On contraction analysis of switched systems with mixed contracting-noncontracting modes via mode-dependent average dwell time

Hao Yin, Bayu Jayawardhana and Stephan Trenn

***Abstract*— This paper studies contraction analysis of switched systems that are composed of a mixture of contracting and non-contracting modes. The first result pertains to the equivalence of the contraction of a switched system and the uniform global exponential stability of its variational system. Based on this equivalence property, sufficient conditions for a mode-dependent average dwell/leave-time based switching law to be contractive are established. Correspondingly, LMI conditions are derived that allow for numerical validation of contraction property of nonlinear switched systems, which include those with all non-contracting modes.**

I. Introduction

For the past two decades, analysis and control of switched systems (as an important and special class of hybrid systems) have been well studied due to their relevance in representing numerous modern engineering systems where an abrupt change of parameters can occur or a jump in systems dynamics can happen as a response to the sudden change in their environment. Some well-known examples of such engineering systems are the dynamics of aircraft [1], of power electronics [2].Typically, switched systems are described by a family of subsystems, which can either be continuous-time or discrete-time dynamics, and a switching signal $\sigma(t)$ with switching sequence $\{t_1, t_2, \ldots\}$ that determines which subsystem is active over each time interval $[t_i, t_{i+1})$ for all $i \geq 0$. Such switching sequence can depend on particular state events [25], or time events [3]–[5]. In the time-dependent switching sequence, the dwell time (DT) [3] and average dwell time (ADT) notions [4] are two basic and important concepts in switched systems, both of which refer to the time interval or the average time interval, respectively, between consecutive switching times being lower bounded by a certain positive constant. A more general and flexible switching sequence, so-called mode dependent average dwell time (MDADT), was introduced in [5], which allows each mode to have its own ADT.

The stability of switched systems has been widely investigated in the literature [3]–[11] with a large body of works concern with switched systems comprising of stable subsystems. The common Lyapunov function technique [6] and multiple Lyapunov function technique [7] are commonly used to analyze the stability of these systems. In recent years, analysis of switched systems has also covered those with both stable and unstable subsystems [10], [11]. The main idea of these studies is to check whether the dwell-time of the stable subsystems is sufficiently large to offset the diverging trajectories caused by the unstable subsystems that are dwelt for a sufficiently short time. This approach of having a trade-off between stable and unstable subsystems is no longer applicable when all subsystems are unstable. In [8], [9], a discretized Lyapunov function technique is presented that can be used to analyze the stability of switched systems with all unstable subsystems. In this paper, we present another approach using contraction analysis to analyze the stability switched systems which encompass all cases including those with all unstable modes.

As one of stability analysis methods that has received a growing interest lately, contraction analysis is concerned with the relative trajectories of a systems than to a particular attractor equilibrium point in standard Lyapunov stability analysis. There are many different methods to analyze the contractivity of non-switched systems in literature, such as [12]–[21] among many others. In [16], the contraction property can be guaranteed if the largest eigenvalue of the symmetric part of the associated variational systems matrix (which is loosely termed as the Jacobian) is uniformly strictly negative. Finsler–Lyapunov functions were introduced in [17] to analyze the incremental stability of the system. A hierarchical approach to study convergence using matrix norm was discussed in [18]. In the context of switched systems, the contraction analysis thereof has recently been presented in [22]–[26]. Using contraction analysis method in [16], sufficient conditions for the convergence behavior of reset control systems have been studied in [24]. The extension of matrix norm-based contraction analysis [18] to piecewise smooth continuous systems is formalized in [22]. In [25], the singular perturbation theory and matrix norm are used to study the contraction property of switched Filippov systems, which include piecewise smooth systems.

In all of above mentioned results on contraction analysis for switched systems, it is assumed that all subsystems are contracting. It remains non-trivial to analyze contractivity of switch systems with all non-contracting subsystems, where in each dwell time interval the trajectories diverge from each other. Following the fact that a switched system with all unstable modes can be made asymptotically stable by an appropriate switching signal, we study in this paper whether the contraction of these systems, as a particular class of switched systems with mixed contracting-noncontracting modes, can be established by using the right switching signals.

