0&0&0&L&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1&R&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&-1&0\\ 0&0&0&0&0&-1&0&1\\ 0&-1&0&-1&0&1&0&0\\ -1&0&0&0&-1&0&1&0\\ 1&0&-1&0&0&0&0&0 \end{bmatrix} x + \begin{bmatrix} 0\\0\\0\\1\\0\\0\\0\\0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0&0&0&0&0&0&0&1 \end{bmatrix} x$$

ODE-model:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} i_L\\ v_c \end{pmatrix} = \begin{bmatrix} 0 & \frac{-1}{L}\\ \frac{1}{C} & \frac{-R}{C} \end{bmatrix}\begin{pmatrix} i_L\\ v_c \end{pmatrix} + \begin{bmatrix} \frac{1}{L}\\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{pmatrix} i_L\\ v_c \end{pmatrix}$$

Transfer function: $g(s) = \dfrac{R + Cs}{CLs^2 + LRs + 1}$

Which is the best?

None! All have advantages and disadvantages.

Pros and Cons of DAE formulation

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & L & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} -1 & R & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

DAE-models: Advantages

- Most natural and intuitive way to model (just write down all first-principal equations)
- Inputs do not need to be specified a priori ($\rightsquigarrow E\dot{x} = Ax$ with rectangular E,A)
- Connecting two DAE models is trivial (just add new algebraic constraints)
- Sudden structural changes (switches or faults) can be modeled easily

DAE-models: Disadvantages

- Solution theory more complicated
- Not so many standard tools available for numerical solutions, control design, ...
- Harder to work with manually

DAEs are not ODEs

$$\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}x + \begin{pmatrix}f_1\\ f_2\\ f_3\end{pmatrix}$$

$$\dot{x}_2 = x_1 + f_1 \longrightarrow x_1 = -f_1 - \dot{f}_2$$

$$0 = x_2 + f_2 \longrightarrow x_2 = -f_2 \nearrow$$

$$0 = f_3 \qquad \text{no restriction on } x_3$$

Key differences to ODEs

- For fixed inhomogeneity, initial values cannot be chosen arbitrarily ($x_1(0) = -f_1(0) - \dot{f}_2(0)$, $x_2(0) = f_2(0)$)
- For fixed inhomogeneity, solution not uniquely determined by initial value (x_3 free)
- Inhomogeneity not arbitrary
  - structural restrictions ($f_3 = 0$)
  - differentiability restrictions ($\dot{f}_2$ must be well defined)

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Solution properties of DAEs

Equivalence and four types of DAEs

Regularity and quasi-Weierstrass form

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Distributional solutions - Dilemma

Review: classical distribution theory

Piecewise smooth distributions

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The single switch result

Calculation of the four subspaces

Summary

Equivalence of matrix pairs and DAEs

Definition (Equivalence of matrix pairs)

(E_1, A_1), (E_2, A_2) are called equivalent $:\iff$ $(E_2, A_2) = (SE_1T, SA_1T)$

short: $(E_1, A_1) \cong (E_2, A_2)$ or $(E_1, A_1) \overset{S,T}{\cong} (E_2, A_2)$

Equivalence and solution behavior

For $(E_1, A_1) \cong (E_2, A_2)$ and $B_2 = SB_1$, $C_2 = C_1T$ we have:

$$(x, u, y) \text{ solves } \begin{cases} E_1\dot{x} = A_1x + B_1u \\ y = C_1x \end{cases} \overset{x=Tz}{\iff} (z, u, y) \text{ solves } \begin{cases} E_2\dot{z} = A_2z + B_2u \\ y = C_2z \end{cases}$$

