Reduced realization for switched linear systems with known mode sequence

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Abstract

We consider switched linear systems with mode-dependent state-dimensions and/or state jumps and propose a method to obtain a switched system of reduced size with identical input-output behavior. Our approach is based in considering time-dependent reachability and unobservability spaces as well as suitable extended reachability and restricted unobservability spaces together with the notion of a weak Kalman decomposition. A key feature of our approach is that only the mode sequence of the switching signal needs to be known and not the exact switching times. However, the size of a minimal realization will in general depend on the mode durations, hence it cannot be expected that our method always leads to minimal realization. Nevertheless, we show that our method is optimal in the sense that a repeated application doesn't lead to a further reduction and we also highlight a practically relevant special case, where minimality is achieved.

Key words: Kalman decomposition, reachability, observability, switched systems.

1 Introduction

Realization theory is a classical topic in control theory and involves finding a (preferably) unique minimal systems which generate the specified input-output behavior of a certain class cf. [10, 25]. Moreover, realization theory provides a theoretical basis for model reduction, system identification and filtering/observer design. In [2], the minimal state space realization problem for (continuous) linear time-invariance systems was first studied based on hidden pole-zero cancellation techniques and in [10], the input-output description is revealed by considering the reachable and observable part of a dynamical system.

Realization theory of switched systems has already been discussed e.g. in [1, 13, 15, 14, 16, 17, 19, 20] and the references therein. In particular, in [13], the author combines the theory of rational formal power series with the classical automata theory to discuss the realization theory of hybrid systems. Specifically, the cases of arbitrary and constrained switching are discussed where the switching signal is considered as an input. This consideration of the switching signal as an "input" is a common viewpoint in most of the existing works, i.e. it is not possible to use these results when trying to find a (minimal) realization for a given switching signal (or given mode sequence). Without discussing realization theory, observability and reachability of switched systems have been studied in [12, 18, 22, 21, 24], the proposed approach is strongly inspired by these results.

In contrast to many of the existing literature on switched system, in this paper we view a switched linear systems as a piecewise-constant time-varying linear systems, in particular, a (minimal) realization in general depends on the specifically given switching signal.

To be more specific, we consider the switched linear systems (SLSs) with a given switching signal of the form

$$\Sigma_\sigma : \begin{cases} \dot{x}_k(t) = A_{\sigma(t)}x_k(t) + B_{\sigma(t)}u(t), \; t \in (s_k, s_{k+1}) \\ x_k(s_k^+) = J_{\sigma(s_k^+),\sigma(s_k^-)}x_{k-1}(s_k^-), \quad k \in \mathcal{Q} \\ y(t) = C_{\sigma(t)}x_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{1}$$

where $\sigma : \mathbb{R} \to \mathcal{Q} = \{0, 1, 2, \ldots, \mathtt{m}\} \subseteq \mathbb{N}$ is the given switching signal with finitely many switching times $s_1 < s_2 < \ldots < s_\mathtt{m}$ in the bounded interval $[t_0, t_f)$ be of interest and $x_k : (s_k, s_{k+1}) \to \mathbb{R}^{n_k}$ is the k-th piece of the state (whose dimension n_k may depend on the mode). For notational convenience let $s_0 := t_0$, $s_{\mathtt{m}+1} := t_f$ and let the duration of mode k be denoted by $\tau_k := s_{k+1} - s_k$, $k \in \{0, 1, \ldots, \mathtt{m}\}$. In the context of realization theory it

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is common to assume that the system starts with a zero initial condition, i.e. set $x_{-1}(t_0^-) := 0$, however, it will turn out that our approach can easily take into account the situation of a nonzero initial value. The input and output are given by $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^m$, respectively. Here, $x(t^-)$ and $x(t^+)$ denote, respectively, the left- and right-sided limit at t, assuming they exist.

For each mode $p \in \{0, 1, 2, \dots, \mathtt{m}\}$, the system matrices A_p, B_p, C_p of appropriate size describe the (continuous) dynamics corresponding to the linear system active on the interval (s_k, s_{k+1}) where $\sigma(t) = p$. Furthermore, $J_{p,q} : \mathbb{R}^{n_q} \to \mathbb{R}^{n_p}$ is the jump map from mode q to mode p. Note that due to the different space dimensions the introduction of a jump map is necessary; on the other hand, in case all state dimensions are equal, the consideration of a jump map is "optional" and leads to so called impulsive systems (in particular, our reduced realization results will also provide novel results for this system class).

It is well known that finding a minimal realization (which can be interpreted as removing unobservable and unreachable states) is a first step towards model reduction (which furthermore reduced difficult to observe and difficult to reach states). In [5], a time-varying model reduction approach is presented for switched linear systems (with identical state-dimensions and without jumps). However, the resulting reduced systems is not switched systems anymore, instead it is fully time-varying and it is difficult to handle numerically. Therefore, the aim is to gain insight into a more suitable model reduction approach by studying the minimal realization problem for switched systems of the form (1) *within* this system class. As already mentioned above, the process of going from a non-minimal representation (with initial value zero) to a minimal one can be seen as removing "unreachable" and "unobservable" states; understanding what the notions "unreachable" and "unobservable" exactly means in this context allows to generalize these ideas to "difficult to reach" and "difficult to observe" which then allows to perform model reduction.

The main goal is to find a reduced size switched system (with the same switching signal σ) of the form

$$\widehat{\Sigma}_\sigma : \begin{cases} \dot{\widehat{x}}_k(t) = \widehat{A}_{\sigma(t)}\widehat{x}_k(t) + \widehat{B}_{\sigma(t)}u(t), \ t \in (s_k, s_{k+1}) \\ \widehat{x}_k(s_k^+) = \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)}\widehat{x}_{k-1}(s_k^-), \quad k \in \mathcal{Q} \\ y(t) = \widehat{C}_{\sigma(t)}\widehat{x}_k(t^+), \quad t \in [s_k, s_{k+1}), \end{cases} \tag{2}$$

which has the same input-output behavior as the original system Σ_σ.

The single switch case was discussed in our conference contributions [7, 6] and a preliminary version of this manuscript is the conference submission [8], which doesn't contain all proofs and less details.

We will assume in the following that the switching signal is fixed, hence by suitable relabeling of the matrices, we can assume that $\sigma(t) = k$ on (s_k, s_{k+1}). Consequently, we can simply write $J_k := J_{\sigma(s_k^+),\sigma(s_k^-)} = J_{k,k-1}$ and $\widehat{J}_k := \widehat{J}_{\sigma(s_k^+),\sigma(s_k^-)} = \widehat{J}_{k,k-1}$ in the following. Furthermore, in some slight abuse of notation, we will speak in the following of the solution $x(\cdot)$ instead of the different solution pieces $x_k(\cdot)$.

This paper is organized as follows. In Section 2, the problem formulation and preliminaries are given, in particular, the concept of a weak Kalman decomposition is presented. In Section 3, the time-varying reachability and observability spaces are discussed for switched systems, and we define suitable extended reachable and restricted unobservable spaces. Section 4 discusses the main results with the proposed reduction algorithm. Finally, some numerical results are shown in Section 5.

2 Preliminaries

2.1 Reduced realization: definition

In this section, we introduce some notions and challenges related to reduced realizations of switched linear systems (1). Let's begin with the formal definition of reduced realization.

Definition 1 (Cf. [13]) *For Σ_σ as in (1) we define its* total dimension *as follows*

$$\dim \Sigma_\sigma := \sum_{q \in \mathcal{Q}} n_q.$$

Furthermore, we define its input-output behavior as follows

$$\mathfrak{B}_\sigma^{io} := \left\{ (u, y) \;\middle|\; \begin{array}{l} \exists x_q : (s_q, s_{q+1}) \to \mathbb{R}^{n_q} \text{ satisfying} \\ (1) \text{ and } x(t_0^-) = 0 \end{array} \right\}.$$

A switched linear system $\widehat{\Sigma}_\sigma$ with corresponding input-output behavior $\widehat{\mathfrak{B}}_\sigma^{io}$ is said to be a reduced realization of switched system Σ_σ if
1) $\mathfrak{B}_\sigma^{io} = \widehat{\mathfrak{B}}_\sigma^{io}$ and
2) $\dim \widehat{\Sigma}_\sigma \leq \dim \Sigma_\sigma$.