As our first main result in this paper, we present contraction analysis for switched systems with mixed contracting-noncontracting modes. We establish that the stability of the corresponding variational dynamics is a sufficient and necessary condition to the contraction of the original switched systems. Subsequently, as our second contribution, we provide sufficient conditions on the time-varying Lyapunov function and on the mode dependent average dwell-time for switched nonlinear systems such that they are contracting. In general, these conditions ensure that the growth of time-varying Lyapunov function due to the noncontracting modes can be compensated by the switching behavior and the decaying Lyapunov function due to the contracting modes. In addition, we also consider all noncontracting subsystems case, where the increment can only be compensated by the switching behavior. Based on these conditions, as our third contribution, we propose a time-varying quadratic Lyapunov function that can be used to establish the contraction of switched systems via LMI conditions. Our result is more general and less conservative than the discretized Lyapunov function technique as proposed and used in [8], [9]. This result implies also that we can establish the stability of switched linear systems with all unstable modes.

The paper is organized as follows. In Section 2, we present

The authors are with the Jan C. Willems Center for Systems and Control, Faculty of Science and Engineering, University of Groningen, 9747AG Groningen, The Netherlands. hao.yin@rug.nl, b.jayawardhana@rug.nl and s.trenn@rug.nl.
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preliminaries and problem formulation. Necessary and sufficient conditions for the contractivity of nonlinear switched systems are presented in Sections 3. The switching law design strategy is provided in Section 4. The numerical simulations are provided in Section 5. The conclusions are given in Section 6.

Notation. The symbols $\mathbb{R}$, $\mathbb{R}_{\geq 0}$, $\mathbb{N}$ denote the set of real, non-negative real, natural numbers, respectively. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space. We denote the identity matrix with appropriate dimension by I. Given a matrix A, $A^\top$ refers to the transpose of A. For a square matrix A, $\lambda(A)$ refers to the set of eigenvalues of A. For symmetric metrics B and C, $B > 0$ ($B \geq 0$) indicates that B is positive definite (positive semidefinite) and $B < 0$ ($B \leq 0$) indicates that B is negative definite (negative semidefinite), $B < C$ ($B \leq C$) means $B - C < 0$ ($B - C \leq 0$). $\overline{\tau}$, $\underline{\tau}$ represent the upper bound, and the lower bound of τ. For vector valued functions $F : x \mapsto F(x)$ with $x \in \mathbb{R}^n$, and $F_p : x \mapsto F_p(x)$ with $x \in \mathbb{R}^n$, we define the Jacobian matrix $\triangledown_x F : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by $\triangledown_x F := \frac{\partial F(x)}{\partial x}$, and $\triangledown_x F_p : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ by $\triangledown_x F_p := \frac{\partial F_p(x)}{\partial x}$, respectively. For a vector or a matrix, $\|\cdot\|$ denotes the Euclidean vector norm or the induced matrix norm, respectively.

II. Preliminaries and problem formulation

Consider switched systems in the form of

$$\dot{x}(t) = f_{\sigma(t)}(x(t), t), \quad x(t_0) = x_0, \tag{1}$$

where $x(t) \in \mathscr{X} \subseteq \mathbb{R}^n$ is the state vector, $t_0 \in \mathbb{R}$ is the initial time and $x_0 \in \mathscr{X}$ is the initial value. Define an index set $\mathcal{M} := \{1, 2, \cdots, N\}$, where N is the number of modes. The signal $\sigma : [t_0, \infty) \to \mathcal{M}$ denotes the switching signal, which is assumed to be a piece-wise constant function continuous from the right. The vector field $f_i : \mathscr{X} \times [t_0, \infty) \to \mathbb{R}^n$, $(x, t) \mapsto f_i(x, t)$, $i \in \mathcal{M}$ is continuous in t and continuously differentiable in x. The switching instants are expressed by a monotonically increasing sequence $\mathscr{S} := \{t_1, t_2, \cdots, t_k, \cdots\}$, where t_k denotes the k-th switching instant. The length between successive switching instants is commonly referred to as the dwell time and given by $\tau_k = t_{k+1} - t_k, k = 0, 1, 2, \cdots$. We assume that (1) is forward complete, which means for each $x_0 \in \mathscr{X}$ there exists a unique solution of (1) and no jump occurs in the state at a switching time.

Definition 2.1: For a switched system given by (1) with a given switching signal $\sigma(t)$, it is called

(i) incrementally uniformly globally asymptotically stable (iUGAS) if there exists a class of KL function β, such that for all solutions $x_1(t)$, $x_2(t)$ of (1) in $t \in [t_0, +\infty)$ we have

$$\|x_1(t) - x_2(t)\| \leq \beta(\|x_1(t_0) - x_2(t_0)\|, t), \tag{2}$$

(ii) uniformly contracting if there exists positive numbers c and α such that for all solutions $x_1(t)$, $x_2(t)$ of (1) we have

$$\|x_1(t) - x_2(t)\| \leq ce^{-\alpha t}\|x_1(t_0) - x_2(t_0)\|. \tag{3}$$

In order to study contractivity of the switched systems (1), as usual, we will analyse the (uniform) stability of the corresponding variational systems, in which case, the following definition is relevant (note that by assumption for each time $t \geq t_0$ the map $x \mapsto f_{\sigma(t)}(x, t)$ is continuously differentiable at all $x \in \mathscr{X}$).