Goal: Reveal inner structure of DAEs

Find S and T such that (SET, SAT) has simple structure

Four types of DAEs

Definition

- (E, A) is of type ODE $:\Longleftrightarrow (E, A) \cong (I, J)$
- (E, A) is of type nDAE $:\Longleftrightarrow (E, A) \cong (N, I)$, N nilpotent (i.e. $N^\nu = 0$)
- (E, A) is of type uDAE $:\Longleftrightarrow (E, A) \cong (\operatorname{diag}(E_1, \ldots, E_k), \operatorname{diag}(A_1, \ldots, A_k))$,

  where $(E_i, A_i) = \left(\begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \right)$ underdetermined prototypes
- (E, A) is of type oDAE $:\Longleftrightarrow (E, A) \cong (\operatorname{diag}(E_1, \ldots, E_k), \operatorname{diag}(A_1, \ldots, A_k))$,

  where $(E_i, A_i) = \left(\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix}, \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} \right)$ overdetermined prototypes

Every DAE can be decoupled in these four types! $\rightsquigarrow$ Quasi-Kronecker form

Quasi-Kronecker form

Theorem (Quasi-Kronecker Form, Berger & T. '12,'13)

For any $E, A \in \mathbb{R}^{\ell \times n}$, $\exists$ invertible $S \in \mathbb{R}^{\ell \times \ell}$ and invertible $T \in \mathbb{R}^{n \times n}$:

$$(E, A) \overset{S,T}{\cong} \left(\begin{bmatrix} E_U & & & \\ & E_J & & \\ & & E_N & \\ & & & E_O \end{bmatrix}, \begin{bmatrix} A_U & & & \\ & A_J & & \\ & & A_N & \\ & & & A_O \end{bmatrix} \right)$$

where

- (E_U, A_U) *is of type uDAE (underdetermined part)*
- (E_J, A_J) *is of type ODE (ODE part)*
- (E_N, A_N) *is of type nDAE (nilpotent part)*
- (E_O, A_O) *is of type oDAE (overdetermined part)*

Regularity

Definition

(E, A) is regular $:\Longleftrightarrow$ $\ell = n$ and $\det(sE - A) \not\equiv 0$

Theorem (Regularity characterizations)

The following statements are equivalent:

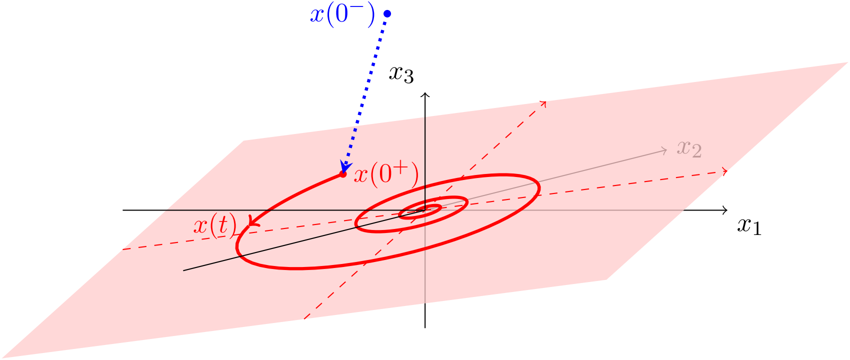
- (E, A) *is regular*
- $(E, A) \cong \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$ *(quasi-Weierstrass form)*
- $E\dot{x} = Ax + Bu$ *has solution for all* u *and is uniquely determined by* $x(0)$

Regularity means existence and uniqueness of solutions

BUT not for all initial conditions $x(0) = x_0$!

Example: $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\rightsquigarrow$ regular, but $x_2(t) = 0$ for all t

Jump and flow



Questions

- How to find consistency space?
- What determines the jump $x(0^-) \mapsto x(0^+)$?