In the following we will also discuss minimal realizations, which are reduced realization of smallest total dimension under all reduced realizations. It should be noted that at this point it is not clear that the sequence of reduced state dimensions is unique for a minimal realization.

For non-switched linear systems, it is well known that a realization is minimal if, and only if, it is reachable and observable, however, for SLSs of the form (1) this is not the case in general as the following example shows:

Example 2 *Consider a switched linear system with two modes*

$$(A_0, B_0, C_0) = \left(\begin{bmatrix} 1&0&0\\0&2&0\\0&0&3 \end{bmatrix}, \begin{bmatrix} 1\\0\\1 \end{bmatrix}, \begin{bmatrix} 1&1&0 \end{bmatrix} \right),$$
$$(A_1, B_1, C_1) = \left(\begin{bmatrix} 0&0&0\\0&2&0\\0&0&1 \end{bmatrix}, \begin{bmatrix} 0\\1\\0 \end{bmatrix}, \begin{bmatrix} 1&0&1 \end{bmatrix} \right)$$

and the switching signal

$$\sigma(t) = \begin{cases} 0, & \text{on } (t_0, s_1), \\ 1, & \text{on } (s_1, t_f). \end{cases} \tag{3}$$

It is easily seen, that each mode is unreachable and unobservable. However, the switched system is reachable in the sense that each value $x(t_f^-) \in \mathbb{R}^3$ can be reached from zero by a suitable input and it is also observable in the sense that (for a vanishing input) only a zero initial value leads to a zero output.
On the other hand, the second state is unreachable in the $1st$ mode and unobservable in the $2nd$ mode. In particular, when starting with a zero initial value, for any input the value of the second state does not effect the output (because in the first mode it is identically zero and in the second mode the corresponding coefficient in the C-matrix is zero). Therefore, we can remove the second state without altering the input-output behavior.

Remark 3 *The above definition of reduced realization is not specifying any* method *how to obtain a reduced realization from a given switched system. In particular, it does not take into account constraints like the requirement that the reduced state is obtained via a uniform projection map (cf. [3, 4] in the context of model reduction). In general, a reduced realization can only be obtained by considering each mode individually (and by properly taking into account the effect from the other modes). Furthermore, Example 4 in [6] shows that by removing locally unreachable and unobservable states in each mode does not preserve the input-output behavior and hence does not lead to a reduced system.*

Another important challenge for obtaining a reduced realization is the fact, that even when we start with a classical switched system (i.e. all states have the same dimensions and the jump map is the identity), a reduced realization may have different state-space dimensions and/or requires the definition of a jump map. This is illustrated with the following example.

Example 4 *Consider a switched linear system with two modes*

$$A_0 = A_1 = \begin{bmatrix} 0&0&0\\1&0&0\\0&0&0 \end{bmatrix}, \quad B_0 = B_1 = \begin{bmatrix} 1\\0\\0 \end{bmatrix},$$
$$C_0 = \begin{bmatrix} 0&1&0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1&0&0 \end{bmatrix},$$

with switching signal (3) *and without jumps. It is easily seen that the first mode corresponds to a double integrator, while the second mode corresponds to a single integrator. Hence a minimal realization is given by the following switched linear system with mode-dependent state-dimensions:*

on $[t_0, s_1)$:	*on* $[s_1, t_f)$:
$\dot{z}_0 = \begin{bmatrix} 0&0\\1&0 \end{bmatrix} z_0 + \begin{bmatrix} 1\\0 \end{bmatrix} u,$	$\dot{z}_1 = 0 \cdot z_1 + u,$
$y = \begin{bmatrix} 0&1 \end{bmatrix} z_0,$	$y = z_1,$

with $z_1(s_1) = \begin{bmatrix} 1&0 \end{bmatrix} z_0(s_1)$.

The possible mode dependence of a reduced realization is our main motivation to study switched systems (1) with mode-dependent state-dimension and jumps, so that both systems (original system and the reduced realization) are from the same overall system class.

2.2 Weak Kalman decomposition

In order to obtain a reduced realization in the following, we will utilize extended reachable and restricted unobservable spaces together with the following *weak Kalman decomposition.*

Let's first recall the classical Kalman decomposition (KD) [10] for a linear system

$$\Sigma : \begin{cases} \dot{x}(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t), \end{cases} \tag{4}$$

with $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$. Based on the reachability and unobservable spaces [1], it is possible to define a coordinate transformation $x = \mathbb{T}z$ which leads to the following block triangular form $(\mathbb{T}^{-1}A\mathbb{T}, \mathbb{T}^{-1}B, C\mathbb{T}) =$

$$\left(\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ 0 & A_{22} & 0 & A_{24} \\ 0 & 0 & A_{33} & A_{34} \\ 0 & 0 & 0 & A_{44} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C_2 & 0 & C_4 \end{bmatrix} \right),$$

where $\left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \right)$ is reachable and $\left(\begin{bmatrix} A_{22} & A_{24} \\ 0 & A_{44} \end{bmatrix}, \begin{bmatrix} C_2 & C_4 \end{bmatrix} \right)$ is observable.

It is then easily seen that a minimal realization of (4) is now given by (A_{22}, B_2, C_2).

It should noted that the above minimal realization is only valid for vanishing initial values; if arbitrary initial values are considered, only the unobservable part can be removed without altering the corresponding input-output behavior.

[1] In fact, $\mathbb{T} = [V^1, V^2, V^3, V^4]$, where $\operatorname{im} V^1$ is the intersection of the reachable and unobservable space, $\operatorname{im}[V^1, V^2]$ is the reachable space and $\operatorname{im}[V^1, V^3]$ is the unobservable space.

In the context of switched systems all, but the first mode, will in general have non-trivial initial states but also not arbitrary initial states, which means that the classical KD cannot directly be used to obtain a reduced realization. In addition to consider an extended reachable space for each mode (due to the–partially–nonzero initial state) also the local unobservable space may need to be restricted, due to the fact, that an unobservable state in the current mode may become observable in the future and hence cannot be removed without altering the overall input-output behavior of the switched system. This motivates us to define a weak KD which takes into account an extended reachable space and restricted unobservable space.

Lemma 5 *Consider a classical LTI system* (4) *and let* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ *and* $\underline{\mathcal{U}} \subseteq \ker C$ *be two* A*-invariant subspaces (an extended reachable and a restricted unobservable space). For any coordinate transformation* $\overline{\mathbb{T}} = [\overline{V}^1, \overline{V}^2, \overline{V}^3, \overline{V}^4]$ *with* $\operatorname{im} \overline{V}^1 := \overline{\mathcal{R}} \cap \underline{\mathcal{U}}$, $\operatorname{im}[\overline{V}^1, \overline{V}^2] := \overline{\mathcal{R}}$, $\operatorname{im}[\overline{V}^1, \overline{V}^3] := \underline{\mathcal{U}}$, *we have* $(\overline{\mathbb{T}}^{-1} A \overline{\mathbb{T}}, \overline{\mathbb{T}}^{-1} B, C\overline{\mathbb{T}}) =$

$$\left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right). \quad (5)$$

In particular, $Ce^{At}B = C^2 e^{A^{22}t} B^2$ *for all* $t \in \mathbb{R}$.