Definition 2.2: The family of (time-varying) linear switched system

$$\dot{\xi}(t) = F_{\sigma(t)}(x(t), t)\xi(t), \quad \xi(t_0) = \xi_0 \in \mathbb{R}^n \tag{4}$$

with $F_p(x(t), t) = \triangledown_x f_p(x(t), t)$ and $x(\cdot) \in \mathbb{R}^n$ be any given solution trajectory of (1) is called

(i) uniformly globally asymptotically stable (UGAS), if there exist a class of KL function β, (independently of the chosen solution $x(\cdot)$) such that for every solution $\xi(t) \in \mathbb{R}^n$ of (4) the following inequality holds,

$$\|\xi(t)\| \leq \beta(\|\xi(t_0)\|, t), \quad \forall t \geq t_0, \tag{5}$$

(ii) uniformly globally exponentially stable (UGES), if there exist positive numbers c, α (independently of the chosen solution $x(\cdot)$) such that for every solution $\xi(t) \in \mathbb{R}^n$ of (4) the following inequality holds,

$$\|\xi(t)\| \leq ce^{-\alpha t}\|\xi(t_0)\|, \quad \forall t \geq t_0. \tag{6}$$

The contraction analysis problem for switched systems with all contracting modes has attracted considerable attentions. For example, in [23], [24], a common contraction region is required between each subsystem. Then, contracting can be achieved by activating the subsystems for a sufficient long time. However, for noncontracting subsystems, you can not find such common contraction region, to be precise, you can not find any contraction region for a noncontracting subsystem. Then, the results in [23], [24] cannot be applied. The objective of this paper is to propose a sufficient condition that guarantees the switched system (1) is contracting with respect to switching law $\sigma(t)$ when not all modes of (1) are contracting, including the case where none of the modes is contracting.

III. A necessary and sufficient condition for the contraction of switched systems

Since switched systems with fixed switching signal can be considered as time-varying systems, tools for time-varying systems can be used to analyse of such switched systems. In this section, inspired by contraction analysis of time-varying systems as presented in [14], [16], we have the following proposition that establish the relations between (1) being iUGAS/contracting and (4) being UGAS/UGES.

Proposition 3.1: For a given switching signal $\sigma(t)$, the following properties hold

(i) the system (1) is iUGAS if the family of systems (4) is UGAS,
(ii) the system (1) is uniformly contracting if, and only if, the family of systems (4) is UGES.

Proof. We first establish a relationship between the solutions of (1) and (4). Let $x(t) = \varphi(t, x_0)$, $\hat{x}(t) = \varphi(t, x_0 + \delta\xi_0)$ be two trajectories of (1) with initial conditions $x(t_0) = \varphi(t_0, x_0) = x_0 \in \mathbb{R}^n$ and $\hat{x}(t_0) = \varphi(t_0, x_0 + \delta\xi_0) = x_0 + \delta\xi_0$, respectively, where δ is a sufficiently small positive constant and ξ_0 will later be related to the initial condition of (4). We will now show that

$$\xi(t) := \lim_{\delta\to 0} \frac{\varphi(t, x_0 + \delta\xi_0) - \varphi(t, x_0)}{\delta} \tag{7}$$

is a solution of (4) with initial value $\xi(t_0) = \xi_0$. For any t, let $i \in \mathbb{N}$ be such that $t \in [t_i, t_{i+1})$, so that the semiflow $\varphi(t, x_0)$ of (1) satisfies

$$\begin{aligned}\varphi(t, x_0) = x_0 + \sum_{k=0}^{i-1} \int_{t_k}^{t_{k+1}} f_{\sigma(t_k)}(\varphi(s, x_0), s)\mathrm{d}s \\ + \int_{t_i}^{t} f_{\sigma(t_i)}(\varphi(s, x_0), s)\mathrm{d}s,\end{aligned} \tag{8}$$

and similarly, the semiflow $\varphi(t, x_0+\delta\xi_0)$ satisfies

$$\varphi(t, x_0+\delta\xi_0) = x_0+\delta\xi_0+\sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} f_{\sigma(t_k)}(\varphi(s, x_0+\delta\xi_0), s)\mathrm{d}s + \int_{t_i}^{t} f_{\sigma(t_i)}(\varphi(s, x_0+\delta\xi_0), s)\mathrm{d}s. \tag{9}$$