Wong-sequences and Wong limits

Definition (Wong sequences)

For $E, A \in \mathbb{R}^{\ell \times n}$ let

$$\mathcal{V}_0 := \mathbb{R}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i), \qquad i = 0,1,2,\dots$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{j+1} := E^{-1}(A\mathcal{W}_j) \quad , \qquad j = 0,1,2,\dots$$

Here $M\mathcal{S} := \{Mx \mid x \in \mathcal{S}\}$ and $M^{-1}\mathcal{S} := \{x \mid Mx \in \mathcal{S}\}$

Wong limits

$$\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \ldots$$
$$\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \ldots$$

Then we can define: $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}$ and $\mathcal{W}^* := \bigcup_{j\in\mathbb{N}} \mathcal{W}_j = \mathcal{W}_{j^*}$

Motivation of first Wong sequence

Definition (Consistency space)

The consistency space of $E\dot{x} = Ax$ is

$$\mathfrak{C}_{(E,A)} := \{x_0 \in \mathbb{R}^n \mid \exists \text{ sol. } x \text{ of } E\dot{x} = Ax \text{ with } x(0) = x_0\}$$

Inductive refinement of consistency space

- Initially no knowledge $\rightsquigarrow \mathcal{V}_0 = \mathbb{R}^n \supseteq \mathfrak{C}_{(E,A)} \rightsquigarrow$ trivial constraint $\dot{x} \in \mathcal{V}_0$
- $E\dot{x} = Ax$ constraints x to $x \in A^{-1}\{E\dot{x}\} \subseteq A^{-1}(E\mathcal{V}_0) =: \mathcal{V}_1 \supseteq \mathfrak{C}_{(E,A)}$
- $\dot{x}(t) := \lim_{h\to 0} \frac{x(t+h)-x(t)}{h} \in \mathcal{V}_1$
- $E\dot{x} = Ax$ constraints x to $x \in A^{-1}\{E\dot{x}\} \subseteq A^{-1}(E\mathcal{V}_1) =: \mathcal{V}_2 \supseteq \mathfrak{C}_{(E,A)}$
- $\dot{x} \in \mathcal{V}_2 \rightsquigarrow x \in A^{-1}(E\mathcal{V}_2) =: \mathcal{V}_3 \subseteq \mathfrak{C}_{(E,A)}$ ...
- $\mathcal{V}^* \supseteq \mathfrak{C}_{(E,A)}$, in fact, it turns out that $\mathcal{V}^* = \mathfrak{C}_{(E,A)}$

Regularity and Wong limits

Theorem (Ilchmann et al. 2012)

- (E, A) *is regular* $\iff$ $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$ *and* $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^\ell$
- $T := [V, W]$, $S = [EV, AW]^{-1}$ *where* $\operatorname{im} V = \mathcal{V}^*$ *and* $\operatorname{im} W = \mathcal{W}^*$ *gives QWF*

$$(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$$

Definition (Consistency projector and differential/impulsive selectors)

- Consistency projector $\Pi_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$
- Differential selector $\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S$
- Impulse selector $\Pi^{\text{imp}}_{(E,A)} := T \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S$

Explicit solution formula for regular DAEs

$$E\dot{x} = Ax + Bu \qquad (E,A) \overset{S,T}{\cong} \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix}\right)$$

Theorem (Solution formula, T. 2012)

(x,u) *is a smooth solution of* $E\dot{x} = Ax + Bu \iff$

$$x(t) = e^{A^{\text{diff}}t}\Pi_{(E,A)}x(0) + \int_0^t e^{A^{\text{diff}}(t-s)}B^{\text{diff}}u(s)\mathrm{d}s - \sum_{i=0}^{\nu-1}(E^{\text{imp}})^i B^{\text{imp}}u^{(i)}(t)$$

$\iff x = x^{\text{diff}} \oplus x^{\text{imp}}$ *where*

$$\dot{x}^{\text{diff}} = A^{\text{diff}}x + B^{\text{diff}}u, \quad x^{\text{diff}}(0) \in \operatorname{im}\Pi_{(E,A)}, \quad x^{\text{diff}}(t) \in \mathcal{V}^*$$

$$E^{\text{imp}}\dot{x}^{\text{imp}} = x^{\text{imp}} + B^{\text{imp}}u, \qquad x^{\text{imp}}(t) \in \mathcal{W}^*$$

Here $\nu > 0$ is smallest number such that $N^\nu = 0$ and is called index of DAE

Consistency projector

Corollary (Response to inconsistent initial value)