Proof. Since $\overline{\mathcal{R}} \cap \underline{\mathcal{U}} = \operatorname{im} \overline{V}^1$ is A-invariant there is a matrix A^{11} of appropriate size such that $A\overline{V}^1 = \overline{V}^1 A^{11}$. The A-invariance of $\overline{\mathcal{R}}$ implies that $A\overline{V}^2 \subseteq \operatorname{im}[\overline{V}^1, \overline{V}^2]$, hence there exists A^{12}, A^{22} such that $A\overline{V}^2 = \overline{V}^1 A^{12} + \overline{V}^2 A^{22}$. Similarly, A-invariance of $\underline{\mathcal{U}}$ implies $A\overline{V}^3 \subseteq \operatorname{im}[\overline{V}^1, \overline{V}^3]$, hence there exists A^{13}, A^{33} such that $A\overline{V}^3 = \overline{V}^1 A^{13} + \overline{V}^3 A^{33}$. Finally, $\operatorname{im}[\overline{V}^1, \overline{V}^2, \overline{V}^3, \overline{V}^4] = \mathbb{R}^n$ implies existence of A^{14}, A^{24}, A^{34}, A^{44} such that $A\overline{V}^4 = \overline{V}^1 A^{14} + \overline{V}^2 A^{24} + \overline{V}^3 A^{34} + \overline{V}^4 A^{44}$. Combining all of the above, we obtain

$$A[\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] \begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix},$$

which shows that $\overline{\mathbb{T}}^{-1} A \overline{\mathbb{T}}$ has the desired block structure. Since $\operatorname{im} B \subseteq \overline{\mathcal{R}} = \operatorname{im}[\overline{V}^1, \overline{V}^2]$, there exists B^1, B^2 such that

$$B = \overline{V}^1 B^1 + \overline{V}^2 B^2 = [\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix},$$

from which the desired block structure of $\overline{\mathbb{T}}^{-1} B$ follows. Finally, $\ker C \supseteq \underline{\mathcal{U}} = \operatorname{im}[\overline{V}^1\ \overline{V}^3]$ implies that $C[\overline{V}^1\ \overline{V}^3] = \{0\}$, and hence, for $C^2 := C\overline{V}^2$ and $C^4 := C\overline{V}^4$,

$$C\overline{\mathbb{T}} = C[\overline{V}^1\ \overline{V}^2\ \overline{V}^3\ \overline{V}^4] = [0\ C^2\ 0\ C^4].$$

With these block structure, simple matrix multiplication leads to $Ce^{At}B = C^2 e^{A^{22}t} B^2$ for all $t \in \mathbb{R}$. □

For the formulation of forthcoming reduction method, we will need the following notations of invariant subspaces.

Definition 6 *For* $A \in \mathbb{R}^{n \times n}$ *and a subspace* $\mathcal{L} \subseteq \mathbb{R}^n$, *let*

$$\langle A \mid \mathcal{L} \rangle := \mathcal{L} + A\mathcal{L} + \ldots + A^{n-1}\mathcal{L}$$

be the smallest A*-invariant subspace containing* $\mathcal{L}$. *Furthermore, let (here* A^{-1} *stands for the preimage, it is not assumed that* A *is invertible)*

$$\langle \mathcal{L} \mid A \rangle := \mathcal{L} \cap A^{-1}\mathcal{L} \ldots \cap A^{-(n-1)}\mathcal{L}$$

be the largest A*-invariant subspace contained in* $\mathcal{L}$. △

Note that for any $C \in \mathbb{R}^{m \times n}$ we have

$$\langle \ker C \mid A \rangle = \ker[C^\top, (CA)^\top, \ldots, (CA^{n-1})^\top]^\top.$$

Furthermore, it is well known that for a linear system (A, B, C), the reachable space $\mathcal{R}$ is given by $\mathcal{R} = \langle A \mid \operatorname{im} B \rangle$ and the unobservable space $\mathcal{U}$ is given by $\langle \ker C \mid A \rangle$.

Remark 7 *Clearly, the choice* $\overline{\mathcal{R}} = \mathcal{R}$ *and* $\underline{\mathcal{U}} = \mathcal{U}$ *in Lemma 5 leads to the well known KD. Furthermore, any* A*-invariant subspace* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ *will be a superset of* $\mathcal{R}$, *because* $\mathcal{R}$ *is the smallest* A*-invariant subspace containing* $\operatorname{im} B$; *analogously, any* A*-invariant subspace* $\underline{\mathcal{U}} \subseteq \ker C$ *will be contained in* $\mathcal{U}$. *This is the motivation to call* $\overline{\mathcal{R}} \supseteq \mathcal{R}$ *an extended reachable space and* $\underline{\mathcal{U}} \subseteq \mathcal{U}$ *a restricted unobservable space in Lemma 5.*

For a linear system (A, B, C) with given extended reachable space $\overline{\mathcal{R}}$ and restricted unobservable space $\underline{\mathcal{U}}$ the weak KD (5) immediately leads to the reduced system (A^{22}, B^2, C^2) which can be obtained from (A, B, C) by suitable left and right projection defined as follows.

Definition 8 *For any coordinate transformation* $\overline{\mathbb{T}} = [\overline{V}^1, \overline{V}^2, \overline{V}^3, \overline{V}^4]$ *as in Lemma 5, let*

$$[(\overline{W}^1)^\top, (\overline{W}^2)^\top, (\overline{W}^3)^\top, (\overline{W}^4)^\top]^\top := \overline{\mathbb{T}}^{-1}$$

such that the sizes of $(\overline{W}^i)^\top$ *matches the size of* $\overline{V}^i$, $i = 1, 2, 3, 4$. *Then* $\overline{W}^2$ *and* $\overline{V}^2$ *are called the* weak-KD left-projector *and* weak KD right-projector, *respectively.* △

By definition of the weak-KD left- and right-projector, we have $\overline{W}^2\overline{V}^2 = I$ and

$$(A^{22}, B^2, C^2) = (\overline{W}^2 A\overline{V}^2, \overline{W}^2 B, C\overline{V}^2).$$

3 Exact (time-varying) reachability / Unobservability spaces

Our reduction approach relies on identifying suitable extended reachable and restricted unobservable spaces for each mode of the switched system (1). Towards this goal, we first provide expression for the exact (time-varying) reachable and unobservable space for (1) in the following. Before doing so, we briefly highlight that the solution of (1) is given recursively by, for $t \in [s_k, s_{k+1})$ and $k = 1, \dots, \mathtt{m}$,

$$x(t) := e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^{t} e^{A_k(t-s)} B_k u(s) ds. \tag{6}$$

and the output equation is given by

$$y(t) := C_k x(t),\ t \in [s_k, s_{k+1}),\ k = 0, 1, \dots, \mathtt{m}. \tag{7}$$

3.1 Exact (time-varying) reachability space

Definition 9 *The reachable space of the switched system* (1) *on time interval* $[t_0, t)$ *is defined by*

$$\mathcal{R}^{\sigma}_{[t_0,t)} := \left\{ x(t^-) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x,u) \text{ of } (1) \\ \text{with } x(t_0^-) = 0 \end{array} \right\}.$$

We call the switched system (1) *reachable (on* $[t_0, t_f)$*) if, and only if,*

$$\mathcal{R}^{\sigma}_{[t_0,t_f)} = \mathbb{R}^{n_{\mathtt{m}}}. \qquad \triangle$$

To calculate the reachability spaces of (1), the known reachability information from the previous modes needs to carry over appropriately to the current mode. Let $\mathcal{R}_k = \langle A_k \mid \operatorname{im} B_k \rangle$ be the local reachable subspace for mode k. We will show then that the reachable space at the end of the k-th mode is defined by the following recursive equation, $k = 1, 2, \dots, \mathtt{m}$:

$$\begin{aligned} \mathcal{M}^{\sigma}_0 &:= \mathcal{R}_0, \\ \mathcal{M}^{\sigma}_k &:= \mathcal{R}_k + e^{A_k \tau_k} J_k \mathcal{M}^{\sigma}_{k-1}. \end{aligned} \tag{8}$$

The intuition behind the sequence (8) is as follows. By starting with a zero initial value in the initial mode, clearly $\mathcal{R}^{\sigma}_{[t_0,s_1)} = \mathcal{R}_0$; continuing recursively, the reachable space at the end of mode k, is obtained by propagating forward the reachable space $\mathcal{M}^{\sigma}_{k-1}$ at the end of the previous mode, i.e. first jump via J_k and then propagate according to the matrix exponential (the time-evolution for a zero input). Finally, to take into account the effect of the input, the local reachable space of mode k is added. This intuition is formalized as follows.

Lemma 10 (Cf. [11]) *For all* $0 \le k \le \mathtt{m}$*,*

$$\mathcal{M}^{\sigma}_k = \mathcal{R}^{\sigma}_{[t_0,s_{k+1})}.$$

In particular, (1) *is reachable if, and only if* $\mathcal{M}^{\sigma}_{\mathtt{m}} = \mathbb{R}^{n_{\mathtt{m}}}$.