Hence,

$$\xi(t) = \xi_0 + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} \lim_{\delta\to 0}\frac{f_{\sigma(t_k)}(\varphi(s, x_0+\delta\xi_0), s) - f_{\sigma(t_k)}(\varphi(s, x_0), s)}{\delta}\mathrm{d}s + \int_{t_i}^{t} \lim_{\delta\to 0}\frac{f_{\sigma(t_i)}(\varphi(s, x_0+\delta\xi_0), s) - f_{\sigma(t_i)}(\varphi(s, x_0), s)}{\delta}\mathrm{d}s. \tag{10}$$

Clearly, for $j \in \{0, 1, \dots, i\}$

$$\begin{aligned}&\lim_{\delta\to 0}\frac{f_{\sigma(t_j)}(\varphi(s, x_0+\delta\xi_0), s) - f_{\sigma(t_j)}(\varphi(s, x_0), s)}{\delta}\\ &= \frac{\partial}{\partial x_0}\left[f_{\sigma(t_j)}(\varphi(s, x_0), s)\right]\cdot\xi_0\\ &= \left[\nabla_x f_{\sigma(t_j)}(\varphi(s, x_0), s)\cdot\nabla_{x_0}\varphi(s, x_0)\right]\cdot\xi_0.\end{aligned}$$

Here we used the fact that the map $x_0 \mapsto \varphi(t, x_0)$ is differentiable for all $t \in [t_0, \infty)$ which is a consequence from the ability to write φ as a concatenation of the smooth solution flows $\varphi_{\sigma(t_i)}(t, t_i, x_i)$ of the (non-switched) differential equations $\dot{x} = f_{\sigma(t_i)}(x, t)$, $x(t_i) = x_i$. In fact, $\varphi(t, x_0) = \varphi_{\sigma(t_j)}(t, t_i, \varphi(t_i, x_0))$ and, recursively for $k = i-1, \dots, 2, 1$, we have $\varphi(t_k, x_0) = \varphi_{\sigma(t_{k-1})}(t_k, t_{k-1}, \varphi(t_{k-1}, x_0))$. Hence

$$\xi(t) = \xi_0 + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} \nabla_x f_{\sigma(t_k)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\xi_0\mathrm{d}s + \int_{t_i}^{t}\nabla_x f_{\sigma(t_i)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\xi_0\mathrm{d}s \tag{11}$$

and consequently

$$\begin{aligned}\dot{\xi}(t) &= \nabla_x f_{\sigma(t_i)}(\varphi(t, x_0), t)\nabla_{x_0}\varphi(t, x_0)\xi_0\\ &= F(t, x(t)))\nabla_{x_0}\varphi(t, x_0)\xi_0,\end{aligned}$$

where the last equality follows from $\sigma(t_i) = \sigma(t)$ for all $t \in [t_i, t_{i+1})$. Furthermore, from (8),

$$\begin{aligned}&\nabla_{x_0}\varphi(t, x_0)\\ &= I + \sum_{k=0}^{i-1}\int_{t_k}^{t_{k+1}} \nabla_x f_{\sigma(t_k)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\mathrm{d}s\\ &\quad + \int_{t_i}^{t}\nabla_x f_{\sigma(t_i)}(\varphi(s, x_0), s)\nabla_{x_0}\varphi(s, x_0)\mathrm{d}s\end{aligned}$$

which when multiplied with ξ_0 and in view of (11) leads to

$$\nabla_{x_0}\varphi(t, x_0)\xi_0 = \xi(t).$$

Altogether this shows that indeed ξ given by (7) is a solution of (4). In particular, $\nabla_{x_0}\varphi(t, x_0)$ is the transition matrix for (4), i.e.

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\nabla_{x_0}\varphi(t, x_0) = \nabla_x f_{\sigma(t)}(x(t), t)\nabla_{x_0}\varphi(t, x_0). \tag{12}$$

Proof of (i) on *UGAS* $\Rightarrow$ *iUGAS.* Let us consider two solutions $x(t) = \varphi(\cdot, x_0)$ and $\hat{x}(t) = \varphi(t, \hat{x}_0)$ of (1). We already highlighted in the first part of the proof that the map $x_0 \mapsto \varphi(t, x_0)$ is differentiable for each fixed $t \in [t_0, \infty)$. Consequently, we can utilize the fundamental theorem of calculus for line integrals to obtain

$$\hat{x}(t) - x(t) = \int_{x_0}^{\hat{x}_0}\nabla_y\varphi(t, y)\mathrm{d}y. \tag{13}$$

From UGAS of (4) and (12) it follows that there exists a class of KL function β, such that

$$\|\nabla_y\varphi(t, y)\| \le \beta(\|\underbrace{\nabla_y\varphi(t_0, y)}_{=I}\|, t) = \beta(1, t), \tag{14}$$

for all $y \in \mathscr{X}$. Using (14) to get the upper bound of (13), we have

$$\|\hat{x}(t) - x(t)\| \le \beta(1, t)\|\hat{x}_0 - x_0\| = \beta'(\|\hat{x}_0 - x_0\|, t), \tag{15}$$

where $\beta'(\|\hat{x}_0 - x_0\|, t)$ is a class of KL function.