For $u = 0$ we have

$$x(0^+) = \Pi_{(E,A)} x(0^-), \qquad \Pi_{(E,A)} = T \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] T^{-1} = \Pi_{\mathcal{V}^*}^{\mathcal{W}^*}$$

Other jump rules

Wong-sequence based jump rule coincides with (Costantini et al. 2013):

- passivity based energy minimization jump rule (Frasca et al. 2010)
- Conservation of charge/flux (Liou 1972)
- Laplace transform approach (Opal & Vlach 1990)

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Distributional solutions - Dilemma
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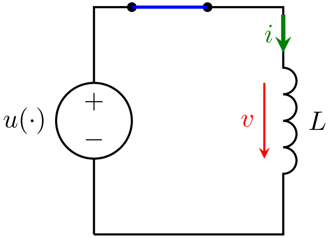
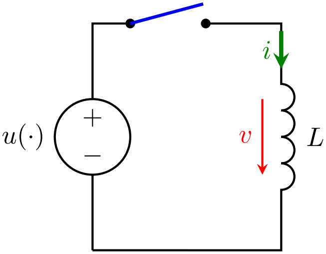
Motivating example

$t < 0$



$t \geq 0$

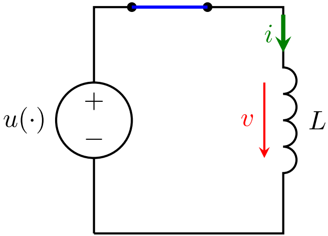


inductivity law: $L\frac{\mathrm{d}}{\mathrm{d}t}i = v$

switch dependent: $0 = v - u$ $\quad$ $0 = i$

Motivating example

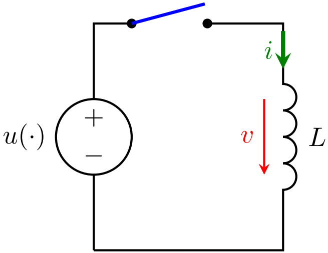
$t < 0$



$x = [i, v]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$t \geq 0$



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Motivating example

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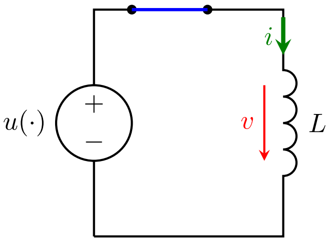
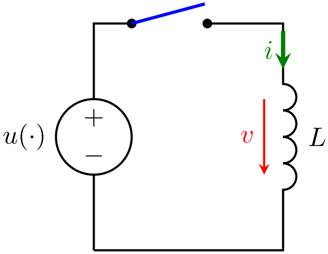


$E_1\dot{x} = A_1 x + B_1 u$
on $(-\infty, 0)$

$t \geq 0$



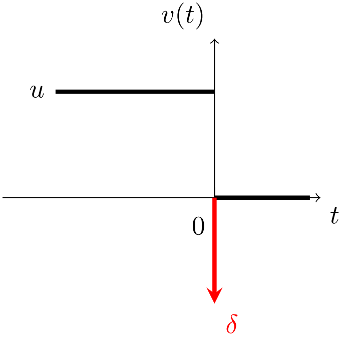
$E_2\dot{x} = A_2 x + B_2 u$
on $[0, \infty)$

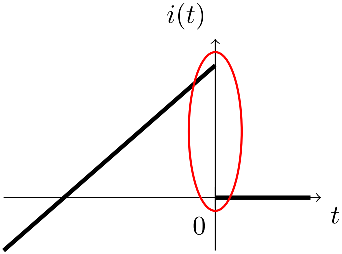
$\rightarrow$ switched differential-algebraic equation

Solution of circuit example

$t < 0$	$t \geq 0$
$v = u$	$i = 0$
$L\frac{\mathrm{d}}{\mathrm{d}t}i = v$	$v = L\frac{\mathrm{d}}{\mathrm{d}t}i$

Solution (assume constant input u):

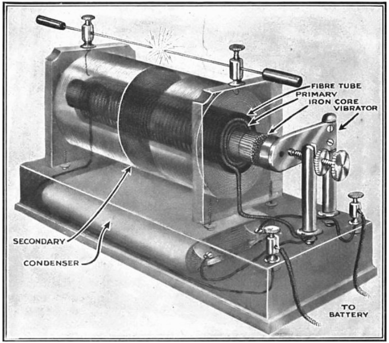




Dirac impulse is "real"

Dirac impulse

Not just a mathematical artifact!