Proof. Clearly, $\mathcal{M}^{\sigma}_0 = \mathcal{R}^{\sigma}_{[t_0,s_1)}$. Inductively, assume that for some $k \in \{1, 2, \dots, \mathtt{m}\}$,

$$\mathcal{M}^{\sigma}_{k-1} = \mathcal{R}^{\sigma}_{[t_0,s_k)},$$

we will then show that $\mathcal{M}^{\sigma}_k = \mathcal{R}^{\sigma}_{[t_0,s_{k+1})}$. Let $x_{k+1} \in \mathcal{M}^{\sigma}_k$, then there exists $x_k \in \mathcal{M}^{\sigma}_{k-1}$ and $x_u \in \mathcal{R}_k$ such that $x_{k+1} = e^{A_k\tau_k} J_k x_k + x_u$. From $\mathcal{M}^{\sigma}_{k-1} = \mathcal{R}^{\sigma}_{[t_0,s_k)}$ it follows that there exists a solution $(\widehat{x}, \widehat{u})$ on $[t_0, s_k)$ with $\widehat{x}(0^-) = 0$ and $\widehat{x}(s_k^-) = x_k$.

In view of (6) the extension of $(\widehat{x}, \widehat{u})$ on the interval $[t_0, s_{k+1})$ via $(\widehat{x}(t), \widehat{u}(t)) := (e^{A_k(t-s_k)} J_k x_k, 0)$ is a solution of (1) on the larger interval $[t_0, s_{k+1})$. Furthermore, there exists a solution $(\widetilde{x}, \widetilde{u})$ of mode k on (s_k, s_{k+1}) with $\widetilde{x}(s_k^+) = 0$ and $\widetilde{x}(s_{k+1}^-) = x_u$.

By setting $(\widetilde{x}(t), \widetilde{u}(t)) = (0, 0)$ for all $t \in [t_0, s_k)$, it is easily seen that $(\widetilde{x}, \widetilde{u})$ is a solution of the switched system (1) on $[t_0, s_{k+1})$ with $\widetilde{x}(t_0^-) = 0$.

Altogether, by linearity we have that $(x, u) := (\widehat{x}, \widehat{u}) + (\widetilde{x}, \widetilde{u})$ is a solution of (1) on $[t_0, s_{k+1})$ with $x(t_0^-) = 0$ and

$$x(s_{k+1}^-) = \widehat{x}(s_{k+1}^-) + \widetilde{x}(s_{k+1}^-) = e^{A_k\tau_k} J_k x_k + x_u = x_{k+1},$$

which implies that $x_{k+1} \in \mathcal{R}^{\sigma}_{[t_0,s_{k+1})}$. Hence,

$$\mathcal{M}^{\sigma}_k \subseteq \mathcal{R}^{\sigma}_{[t_0,s_{k+1})}.$$

To show the converse subspace relationship, let $x_{k+1} \in \mathcal{R}^{\sigma}_{[t_0,s_{k+1})}$, then there exists a solution (x, u) of (1) with $x(s_{k+1}) = x_{k+1}$.

From $x(s_k^-) \in \mathcal{R}^{\sigma}_{[t_0,s_k)} = \mathcal{M}^{\sigma}_{k-1}$ and

$$x_u := \int_{s_k}^{s_{k+1}} e^{A_k(s_{k+1}-s)} B_k u(s) ds \in \mathcal{R}_k,$$

it follows immediately from (6) that $x_{k+1} = x(s_{k+1}) = e^{A_k\tau_k} J_k x(s_k^-) + x_u \in e^{A_k\tau_k} J_k \mathcal{M}^{\sigma}_{k-1} + \mathcal{R}_k = \mathcal{M}^{\sigma}_k$.

Now if the system (1) is reachable then

$$\mathcal{R}^\sigma_{[t_0,s_{\mathtt{m}+1})} = \mathbb{R}^{n_\mathtt{m}},$$

and consequently,

$$\mathcal{M}^\sigma_\mathtt{m} = \mathbb{R}^{n_\mathtt{m}}.$$

This completes the proof. □

From (8), it is clear that the reachable spaces depend on the switching times (in fact, on the mode duration τ_k) and this dependency cannot be avoided in general as the following example shows. In particular, the overall reachability of the switched system (1) on $[t_0, t_f)$ depends on the switching times and how long each mode is active.

Example 11 (Dependency on the switching times) *Consider the switched system* (1) *given by*

$$A_0 = A_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], A_1 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right],$$
$$B_0 = B_2 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], B_1 = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right],$$

with $J_{1,0} = J_{2,1} = I$. It is noted that none of the pairs (A_i, B_i) are reachable. Consider the switching signal σ with the mode sequence $0 \to 1 \to 2$ and switching times s_1, s_2. Let $\{e_1, e_2\}$ denote the natural basis vectors for $\mathbb{R}^2$.

Clearly, $\mathcal{R}_0 = \mathcal{R}_2 := \operatorname{span}\{e_1\}$, $\mathcal{R}_1 := \{0\}$, $e^{A_1\tau} = \left[\begin{smallmatrix} \cos\tau & -\sin\tau \\ \sin\tau & \cos\tau \end{smallmatrix}\right]$ and $e^{A_2\tau} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Hence

$$\mathcal{M}^\sigma_0 = \mathcal{R}_0 = \operatorname{span}\{e_1\},$$
$$\mathcal{M}^\sigma_1 = \mathcal{R}_1 + e^{A_1\tau_1} J_{1,0}\mathcal{M}^\sigma_0 = \operatorname{span}\left\{\left[\begin{smallmatrix} \cos\tau_1 \\ \sin\tau_1 \end{smallmatrix}\right]\right\},$$
$$\mathcal{M}^\sigma_2 = \mathcal{R}_2 + e^{A_2\tau_2} J_{2,1}\mathcal{M}^\sigma_1 = \operatorname{span}\{e_1\} + \operatorname{span}\left\{\left[\begin{smallmatrix} \cos\tau_1 \\ \sin\tau_1 \end{smallmatrix}\right]\right\}.$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$ then $\mathcal{M}^\sigma_2 = \operatorname{span}\{e_1\}$, otherwise, $\mathcal{M}^\sigma_2 = \mathbb{R}^2$. This clearly shows that the overall reachability of a switched system depends on the switching times. △

Note that although $\mathcal{M}^\sigma_k \supseteq \mathcal{R}_k \supseteq \operatorname{im} B_k$, the space $\mathcal{M}^\sigma_k$ is not a suitable extended reachable space for the mode (A_k, B_k, C_k) in the sense of Lemma 5, because it is *not* A_k-invariant in general. Before addressing this problem in Section 3.3, we recall first the "dual" space of the reachability spaces: the unobservable spaces.

3.2 Exact (time-varying) unobservability space

Definition 12 *The unobservable space of the switched system* (1) *on time interval $[t, t_f)$ is defined by*

$$\mathcal{U}^\sigma_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \begin{array}{l} \exists \text{ solution } (x, u = 0) \text{ such that} \\ y = 0 \text{ of (1) on } [t, t_f) \end{array} \right\}.$$

We call the switched system (1) *observable (on $[t_0, t_f)$) if, and only if,*

$$\mathcal{U}^\sigma_{[t_0,t_f)} = \{0\}.$$

△

Similar as for the reachable spaces, we aim to express the unobservable spaces recursively. Starting from the last mode it is clear that the unobservable space is the same as the classical unobservable space $\mathcal{U}_\mathtt{m} = \langle \ker C_\mathtt{m} \mid A_\mathtt{m} \rangle$. Recursively, the unobservable space at switch number $k+1$ can now be propagated backwards in time by first taking the preimage under the jump J_{k+1} and then further propagating it back with the continuous flow of mode k, i.e. by $e^{-A_k\tau_k}$. Finally, this propagated space needs to be combined with the local unobservable space of mode k given by $\mathcal{U}_k = \langle \ker C_k \mid A_k \rangle$. This motivates the definition of the following sequence of subspaces, $k = \mathtt{m}-1, \mathtt{m}-2, \ldots, 0$:

$$\begin{aligned} \mathcal{N}^\sigma_\mathtt{m} &:= \mathcal{U}_\mathtt{m}, \\ \mathcal{N}^\sigma_k &:= \mathcal{U}_k \cap \left(e^{-A_k\tau_k} J^{-1}_{k+1} \mathcal{N}^\sigma_{k+1}\right). \end{aligned} \tag{9}$$

Lemma 13 (Cf. [23, 11]) . *For all $0 \le k \le \mathtt{m}$,*

$$\mathcal{N}^\sigma_k = \mathcal{U}^\sigma_{[s_k,t_f)}.$$

In particular, (1) *is observable if, and only if $\mathcal{N}^\sigma_0 = \{0\}$.*

Proof. For $k = \mathtt{m}$, clearly $\mathcal{N}^\sigma_\mathtt{m} = \mathcal{U}^\sigma_{[s_\mathtt{m},t_f)}$. Inductively, assume now that for $k \in \{\mathtt{m}-1, \mathtt{m}-2, \ldots, 0\}$

$$\mathcal{N}^\sigma_{k+1} = \mathcal{U}^\sigma_{[s_{k+1},t_f)}$$

and we want to show that then $\mathcal{N}^\sigma_k = \mathcal{U}^\sigma_{[s_k,t_f)}$.