Proof of (ii) on *Contracting* $\Leftrightarrow$ *UGES.* As we show $\varphi(t, x_0)$ is differentiable respect to x_0 for each fixed $t \in [t_0, \infty)$. The rest of the proof follows Proposition 1 in [14]. □

In Proposition 3.1 we establish the concept of UGAS for variational system, which is not presented before in [14], [16]. Note that the variational system (4) being UGAS is only a sufficient condition for system (1) being iUGAS. The reverse implication is not trivial to establish and it cannot follow the same line of proof as in [14]. Particularly, we can not conclude that $\delta\beta'(\|\xi(T)\|, T) \ge \beta'(\delta\|\xi(T)\|, T) = \beta'(\|\xi(t_0)\|, T)$ holds.

IV. Switching law design

In general, when individual systems are contracting, the switched systems can be made contracting by activating each subsystem sufficiently long. Instead of considering this situation, in this section, we study the property of contraction of switched systems whose modes are composed of a mixture of contracting and non-contracting modes. The switched systems under study include also the worst case, where all individual systems are not contracting[1], and we provide sufficient conditions on MDADT/MDALT (whose precise definition will shortly be given below) that guarantee the contraction of the switched systems. The use of MDADT/MDALT property in this paper is in contrast to the existing results in literature that are based on common dwell time. For this purpose, we define $\mathcal{S}$ as the set of all stable modes and $\mathcal{U}$ as the set of all unstable modes. In our main result, we propose a new class of switching signals that is suited for switched systems with stable and unstable modes.

Denoting $N_{\sigma p}(t_1, t_2)$ as the number of times that the p^{th} mode is activated in the interval $[t_1,\ t_2)$, and $T_p(t_1, t_2)$ as the sum of the running time of the p^{th} mode in the interval $[t_1,\ t_2)$, $p \in \mathcal{M} = \{1, 2, ..., N\}$. We revisit the following definitions of mode dependent average dwell time in [5].

Definition 4.1: A constant $\tau_{ap} > 0$ is called (slow) mode dependent average dwell time (MDADT) for mode $p \in \mathcal{M}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{M}$, if there exist a constant N_{0p} such that for all finite time intervals $[t_1, t_2) \subseteq [t_0, \infty)$ we have

$$N_{\sigma p}(t_1, t_2) \le N_{0p} + \tfrac{T_p(t_1, t_2)}{\tau_{ap}}. \tag{16}$$

Definition 4.2: A constant $\tau_{ap} > 0$ is called mode dependent average leave time (MDALT) for mode $p \in \mathcal{M}$ of a switching signal $\sigma : [t_0, \infty) \to \mathcal{M}$, if there exist a constant N_{0p} such that for all finite time intervals $[t_1, t_2) \subseteq [t_0, \infty)$,

$$N_{\sigma p}(t_1, t_2) \ge N_{0p} + \tfrac{T_p(t_1, t_2)}{\tau_{ap}}. \tag{17}$$

[1]Equivalently, the corresponding variational system (4) is not UGES [12], [14].

Remark 4.3: In Definition 4.2 we refer to τ_{ap} as the mode dependent average *leave* time (MDALT) instead of *fast* mode dependent average *dwell* time as e.g. in [5]. We prefer the former, because τ_{ap} in Definition 4.2 is not related to how long (at least, on average) the system dwells (remains) in a certain mode, but when the system has to leave a certain mode at the latest (on average). So "leave time" seems a better naming choice for τ_{ap} than "fast dwell time".

We present now the following theorem on the contracting properties of switched systems (1) with MDADT and/or MDALT.