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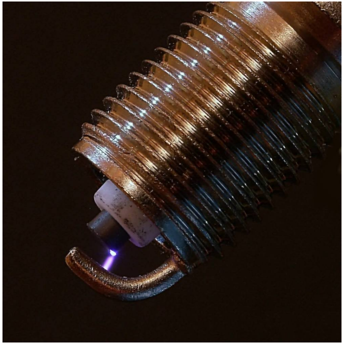

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Definition

Switch $\rightarrow$ Different DAE models (=modes)
depending on time-varying position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \rightarrow \{1, \ldots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \qquad \text{(swDAE)}$$

Attention

Each mode might have different consistency spaces
$\Rightarrow$ inconsistent initial values at each switch
$\Rightarrow$ Dirac impulses, in particular distributional solutions

Definition

Switch $\to$ Different DAE models (=modes)
depending on time-varying position of switch

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Switching signal $\sigma : \mathbb{R} \to \{1, \dots, N\}$ picks mode at each time $t \in \mathbb{R}$:

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Attention

Each mode might have different consistency spaces
$\Rightarrow$ inconsistent initial values at each switch
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Distribution theory - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ is “derivative” of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1) Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2) Axiomatic: Space of all “derivatives” of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi$ is smooth with compact support$\}$

Definition (Distributions)

$\mathbb{D} := \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D$ is linear and continuous$\}$

Definition (Regular distributions)

$f \in \mathcal{L}_{1,\text{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_\mathbb{D} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_\mathbb{R} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

$$(\mathbb{1}_{[0,\infty)\mathbb{D}})'(\varphi) = -\int_\mathbb{R} \mathbb{1}_{[0,\infty)}\varphi' = -\int_0^\infty \varphi' = -(\varphi(\infty) - \varphi(0)) = \varphi(0)$$

Multiplication with functions

Definition (Multiplication with smooth functions)

$\alpha \in \mathcal{C}^\infty : \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \tag{swDAE}$$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, C_\sigma \notin \mathcal{C}^\infty$

Observation, for $\sigma_{[t_i,t_{i+1})} \equiv p_i$, $i \in \mathbb{Z}$:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z} : \begin{aligned} (E_{p_i}\dot{x})_{[t_i,t_{i+1})} &= (A_{p_i}x + B_{p_i}u)_{[t_i,t_{i+1})} \\ y_{[t_i,t_{i+1})} &= (C_{p_i}x + D_{p_i}u)_{[t_i,t_{i+1})} \end{aligned}$$

BUT: Distributional restriction impossible to define (T. 2022)

Dilemma

Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

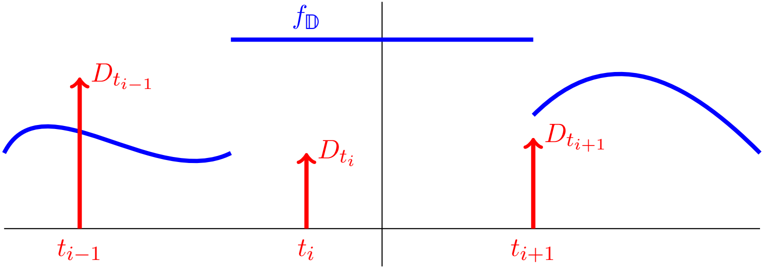
Space of distributions too big.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$, T. 2009)

$$\mathbb{D}_{\text{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\text{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T: D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\text{pw}}$ "$\subseteq$" $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ and $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Well definded restriction $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

$$D = f_{\mathbb{D}} + \sum_{t\in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t\in T\cap M} D_t$$