Let $x_k \in \mathcal{N}^\sigma_k$, then $x_k \in \mathcal{U}_k$ and there exists $x_{k+1} \in \mathcal{N}^\sigma_{k+1} = \mathcal{U}^\sigma_{[s_{k+1},t_f)}$ such that $x_{k+1} = J_{k+1}e^{A_k\tau_k}x_k$. Consequently, the unique solution $(x, u = 0)$ of (1) on $[s_k, t_f)$ with $x(s_k^+)$ satisfies $y = 0$ on $[s_k, s_{k+1})$ because $x_k \in \mathcal{U}_k$ and $y = 0$ on $[s_{k+1}, t_f)$ because $x(s_{k+1}) = x_{k+1} \in \mathcal{U}^\sigma_{[s_{k+1},t_f)}$. This shows that $x_k \in \mathcal{U}^\sigma_{[s_k,t_f)}$.

Now, let $x_k \in \mathcal{U}^\sigma_{[s_k,t_f)}$, then the unique solution $(x, u = 0)$ of (1) on $[s_k, t_f)$ with $x(s_k^+) = x_k$ has zero output. Consequently, $x_{k+1} := x(s_{k+1}^+) \in \mathcal{U}^\sigma_{[s_{k+1},t_f)} = \mathcal{N}^\sigma_{k+1}$. From $x_{k+1} = J_{k+1}e^{A_k\tau_k}x_k$, it follows that $x_k \in e^{-A_k\tau_k}J^{-1}_{k+1}\{x_{k+1}\} \subseteq e^{-A_k\tau_k}J^{-1}_{k+1}\mathcal{N}^\sigma_{k+1} = \mathcal{N}^\sigma_k$, which concludes the proof. □

Similar as for the reachability, the observability of the switched system in general depends on the switching time. This is illustrated by considering again Example 11 with an additional output.

Example 14 (Dependency on the switching times) *Recall Example 11 with output submatrices*

$$C_0 = C_2 = [\,0\ 1\,], C_1 = [\,0\ 0\,].$$

It is noted that none of the pairs (A_i, C_i) are observable.

Clearly, $\mathcal{U}_0 = \mathcal{U}_2 = \operatorname{span}\{e_1\}$, $\mathcal{U}_1 = \mathbb{R}^2$, $e^{-A_1\tau} = \left[\begin{smallmatrix} \cos\tau & \sin\tau \\ -\sin\tau & \cos\tau \end{smallmatrix}\right]$ *and* $e^{-A_2\tau} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. *Hence*

$$\begin{aligned}
\mathcal{N}_2^\sigma &= \mathcal{U}_2 = \operatorname{span}\{e_1\},\\
\mathcal{N}_1^\sigma &= \mathcal{U}_1 \cap e^{-A_1\tau_1} J_2^{-1} \mathcal{N}_2^\sigma = \mathbb{R}^2 \cap \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{smallmatrix}\right] \right\},\\
\mathcal{N}_0^\sigma &= \mathcal{U}_0 \cap e^{-A_0\tau_0} J_1^{-1} \mathcal{N}_1^\sigma = \operatorname{span}\{e_1\} \cap \operatorname{span}\left\{ \left[\begin{smallmatrix} \cos\tau_1 \\ -\sin\tau_1 \end{smallmatrix}\right] \right\}.
\end{aligned}$$

If $\tau_1 = k\pi$ for any $k \in \mathbb{N}$, then $\mathcal{N}_0^\sigma := \operatorname{span}\{e_1\}$, otherwise $\mathcal{N}_0^\sigma = \{0\}$. Therefore, the overall observability of (1) *depends on the switching time.* △

Note that similar to the reachability spaces, although the unobservable spaces $\mathcal{N}_k^\sigma$ satisfy $\mathcal{N}_k^\sigma \subseteq \mathcal{U}_k \subseteq \ker C$, they are not A_k-invariant and hence, they are not restricted unobservable spaces in the sense of Lemma 5.

3.3 Extended reachable / restricted unobservable spaces

So far, we have seen that the reachability spaces and observability spaces of (1) depend on the switching time. Even worse, when looking at the reachable / unobservable space at a particular time $t \in (s_k, s_{k+1})$ between two switches, then it is easily seen that these spaces in general also depend on the considered time t and a reduction method based on the exact reachability / observability spaces will necessarily result in general time-varying coordinate transformations / projections (cf. our previously proposed reduction method [5]) and would not lead to a reduced system of the desired form (2).

To circumvent this problem, we introduce suitable extended reachable and restricted unobservable spaces for the switched system (1). The key idea is based on the fact that for any subspace $\mathcal{H} \subseteq \mathbb{R}^n$, any matrix $A \in \mathbb{R}^{n\times n}$ and any $t \in \mathbb{R}$ the following subspace relationship holds:

$$\langle \mathcal{H} \mid A \rangle \subseteq e^{At}\mathcal{H} \subseteq \langle A \mid \mathcal{H} \rangle. \tag{10}$$

By replacing the matrix-exponentials in the constructions of the reachable / unobservable spaces by the corresponding A-invariant subspace we arrive at the following sequences (cf. [23] for the unobservable spaces):

$$\begin{aligned}
\overline{\mathcal{R}}_0 &:= \mathcal{R}_0,\\
\overline{\mathcal{R}}_k &:= \mathcal{R}_k + \langle A_k \mid J_k \overline{\mathcal{R}}_{k-1} \rangle,\ k = 1, \ldots, \mathtt{m};
\end{aligned} \tag{11}$$

$$\begin{aligned}
\underline{\mathcal{U}}_{\mathtt{m}} &:= \mathcal{U}_{\mathtt{m}},\\
\underline{\mathcal{U}}_k &:= \mathcal{U}_k \cap \langle J_{k+1}^{-1} \underline{\mathcal{U}}_{k+1} \mid A_k \rangle,\ k = \mathtt{m}-1, \ldots, 0.
\end{aligned} \tag{12}$$

In view of (10), it is easy to see that

$$\overline{\mathcal{R}}_k \supseteq \mathcal{M}_k^\sigma \supseteq \mathcal{R}_k \quad \text{and} \quad \underline{\mathcal{U}}_k \subseteq \mathcal{N}_k^\sigma \subseteq \mathcal{U}_k.$$

In particular, $\overline{\mathcal{R}}_{\mathtt{m}} = \mathbb{R}^{n_{\mathtt{m}}}$ and $\underline{\mathcal{U}}_0 = \{0\}$ respectively, are necessary conditions for reachability and observability of the overall switched system (5).

Finally, observe that by construction both $\overline{\mathcal{R}}_k$ and $\underline{\mathcal{U}}_k$ are A_k-invariant, i.e. they are extended reachable / restricted unobservable spaces in the sense of Lemma 5 and we are now ready to propose our main result about the reduction of switched systems of the form (1).

We conclude this section by highlighting an interesting special case, which is motivated by the following "application": Consider a large scale network whose dynamics can be described by a linear ODE. The network can be controlled through several actuators at different locations and several sensors are distributed throughout the network. However, due to resource limitation at any given time only one or a limited number of actuators can be used and the data of only one or a limited number of sensors is available. This situation can be modelled by the following switched system (without jumps)

$$\begin{aligned}
\dot{x} &= Ax + B_\sigma u,\\
y &= C_\sigma x,
\end{aligned} \tag{13}$$

where the switching signal is determined by the schedule of the actuator and sensor usages. In this scenario it seems rather natural that the mode sequence is fixed a priori (e.g. to make sure that all sensors and actuators are equally used), while the time duration may depend on the actual measured outputs. For this setup we have the following result:

Proposition 15 (Constant A-case) *Consider the switched linear systems* (13) *with corresponding time-dependent reachability space $\mathcal{R}^\sigma_{[t_0,t)}$ and unobservable space $\mathcal{U}^\sigma_{[t,t_f)}$. Then for all $t \in (s_k, s_{k+1})$ we have*

$$\mathcal{R}^\sigma_{[t_0,t)} = \overline{\mathcal{R}}_k \quad \text{and} \quad \mathcal{U}^\sigma_{[t,t_f)} = \underline{\mathcal{U}}_k,$$

i.e. the time-varying reachable and unobservable spaces are piecewise constant and can be calculated recursively via (11) *and* (12).