Theorem 4.4: Consider switched nonlinear system (1) with switching signal $\sigma : [0,\infty) \to \mathcal{M}$ and corresponding switching times $\mathscr{S} := \{t_0, t_1, \dots, t_i, \dots\}$. Assume that we can classify each mode p as being either stable or unstable, i.e. assume $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$ and, correspondingly, assume the switching signal σ has a MDADT $\tau_{ap} > 0$ for each stable mode $p \in \mathcal{S}$ and a MDALT $\tau_{ap} > 0$ for each unstable mode $p \in \mathcal{U}$. Furthermore, assume that for each mode $p \in \mathcal{M}$ there exist a continuously differentiable function $V_p : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with

$$\dot V_p(x,\xi,t) := \nabla_{(x,\xi)} V_p(x,\xi,t) \begin{pmatrix} f_p(x,t) \\ F_p(x,t)\xi \end{pmatrix} + \nabla_t V_p(x,\xi,t)$$

such that for all $(x,\xi,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_{\geq 0}$

$$\dot V_p(x,\xi,t) \leq \eta_p V_p(x,\xi,t), \qquad \forall p \in \mathcal{M}, \tag{18}$$

with $\eta_p \geq 0$ if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Finally, assume that for every $p \in \mathcal{M}$, there exists $\mu_p > 0$ such that

$$V_{\sigma(t_i)}(x,\xi,t_i) \leq \mu_{\sigma(t_i^-)} V_{\sigma(t_i^-)}(x,\xi,t_i), \quad \forall t_i \in \mathscr{S}. \tag{19}$$

Without loss of generality, we let $\mu_p > 1$ for $p \in \mathcal{S}$. Then, with the following switching law

$$\left.\begin{aligned} \tau_{ap} > \underline{\tau}_{ap} := -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathcal{S}, \\ \tau_{ap} < \overline{\tau}_{ap} := -\frac{\ln \mu_p}{\eta_p}, \quad \forall p \in \mathcal{U}. \end{aligned}\right\} \tag{20}$$

the switched nonlinear system (1) is

(i) incrementally uniformly globally asymptotically stable (iUGAS) if there exist class K_∞ functions $\underline{v}_p$, $\overline{v}_p$, such that $V_p(x,\xi,t)$ satisfies

$$\underline{v}_p(\|\xi\|) \leq V_p(x,\xi,t) \leq \overline{v}_p(\|\xi\|), \quad \forall p \in \mathcal{M}, \tag{21}$$

(ii) uniformly contracting if there exist $\overline{v}_p \geq \underline{v}_p \geq 0$, such that $V_p(x,\xi,t)$ satisfies

$$\underline{v}_p\|\xi\|_2^2 \leq V_p(x,\xi,t) \leq \overline{v}_p\|\xi\|_2^2, \quad \forall p \in \mathcal{M}. \tag{22}$$

We note that $\tau_{ap} < \overline{\tau}_{ap}$ in (20) can only be satisfied if $\mu_p \in (0,1)$ for $p \in \mathcal{U}$.

PROOF. Let $x(\cdot)$ be a solution of (1) and let $\xi(\cdot)$ be a solution of the corresponding system (4). We will show in the following that there exists $k > 0$ and $\lambda > 0$ (independent from $x(\cdot)$ and $\xi(\cdot)$) such that

$$V_{\sigma(t)}(x(t),\xi(t),t) \leq k e^{-\lambda(t-t_0)} V_{\sigma(t_0)}(x_0,\xi_0,t_0). \tag{23}$$

From From (21) we can then conclude that

$$\begin{aligned} \|\xi(t)\| &\leq \underline{v}_{\sigma(t_n)}^{-1} \circ V_{\sigma(t_n)}(x(t),\xi(t),t) \\ &\leq \underline{v}_{\sigma(t_n)}^{-1}\big(k e^{-\lambda(t-t_0)} V_{\sigma(t_0)}(x_0,\xi_0,t_0)\big) \\ &\leq \underline{v}_{\sigma(t_n)}^{-1}\big(k e^{-\lambda(t-t_0)} \underline{v}_{\sigma(t_0)}^{-1}(\|\xi_0\|)\big). \end{aligned} \tag{24}$$

It is easy to see that $\underline{v}_{\sigma(t_n)}^{-1}\big(k e^{-\lambda(t-t_0)} \underline{v}_{\sigma(t_0)}^{-1}(\|\xi_0\|)\big)$ is a class KL function.

From (22) we can then conclude that

$$\begin{aligned} \|\xi(t)\| &\leq \frac{1}{\sqrt{\underline{v}_{\sigma(t_n)}}} V_{\sigma(t_n)}^{\frac{1}{2}}(x(t),\xi(t),t) \\ &\leq \sqrt{\frac{k}{\underline{v}_{\sigma(t_n)}}} e^{-\frac{\lambda}{2}(t-t_0)} V_{\sigma(t_0)}^{\frac{1}{2}}(x_0,\xi_0,t_0) \\ &\leq \sqrt{k\frac{\overline{v}_{\sigma(t_0)}}{\underline{v}_{\sigma(t_n)}}} e^{-\frac{\lambda}{2}(t-t_0)} \|\xi_0\|. \end{aligned} \tag{25}$$

According to (24), (25), Proposition 3.1 and Definition 2.2, we can then conclude that (i) system (4) is UGAS the system (1) is iUGAS, (ii) system (4) is UGES the system (1) is contracting.