- Multiplication with $\alpha = \sum_{i\in\mathbb{Z}} \alpha_{i[t_i,t_{i+1})} \in \mathcal{C}^\infty_{\text{pw}}$ well defined:

$$\alpha D := \sum_{i\in\mathbb{Z}} \alpha_i D_{[t_i,t_{i+1})}$$

- Evaluation at $t \in \mathbb{R}$: $D(t^-) := f(t^-)$, $D(t^+) := f(t^+)$
- Impulses at $t \in \mathbb{R}$: $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$

Application to (swDAE)

(x,u) solves (swDAE) $:\Leftrightarrow$ (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

Piecewise-smooth distributional solution framework

$x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

- Existence and uniqueness of solutions?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?
- Control theoretical questions
  - Stability and stabilization
  - Observability and observer design
  - Controllability and controller design

Existence and uniqueness of solutions for (swDAE)

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \qquad \text{(swDAE)}$$

Basic assumptions

- $\sigma \in \Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty,0)} \text{ is constant} \end{array} \right\}.$
- (E_p, A_p) is regular $\forall p \in \{1, \ldots, N\}$, i.e. $\det(sE_p - A_p) \not\equiv 0$

Theorem (T. 2009)

Consider (swDAE) *satisfying the basic assumptions. Then*

$$\forall\, u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}\; \forall\, \sigma \in \Sigma_0\; \exists \text{ solution } x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$$

and $x(0^-)$ *uniquely determines* x.

Inconsistent initial values

$$E\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n$$

Initial trajectory problem = special switched DAE

$$\begin{aligned} x_{(-\infty,0)} &= x^0_{(-\infty,0)} \\ (E\dot{x})_{[0,\infty)} &= (Ax + Bu)_{[0,\infty)} \end{aligned} \qquad \text{(ITP)}$$

Corollary (Consistency projector and Dirac impulses)

Unique jumps and impulses for ITP, in particular, for $u = 0$,

$$x(0^+) = \Pi_{(E,A)} x^0(0^-)$$

$$x[0] = -\sum_{i=0}^{\nu-2} (E^{\mathsf{imp}})^{i+1} x^0(0^-) \delta^{(i)}$$

Sufficient conditions for impulse-freeness

Question

When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ impulse free?

Note: Jumps are OK.

Lemma (Sufficient conditions)

- *(E_p, A_p) all have index one (i.e. $(sE_p - A_p)^{-1}$ is proper)*
  $\Rightarrow$ (swDAE) impulse free
- *all consistency spaces of (E_p, A_p) coincide*
  $\Rightarrow$ (swDAE) impulse free

Characterization of impulse-freeness

Theorem (Impulse-freeness, T. 2009)

The switched DAE $E_\sigma \dot{x} = A_\sigma x$ *is impulse free* $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \ldots, N\}$$

where $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \ldots, N\}$ *is the* p*-th consistency projector.*

Remark

- Index-1-case $\Rightarrow$ $E_q(I - \Pi_q) = 0 \ \forall q$
- Consistency spaces equal $\Rightarrow$ $(I - \Pi_q)\Pi_p = 0 \ \forall p, q$

Contents

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Solution properties of DAEs
- Equivalence and four types of DAEs
- Regularity and quasi-Weierstrass form
- Wong sequences

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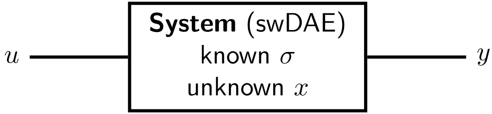
- Piecewise smooth distributions
- Distributional solutions
- Impulse-freeness

Observability
- Definition
- The single switch result
- Calculation of the four subspaces

Summary

Global Observability of Switched DAEs



Definition (Global observability)

(swDAE) with given σ is (globally) observable $:\Leftrightarrow$
$\forall$ solutions $(u_1, x_1, y_1), (u_2, x_2, y_2)$: $\quad (u_1, y_1) \equiv (u_2, y_2) \;\Rightarrow\; x_1 \equiv x_2$