Proof. Inductively, it is easily seen that $\mathcal{R}^\sigma_{[t_0,t)}$ and $\mathcal{U}^\sigma_{[t,t_f)}$ are A-invariant, from which the claim follows.

4 Main result: Proposed reduction method

We now propose a method to compute a reduced realization (2) of (1) for a given switching signal.

Step 1. Compute the sequence of extended reachable $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \cdots, \overline{\mathcal{R}}_{\mathtt{m}}$ and restricted unobservable subspaces $\underline{\mathcal{U}}_0, \underline{\mathcal{U}}_1, \cdots, \underline{\mathcal{U}}_{\mathtt{m}}$ as in (11) and (12).
Step 2. Apply Lemma 5 to (A_k, B_k, C_k) with $(\overline{\mathcal{R}}_k, \underline{\mathcal{U}}_k)$ to compute the weak-KD left- and right-projectors $\overline{W}_k^2, \overline{V}_k^2$, and let

$$\left(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k\right) = \left(\overline{W}_k^2 A_k \overline{V}_k^2, \overline{W}_k^2 B_k, C_k \overline{V}_k^2\right).$$

Step 3. Calculate the reduced jump map

$$\widehat{J}_k := \overline{W}_k^2 J_k \overline{V}_{k-1}^2.$$

Before showing that the resulting reduced system (2) is indeed a realization of (1), we first highlight an important connection between the solutions of both systems.

Lemma 16 *Consider the switched system Σ_σ as in* (1) *and the reduced system $\widehat{\Sigma}_\sigma$ as in* (2) *obtained by the left- and right-projectors $\overline{W}_{\sigma(\cdot)}^2, \overline{V}_{\sigma(\cdot)}^2$. If $x(\cdot)$ is a solution of Σ_σ then $\widehat{x}(\cdot) := \overline{W}_{\sigma(\cdot)}^2 x(\cdot)$ is a solution of $\widehat{\Sigma}_\sigma$.*

Proof. Consider any time interval (s_k, s_{k+1}) between two switches, then, for $t \in (s_k, s_{k+1})$,

$$\begin{aligned}\dot{\widehat{x}}(t) &= \overline{W}_k^2 \dot{x} = \overline{W}_k^2 A_k x(t) + \overline{W}_k^2 B u(t) \\ &= [0, \widehat{A}_k, 0, *] \overline{T}_k^{-1} x(t) + B_k^2 u(t),\end{aligned}$$

where $\overline{T}_k = [\overline{V}_k^1, \overline{V}_k^2, \overline{V}_k^3, \overline{V}_k^4]$ is the coordinate transformation according to Lemma 5 for mode k. Since $x(t) \in \mathcal{R}_{[t_0,t)}^\sigma \subseteq \overline{\mathcal{R}}_k = \operatorname{im}[\overline{V}_k^1, \overline{V}_k^2]$, it follows that $\overline{T}_k^{-1} x(t) = [*, \widehat{x}(t)^\top, 0, 0]^\top$ and hence, as claimed, for all $t \in (s_k, s_{k+1})$

$$\dot{\widehat{x}}(t) = \widehat{A}_k \widehat{x}(t) + \widehat{B}_k u(t).$$

In particular, due to unique solvability of linear ODEs, for any solutions x of Σ_σ and $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ the following implication holds:

$$\overline{W}_k^2 x(s_k^+) = \widehat{x}(s_k^+) \implies \forall t \in (s_k, s_{k+1}) : \overline{W}_k^2 x(t) = \widehat{x}(t).$$

To show that $\widehat{x} = \overline{W}_\sigma^2 x$ is indeed a global solution of $\widehat{\Sigma}_\sigma$ it therefore remains to be shown that

$$\overline{W}_k^2 x(s_k^+) = \widehat{J}_k \overline{W}_{k-1}^2 x(s_k^-). \tag{14}$$

In fact,

$$\begin{aligned}\overline{W}_k^2 x(s_k^+) &= \overline{W}_k^2 J_k x(s_k^-) = \overline{W}_k^2 J_k \overline{T}_{k-1} \overline{T}_{k-1}^{-1} x(s_k^-) \\ &= \overline{W}_k^2 J_k [\overline{V}_{k-1}^1, \overline{V}_{k-1}^2, \overline{V}_{k-1}^3, \overline{V}_{k-1}^4] \begin{pmatrix} * \\ \overline{W}_{k-1}^2 x(s_k^-) \\ 0 \\ 0 \end{pmatrix}.\end{aligned}$$

From (12) it is easily seen that $J_k \underline{\mathcal{U}}_{k-1} \subseteq \underline{\mathcal{U}}_k$, hence $\operatorname{im} J_k \overline{V}_{k-1}^1 \subseteq \operatorname{im} J_k [\overline{V}_{k-1}^1, \overline{V}_{k-1}^3] = J_k \underline{\mathcal{U}}_{k-1} \subseteq \underline{\mathcal{U}}_k = \operatorname{im}[\overline{V}_k^1 \ \overline{V}_k^3] \subseteq \ker \overline{W}_k^2$, i.e. $\overline{W}_k^2 J_k \overline{V}_{k-1}^1 = 0$, from which it follows that

$$\overline{W}_k^2 x(s_k^+) = \overline{W}_k^2 J_k \overline{V}_{k-1}^2 \overline{W}_{k-1}^2 x(s_k^-)$$

as desired. □

As a consequence of the above and of the uniqueness of solutions it follows that every solution $\widehat{x}$ of $\widehat{\Sigma}_\sigma$ with zero initial value and given input u satisfies $\widehat{x} = \overline{W}_\sigma^2 x$ where x is the solution of Σ_σ with zero initial value and the same input u. We will now prove that the corresponding outputs are indeed equal.

Theorem 17 *Consider the switched system Σ_σ as in* (1) *and the reduced system $\widehat{\Sigma}_\sigma$ as in* (2) *obtained by the above reduction method. Then Σ_σ and $\widehat{\Sigma}_\sigma$ are input-output equivalent in the sense that for all inputs u the output y of* (1) *with initial condition $x(t_0^-) = 0$ equals the output $\widehat{y}$ of* (2) *with initial condition $\widehat{x}(t_0^-) = 0$.*

Proof. The output of Σ_σ on $[s_k, s_{k+1})$ is given by

$$\begin{aligned}y(t) &= C_k e^{A_k(t-s_k)} J_k x(s_k^-) + \int_{s_k}^t C_k e^{A_k(t-s)} B_k u(s) ds \\ &=: y_{\mathtt{J}}(t) + y_{\mathtt{u}}(t).\end{aligned}$$

Inserting suitable identity matrices we have that

$$y_{\mathtt{J}} = C_k \overline{\mathbb{T}}_k e^{\overline{\mathbb{T}}_k^{-1} A_k \overline{\mathbb{T}}_k (t-s_k)} \overline{\mathbb{T}}_k^{-1} J_k \overline{\mathbb{T}}_{k-1} \overline{\mathbb{T}}_{k-1}^{-1} x(s_k^-),$$

$$y_{\mathtt{u}}(t) = \int_{s_k}^t C_k \overline{\mathbb{T}}_k e^{\overline{\mathbb{T}}_k^{-1} A_k \overline{\mathbb{T}}_k (t-s)} \overline{\mathbb{T}}_k^{-1} B_k u(s) ds,$$

where $\overline{\mathbb{T}}_k = [\overline{V}_k^1, \overline{V}_k^2, \overline{V}_k^3, \overline{V}_k^4]$ is the coordinate transformation according to Lemma 5 for mode k. The special block structure of the matrices $\overline{\mathbb{T}}_k^{-1} A_k \overline{\mathbb{T}}_k$, $\overline{\mathbb{T}}_k^{-1} B_k$, $C_k \overline{\mathbb{T}}_k$ implied by Lemma 5 immediately leads to

$$y_{\mathtt{u}}(t) = \int_{s_k}^t \widehat{C}_k e^{\widehat{A}_k (t-s)} \widehat{B}_k u(s) ds.$$