Towards showing (23) first observe that for any $t \in [t_{i-1}, t_i)$ and $p := \sigma(t_i^-)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t} V_p(x(t),\xi(t),t) = \dot V_p(x(t),\xi(t),t).$$

Consequently, in view of (19) and (18),

$$\begin{aligned} &V_{\sigma(t_i)}(x(t_i),\xi(t_i),t_i) \\ &\leq \mu_{\sigma(t_i^-)} V_{\sigma(t_i^-)}(x(t_i),\xi(t_i),t_i) \\ &= \mu_{\sigma(t_{i-1})} V_{\sigma(t_{i-1})}(x(t_i),\xi(t_i),t_i) \\ &\leq \mu_{\sigma(t_{i-1})} e^{\eta_{\sigma(t_{i-1})}(t_i - t_{i-1})} V_{\sigma(t_{i-1})}(x(t_{i-1}),\xi(t_{i-1}),t_{i-1}). \end{aligned}$$

Recursively applying this inequality, we arrive at, for $t \in [t_i, t_{i+1})$,

$$V_{\sigma(t_i)}(x(t),\xi(t),t) \leq c_\sigma(t) V_{\sigma(t_0)}(x,\xi,t_0), \tag{26}$$

with

$$\begin{aligned} c_\sigma(t) &= e^{\eta_{\sigma(t_i)}(t-t_i)} \prod_{k=0}^{i-1} \mu_{\sigma(t_k)} e^{\eta_{\sigma(t_k)}(t_{k+1}-t_k)} \\ &= \prod_{p\in\mathcal{M}} \mu_p^{N_{\sigma p}(t,t_0)} e^{\eta_p T_p(t,t_0)} \\ &= \prod_{p\in\mathcal{M}} e^{N_{\sigma p}(t,t_0)\ln\mu_p + \eta_p T_p(t,t_0)}. \end{aligned}$$

By assumption, we have for $p \in \mathcal{S}$ that $\ln \mu_p > 0$ and hence by (16)

$$N_{\sigma p}(t,t_0)\ln\mu_p + \eta_p T_p(t,t_0) \leq N_{0p}\ln\mu_p + (\eta_p + \tfrac{\ln\mu_p}{\tau_{ap}})T_p(t,t_0);$$

and for $p \in \mathcal{U}$ we have $\ln \mu_p < 0$ and hence by (17) we arrive at the same inequality as above. Let $\lambda_p := \eta_p + \frac{\ln\mu_p}{\tau_{ap}}$, then from (20) together with $\ln\mu_p > 0$ for $p \in \mathcal{S}$ and $\ln\mu_p < 0$ for $p \in \mathcal{U}$, we have that $\lambda_p < 0$ for all $p \in \mathcal{M}$. With $k = \prod_{p\in\mathcal{M}} \mu_p^{N_{0p}}$ and $\lambda := \min_{p\in\mathcal{M}}(-\lambda_p) > 0$ we obtain

$$c_{\sigma(t)} \leq k \prod_{p\in\mathcal{M}} e^{-\lambda T_p(t,t_0)} = k e^{-\lambda(t-t_0)},$$

where we used the fact that $\sum_{p\in\mathcal{M}} T_p(t,t_0) = t - t_0$. This concludes the proof.

□

Different from Corollary 1 in [5], we do not need here to consider the ordering of stable and unstable subsystems. Some Lyapunov methods of incremental stability have recently appeared in the literature. Let us compare our results to these works. In this paper we do not exclude the case that the system switches from a non-contracting mode q to another non-contracting mode p and then back to mode q again (Example 5.2). In this case, according to (18), the variational system of each subsystem is divergent with a bounded rate η_p. Therefore, we need condition (19) to compensate for the divergent trajectory by having $\mu_p < 1$. This is not possible if $V_p(x,\xi,t)$ is time independent. Indeed, otherwise we have $V_p(x,\xi) < \mu_q V_q(x,\xi) < \mu_q\mu_p V_p(x,\xi) < V_p(x,\xi)$, which is a contradiction. In [17], the authors study the incremental stability of time-varying system based on the Finsler distance. A sufficient and necessary condition for incremental stability of time-invariant system with input is given in [19], which shows that the time-invariant system is incrementally stable if and only if there exists an incremental Lyapunov function with respect to the manifold $\{x_1 = x_2\}$). Neither [17] nor [19] study the stability properties of the variational systems. In addition, the Lyapunov functions in [17], [19] are all time-independent, which cannot solve switched systems with all non-contracting subsystems. By means of Proposition 3.1 and Theorem 4.4, we can analyze the contraction of switched systems with all non-contracting subsystems by finding multiple time-dependent Lyapunov functions for its variational system. Since constructing time-dependent Lyapunov functions is much more difficult than constructing time-independent Lyapunov functions, a LMI method is established in Theorem 4.10 to construct time-dependent Lyapunov functions for a family of nonlinear switched systems.