Lemma (0-distinguishability)

(swDAE) *is observable if, and only if,* $\boxed{y \equiv 0 \text{ and } u \equiv 0 \quad \Rightarrow \quad x \equiv 0}$

Hence consider in the following (swDAE) without inputs:

$$\boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}} \quad \text{and observability question:} \quad \boxed{y \equiv 0 \overset{?}{\Rightarrow} x \equiv 0}$$

Motivating example

System 1:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\dot x = \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}x$$

$$y = \begin{bmatrix}0&0&1\end{bmatrix}x$$

$$y = x_3,\ \dot y = \dot x_3 = 0,\ x_2 = 0,\ \dot x_1 = 0$$

$\Rightarrow x_1$ unobservable

System 2:

$$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}\dot x = \begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}x$$

$$y = \begin{bmatrix}0&0&1\end{bmatrix}x$$

$$y = x_3 = \dot x_1,\ x_1 = 0,\ \dot x_2 = 0$$

$\Rightarrow x_2$ unobservable

$\sigma(\cdot) : 1 \to 2$

Jump in x_1 produces impulse in y
$\Rightarrow$ Observability

$\sigma(\cdot) : 2 \to 1$

Jump in x_2 no influence in y
$\Rightarrow x_2$ remains unobservable

Question

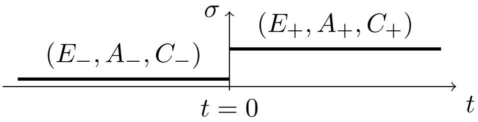
$$E_p\dot x = A_p x + B_p u,\quad y = C_p x + D_p u$$

not observable $\overset{?}{\Rightarrow}$

$$E_\sigma\dot x = A_\sigma x + B_\sigma u,\quad y = C_\sigma x + D_\sigma u$$

observable

The single switch result



Theorem (Unobservable subspace, Tanwani & T. 2010)

For (swDAE) *with* ***a single switch*** *the following equivalence holds*

$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathcal{M}$$

where

$$\boxed{\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}}$$

In particular: (swDAE) *observable* $\Leftrightarrow$ $\mathcal{M} = \{0\}$.

What are these four subspace?

The four subspaces

Unobservable subspace: $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$, i.e.

$$x(0^-) \in \mathcal{M} \quad \Leftrightarrow \quad y_{(-\infty,0)} \equiv 0 \ \wedge \ y[0] = 0 \ \wedge \ y_{(0,\infty)} \equiv 0$$

The four spaces

- Consistency: $x(0^-) \in \mathfrak{C}_-$
- Left unobservability: $y_{(-\infty,0)} \equiv 0 \ \Leftrightarrow \ x(0^-) \in \ker O_-$
- Right unobservability: $y_{(0,\infty)} \equiv 0 \ \Leftrightarrow \ x(0^-) \in \ker O_+^-$
- Impulse unobservability: $y[0] = 0 \ \Leftrightarrow \ x(0^-) \in \ker O_+^{\text{imp}}$

Question

How to calculate these four spaces?

Consistency space

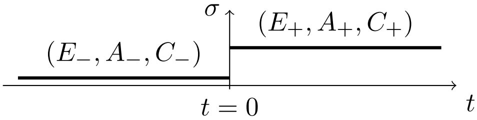
$$x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathsf{imp}-} \quad \Leftrightarrow \quad y \equiv 0$$

Corollary from QWF

$$\mathfrak{C}_- = \mathcal{V}_-^*$$

where $\mathcal{V}_-^*$ is the first Wong limit of (E_-, A_-).