Hence, for showing $\widehat{y}(t) = y(t) = y_{\mathtt{J}}(t) + y_{\mathtt{u}}(t)$ it remains to be shown that

$$y_{\mathtt{J}}(t) = \widehat{C}_k e^{\widehat{A}_k(t-s_k)} \widehat{J}_k \widehat{x}(s_k^-). \tag{15}$$

With similar arguments as used to establish (14) in Lemma 16 we can show that

$$\overline{\mathbb{T}}_k^{-1} J_k \overline{\mathbb{T}}_{k-1} \overline{\mathbb{T}}_{k-1}^{-1} x(s_k^-) = \begin{pmatrix} * \\ \widehat{J}_k \overline{W}_k^2 x(s_k^-) \\ 0 \\ 0 \end{pmatrix}.$$

Using the already established fact in Lemma 16, that $\overline{W}_k^2 x(s_k^-) = \widehat{x}(s_k^-)$ together with the special block structures of $\overline{\mathbb{T}}_k^{-1} A_k \overline{\mathbb{T}}_k$, $\overline{\mathbb{T}}_k^{-1} B_k$, $C_k \overline{\mathbb{T}}_k$ we can conclude that (15) holds. $\square$

Remark 18 (Non-zero initial values) *Our method can easily be adjusted to account for non-zero initial values. Assume $x(t_0^-) \in \mathcal{X}_0$ for some subspace $\mathcal{X}_0 \subseteq \mathbb{R}^n$, then in* (8) *we just have to replace the initial definition by*

$$\mathcal{M}_0^\sigma := \mathcal{R}_0 + e^{A_0 \tau_0} J_0 \mathcal{X}_0$$

and in (11) *the initial space needs to be adjusted to*

$$\overline{\mathcal{R}}_0 := \mathcal{R}_0 + \langle A_0 \mid J_0 \mathcal{X}_0 \rangle,$$

while the definition of the other subspaces remain unchanged.

A key feature of our method is that it is independent of the actual switching times (or mode durations) and only requires knowledge of the mode sequence. The following example shows however that the size of a minimal realization depends on the mode durations, hence we cannot expect that our method results in a minimal realization in general.

Example 19 *Consider a switched system with modes*

$$A_0 = A_2 = \begin{bmatrix} 0&0&0 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}, A_1 = \begin{bmatrix} 0&0&0 \\ 0&0&-1 \\ 0&1&0 \end{bmatrix}, B_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},$$
$$B_1 = B_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_0 = C_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, C_2 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}.$$

with $J_{1,0} = J_{2,1} = I$. Assume the mode sequence $0 \to 1 \to 2$. Fix the switching time duration $\tau_1 = \pi/2$ for mode 1*. Then the original solution x and output y of each time interval can be characterized as follows:*

$$t \in (t_0, s_1): \; x(t) = \begin{bmatrix} * \\ * \\ 0 \end{bmatrix}, \quad y(t) = C_0 x(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ 0 \end{bmatrix},$$
$$t \in (s_1, s_1 + \tfrac{\pi}{2}): \; x(t) = \begin{bmatrix} * \\ * \\ * \end{bmatrix}, \quad y(t) = C_1 x(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} * \\ * \\ * \end{bmatrix},$$
$$x(s_2) = x(s_1 + \tfrac{\pi}{2}) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}$$
$$t \in (s_2, t_f): \; x(t) = \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}, \quad y(t) = C_2 x(t) = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} * \\ 0 \\ * \end{bmatrix}.$$

Clearly, the second and third states do not affect the output for this specific switching signal. In particular, it is easily seen that the overall input-output behavior is described by the (nonswitched) system $\dot{\widehat{x}} = u$, $y = \widehat{x}$. However, if we apply our proposed method, then the sequence of reachable and unobservable spaces are given by

$$\begin{aligned} \mathcal{M}_1^\sigma &= \operatorname{im} B_0, & \mathcal{N}_0^\sigma &= \{0\}, \\ \mathcal{M}_2^\sigma &= \mathbb{R}^3, & \mathcal{N}_1^\sigma &= \{0\}, \\ \mathcal{M}_3^\sigma &= \mathbb{R}^3, & \mathcal{N}_2^\sigma &= \operatorname{span}\{e_3\}. \end{aligned}$$

Indeed, the sequences produce a switched system with modes in dimensions 1, 3 *and* 2*, respectively, instead of a one dimensional minimal systems. Nevertheless, one should note that for $\tau_1 \neq k\pi/2$, our method actually produces a minimal realization.* $\triangle$

The previous example however leads to our believe that our method results in a minimal realization for *almost all* switching times. While we have not been able to prove this conjecture, we are able to show that our method is optimal in the sense that a repeated application doesn't lead to a further reduction.

Theorem 20 *Consider the switched system Σ_σ and the reduced switched system $\widehat{\Sigma}_\sigma$ resulting from our proposed method. Let $\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)}$ and $\underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)}$ be the sequences of reachability and unobservability spaces, respectively, of $\widehat{\Sigma}_\sigma$. Then*

$$\overline{\widehat{\mathcal{R}}}_{\sigma(\cdot)} = \mathbb{R}^{\widehat{n}_{\sigma(\cdot)}}, \quad \underline{\widehat{\mathcal{U}}}_{\sigma(\cdot)} = \{0\}.$$

In particular, the left- and right-projectors for a potential further reduction are given by identity matrices, i.e. no further reduction occurs.

Proof. Our proposed methods yields for each mode k a coordinate transformation $\overline{\mathbb{T}}_k$ such that (A_k, B_k, C_k) is transformed to

$$\left(\begin{bmatrix} A_k^{11} & A_k^{12} & A_k^{13} & A_k^{14} \\ 0 & \widehat{A}_k & 0 & A_k^{24} \\ 0 & 0 & A_k^{33} & A_k^{34} \\ 0 & 0 & 0 & A_k^{44} \end{bmatrix}, \begin{bmatrix} B_k^1 \\ \widehat{B}_k \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & \widehat{C}_k & 0 & C_k^4 \end{bmatrix} \right), \tag{16}$$

where $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k)$ is the input-output equivalent reduced system for mode k. By construction, the extended reachable and restricted unobservable spaces of (A_k, B_k, C_k) are given by

$$\overline{\mathcal{R}}_k = \mathbb{T}_k \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \underline{\mathcal{U}}_k = \mathbb{T}_k \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix},$$

respectively.

Seeking a contradiction assume $\overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}$ (Case I), or $\underline{\widehat{\mathcal{U}}}_k \neq \{0\}$ (Case II) for some k.

Case I: For $k = 0$ we see that from $\overline{\mathcal{R}}_0 = \mathcal{R}_0$ it follows that the pair $(\widehat{A}_0, \widehat{B}_0)$ must be reachable and hence $\overline{\widehat{\mathcal{R}}}_0 = \widehat{\mathcal{R}}_0 = \mathbb{R}^{\widehat{n}_0}$. Assume now inductively that for some k we have $\overline{\widehat{\mathcal{R}}}_{k-1} = \mathbb{R}^{\widehat{n}_{k-1}}$ and $\overline{\widehat{\mathcal{R}}}_k \subsetneq \mathbb{R}^{\widehat{n}_k}$. Since $\overline{\widehat{\mathcal{R}}}_k$ is $\widehat{A}_k$-invariant and contains $\operatorname{im} \widehat{B}_k$ we can choose a coordinate transformation $\overline{\widehat{\mathbb{T}}}_k$ such that $(\widehat{A}_k, \widehat{B}_k)$ is transformed to

$$\left(\begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix}, \begin{bmatrix} \widehat{B}_k^1 \\ 0 \end{bmatrix} \right) \tag{17}$$

and $\operatorname{im} \overline{\widehat{\mathbb{T}}}_k \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \overline{\widehat{\mathcal{R}}}_k$. By adjusting the original coordinate transformation $\overline{\mathbb{T}}_k$ we can assume in the following that $(\widehat{A}_k, \widehat{B}_k)$ is actually equal to (17). In particular, we then have

$$\operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] = \overline{\widehat{\mathcal{R}}}_k = \widehat{\mathcal{R}}_k + \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle.$$

Since $\widehat{\mathcal{R}}_k = \langle \widehat{A}_k \mid \widehat{B}_k \rangle \subseteq \operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ we can conclude that, $\operatorname{im} \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right] \supseteq \langle \widehat{A}_k \mid \widehat{J}_k \overline{\widehat{\mathcal{R}}}_{k-1} \rangle = \langle \widehat{A}_k \mid \operatorname{im} \widehat{J}_k \rangle \supseteq \operatorname{im} \widehat{J}_k$. Therefore (A_k, B_k, J_k) is actually transformed to

$$\left(\begin{bmatrix} * & * & * & * \\ 0 & \begin{bmatrix} \widehat{A}_k^1 & * \\ 0 & \widehat{A}_k^2 \end{bmatrix} & 0 & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \begin{bmatrix} * \\ \begin{bmatrix} \widehat{B}_k^1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} * \\ \begin{bmatrix} J_k^1 \\ 0 \end{bmatrix} \\ 0 \\ 0 \end{bmatrix} \right).$$

From this we arrive at the following contradiction:

$$\operatorname{im} \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \overline{\mathcal{R}}_k = \mathcal{R}_k + \langle A_k \mid J_k \overline{R}_{k-1} \rangle \subseteq \operatorname{im} \begin{bmatrix} I & 0 \\ 0 & \left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right] \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence we have inductively shown that $\overline{\widehat{\mathcal{R}}}_k = \mathbb{R}^{\widehat{n}_k}$ for all mode k.

Case II: Assume $\underline{\widehat{\mathcal{U}}}_k \neq \{0\}$. Analogously as in Case I, the contradiction

$$\underline{\mathcal{U}}_k \neq \operatorname{im} \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix},$$

arises, the details are omitted. □

For the special case of constant A-matrices, our method does in fact result in a minimal realization.

Corollary 21 *Consider the switched system* (13) *with mode-independent A-matrix. Then the reduced switched system obtained via our proposed reduction method is minimal.*

Proof. This is a simple consequence from Proposition 15, because in any mode a smaller reduced model would necessarily remove some reachable and observable states and hence cannot lead to the same input-output behavior.

5 Numerical results

In this section, we demonstrate the operation of the proposed reduction method for the switched linear system. The proposed method is illustrated by means of numerical examples. The source code for the numerical examples is available from [9].

Example 22 *Consider a switched linear system with modes:*

$$\begin{aligned}
(A_0, B_0, C_0) &= \left(\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \right), \\
(A_1, B_1, C_1) &= \left(\begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \right), \\
(A_2, B_2, C_2) &= \left(\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \right), \\
J_{1,0} &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad J_{2,1} = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix}.
\end{aligned}$$

Assume the mode sequence $0 \to 1 \to 2$. We apply the proposed reduction method and the reduced systems can be obtained as follows.

Step 1. *Here,* $\mathcal{R}_0 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right]$, $\mathcal{R}_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, $\mathcal{R}_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]$, $\mathcal{U}_0 = \left[\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right]$, $\mathcal{U}_1 = \left[\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{smallmatrix}\right]$, $\mathcal{U}_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. *Now we compute the sequence of reachable and unobservable spaces:*

$$\begin{aligned}
\overline{\mathcal{R}}_0 &= \mathcal{R}_0 = \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right], \\
\overline{\mathcal{R}}_1 &= \mathcal{R}_1 + \langle A_1 \mid J_{1,0} \overline{\mathcal{R}}_0 \rangle = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right], \\
\overline{\mathcal{R}}_2 &= \mathcal{R}_2 + \langle A_2 \mid J_{2,1} \overline{\mathcal{R}}_1 \rangle = \mathbb{R}^3, \\
\underline{\mathcal{U}}_2 &= \mathcal{U}_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \\
\underline{\mathcal{U}}_1 &= \mathcal{U}_1 \cap \langle J_{2,1}^{-1} \, \underline{\mathcal{U}}_2 \mid A_1 \rangle = \left[\begin{smallmatrix} 0 \\ 0 \\ 1 \\ 0 \end{smallmatrix}\right], \\
\underline{\mathcal{U}}_0 &= \mathcal{U}_0 \cap \langle J_{1,0}^{-1} \, \underline{\mathcal{U}}_1 \mid A_0 \rangle = \{0\}.
\end{aligned}$$

Step 2. *Via the proposed method, the sequence of left- and right-projectors are obtained by*

$$\begin{aligned}
(\overline{W}_0^2, \overline{V}_0^2) &= \left(\left[\begin{smallmatrix} 1 \\ -2 \\ -2 \end{smallmatrix}\right]^\top, \left[\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}\right] \right), \\
(\overline{W}_1^2, \overline{V}_1^2) &= \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]^\top, \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{smallmatrix}\right] \right), \\
(\overline{W}_2^2, \overline{V}_2^2) &= \left(\left[\begin{smallmatrix} 0 \\ -1 \\ 0 \end{smallmatrix}\right]^\top, \left[\begin{smallmatrix} 0 \\ -1 \\ -1 \end{smallmatrix}\right] \right).
\end{aligned}$$

The reduced switched system is given by

$$
\begin{aligned}
(\widehat{A}_0, \widehat{B}_0, \widehat{C}_0) &= (\overline{W}_0^2 A_0 \overline{V}_0^2, \overline{W}_0^2 B_0, C_0 \overline{V}_0^2) = (2, 1, 1)\,, \\
(\widehat{A}_1, \widehat{B}_1, \widehat{C}_1) &= \left(\overline{W}_1^2 A_1 \overline{V}_1^2, \overline{W}_1^2 B_1, C_1 \overline{V}_1^2\right) \\
&= \left(\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \end{smallmatrix}\right]\right), \\
(\widehat{A}_2, \widehat{B}_2, \widehat{C}_2) &= \left(\overline{W}_2^2 A_2 \overline{V}_2^2, \overline{W}_2^2 B_2, C_2 \overline{V}_2^2\right) = (1, -1, -1)\,.
\end{aligned}
$$

Step 3. *The reduced jump maps are given by*

$$
\widehat{J}_1 = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \quad \widehat{J}_2 = \left[\begin{smallmatrix} 0 & -1 \end{smallmatrix}\right].
$$

Figure 1 shows the output of the original and its minimal switched linear system for input $u(t) = 1$ *with switching times* $s_1 = 2$ *and* $s_2 = 5$ *over* $[0, 6]$ *and clearly both outputs coincide.*

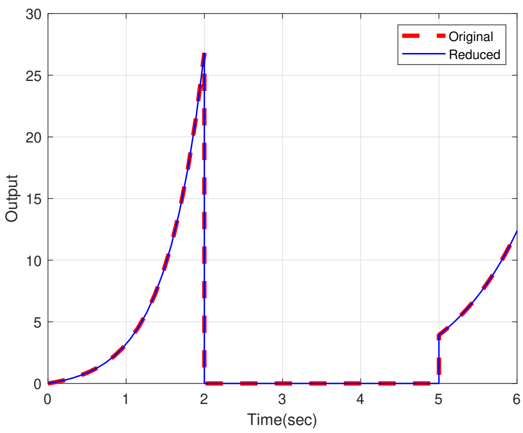

Fig. 1. Outputs of original system and the proposed reduced system.

6 Conclusions

In this paper, we have proposed a method for obtaining a reduced realization for switched linear systems with jumps and mode-dependent state-dimensions; the switching signal is assumed to be fixed with known mode sequence. Our reduction method is independent of the switching times and hence in principle also applicable for state-dependent switched systems if a certain mode sequence is known a-priori. The proposed reduction method is based on a weak Kalman decomposition of each mode by defining suitable extended reachable and restricted unobservable spaces. We believe, that our method results in a minimal realization for almost all switching times, however, a definite answer to this question is still ongoing research. It cannot be expected that our method will result in a minimal realization for *all* switching times, we provided an example for which the dimension of the minimal realization depends on the specific switching times. We have so far assumed that all subspace related operations (intersections, sums, . . .) can be carried out with exact arithmetics, however, for large scale systems and/or for systems with numerical coefficient matrices the involved subspace calculations are in general ill-posed. A suitable adaption of our algorithm utilizing e.g. the singular value decomposition to carry out the subspace calculations approximately is a topic of future research.

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