The spaces O_-, O_+ and O_+^-



$$\dot{x} = A_-^{\text{diff}} x \quad \dot{x} = A_+^{\text{diff}}$$
$$y = C_- x \quad y \quad = C_+ x$$

Hence

$$y_{(-\infty,0)} \equiv 0 \quad \Rightarrow \quad x(0^-) \in \ker \underbrace{[C_-/C_-A_-^{\text{diff}}/C_-(A_-^{\text{diff}})^2/\cdots/C_-(A_-^{\text{diff}})^{n-1}]}_{:= O_-}$$

and

$$y_{(0,\infty)} \equiv 0 \quad \Rightarrow \quad x(0^+) \in \ker \underbrace{[C_+/C_+A_+^{\text{diff}}/C_+(A_+^{\text{diff}})^2/\cdots/C_+(A_+^{\text{diff}})^{n-1}]}_{:= O_+}$$

$$\ker O_+ \ni x(0^+) = \Pi_+ x(0^-) \quad \Longrightarrow \quad x(0^-) \in \Pi_+^{-1} \ker O_+ = \ker \underbrace{O_+\Pi_+}_{=: O_+^-}$$

The impulsive effect

Assume $(S_+E_+T_+, S_+A_+T_+) = \left(\begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix}\right)$:

Definition (Impulse “projector”)

$$\Pi_+^{\text{imp}} := T_+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S_+ \quad \text{and} \quad \boxed{E_+^{\text{imp}} := \Pi_+^{\text{imp}} E_+}$$

Impulsive part of solution:

$$x[0] = -\sum_{i=0}^{n-1} (E_+^{\text{imp}})^{i+1} x(0^-)\, \delta_0^{(i)}$$

Dirac impulses

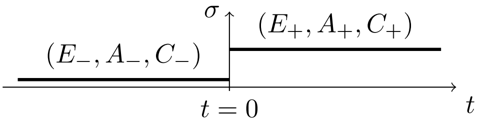
Conclusion:

$$y[0] = 0 \quad \Rightarrow \quad C_+ x[0] = 0 \quad \Rightarrow \quad \boxed{x(0^-) \in \ker O_+^{\text{imp}}}$$

where

$$O_+^{\text{imp}} := [C_+ E_+^{\text{imp}} \,/\, C_+ (E_+^{\text{imp}})^2 \,/\, \cdots \,/\, C_+ (E_+^{\text{imp}})^{n-1}]$$

Observability summary



$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$$

with

- $\mathfrak{C}_- = \mathcal{V}_-^*$ (first Wong limit)
- $O_- = [C_-/C_-A_-^{\text{diff}}/C_-(A_-^{\text{diff}})^2/\cdots/C_-(A_-^{\text{diff}})^{n-1}]$
- $O_+^- = [C_+/C_+A_+^{\text{diff}}/C_+(A_+^{\text{diff}})^2/\cdots/C_+(A_+^{\text{diff}})^{n-1}]\Pi_+$
- $O_+^{\text{imp}} = [C_+E_+^{\text{imp}}\,/\,C_+(E_+^{\text{imp}})^2\,/\,\cdots\,/\,C_+(E_+^{\text{imp}})^{n-1}]$

Example revisited

System 1:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\dot{x}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}x$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$\sigma(\cdot): 1 \to 2$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_1,e_3\},\\ \ker O_- &= \operatorname{span}\{e_1,e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1,e_2,e_3\},\\ \ker O_+^{\text{imp}} &= \operatorname{span}\{e_2,e_3\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M}=\{0\}$$

System 2:

$$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}\dot{x}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}x$$

$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$\sigma(\cdot): 2 \to 1$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_2\},\\ \ker O_- &= \operatorname{span}\{e_1,e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1,e_2\},\\ \ker O_+^{\text{imp}} &= \operatorname{span}\{e_1,e_2,e_3\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M}=\operatorname{span}\{e_2\}$$

Overall summary

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

Piecewise-smooth distributional solution framework

$x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

- Existence and uniqueness of solutions? ✓
- Jumps and impulses in solutions? ✓
- Conditions for impulse free solutions? ✓
- Control theoretical questions
  - Stability ✓ and stabilization ?
  - Observability ✓ and observer design ✓
  - Controllability ✓ and controller design ?
  - Extension to nonlinear case ?