Remark 4.5: The results for all modes are contracting in [23], [26], can be considered as a particular case of Theorem 4.4. In particular, if we assume that $\mathcal{M} = \mathcal{S}$ in Theorem 4.4 then the switched nonlinear system (1) is contracting for any MDADT switching signals satisfying $\tau_{ap} > \underline{\tau}_{ap} = -\frac{\ln \mu_p}{\eta_p}, \forall p \in \mathcal{M}$, which recovers the results of Theorem 1 in [23] and Proposition 1 in [26].

For switched system (1), if all subsystems are non-contracting, which represents the worst case scenario, the distance increment between two trajectories will not be contracting in each mode and it can only be compensated by at the switching events. In this case, we have the following corollary from Theorem 4.4.

Corollary 4.6: Using the notation of Theorem 4.4, assume that $\mathcal{M} = \mathcal{U}$, i.e. we assume all modes are non-contracting. Then the switched nonlinear system (1) is contracting for any MDALT switching signals satisfying

$$\tau_{ap} \leq \overline{\tau}_{ap} = -\frac{\ln \mu_p}{\eta_p}, \qquad \forall p \in \mathcal{M}. \tag{27}$$

Although Theorem 4.4 provides a general framework to handle the contraction analysis problem, it is impractical for actual use, since it does not provide means to construct the Lyapunov functions $V_p(x, \xi, t)$ using existing computational techniques. In addition, when noncontracting subsystems are involved, we cannot find a monotonically decreasing Lyapunov function for each subsystem. Inequality (18) implies that the value of $V_p(x, \xi, t)$ may increase in some time interval with a bounded rate $\eta_p > 0$. The same as switched systems with all subsystems unstable, it is not easy to find a Lyapunov function and the corresponding parameter η_p satisfying (18). Different from [8, Thm. 1] that uses the DT to ensure asymptotic stability for all unstable mode switching systems, we consider here the use of MDALT to ensure exponentially stability of the switched systems. Based on Theorem 4.4, we will establish a sufficient condition that is easily verifiable for analysing the contraction property of switched systems.

As pursued in recent literature, the contraction analysis pertains to the stability analysis of nonlinear system using linear systems theory via its variational system (4). As the variational system can be regarded as a state-dependent linear system with the state ξ, quadratic Lyapunov function can directly be used to prove the stability. Hence let us consider a time dependent Lyapunov function of the quadratic form $V_p(x, \xi, t) = \xi^\top M_p(t)\xi$ for some matrix function $M_p : [0, \infty) \to \mathbb{R}^{n \times n}$ with symmetric, positive definite values. The following lemma provides conditions on such Lyapunov functions to ensure the contracting property of switched system (1).

Lemma 4.7: Consider a switched nonlinear system (1) with given switching times $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots t_n, \ldots\}$ generated by $\sigma : [0, \infty) \to \mathcal{M}$. Let each mode p be classified as either stable or unstable, i.e. $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$ and correspondingly assume that there exists $\tau_{ap} > 0$ such that (16) holds for the stable mode $p \in \mathcal{S}$ or (17) holds for the unstable mode $p \in \mathcal{U}$. Suppose that for each mode $p \in \mathcal{M}$ there exist $\overline{m}_p \geq \underline{m}_p \geq 0$ and a time dependent symmetric matrix $M_p(t)$ such that

$$\underline{m}_p I \leq M_p(t) \leq \overline{m}_p I, \quad \forall p \in \mathcal{M}, \tag{28}$$

$$F_p(x,t)^\top M_p(t) + \dot{M}_p(t) + M_p(t) F_p(x,t) \leq \eta_p M_p(t), \forall p \in \mathcal{M}, \tag{29}$$

with $\eta_p \geq 0$ if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Assume that for every $p \in \mathcal{M}$, there exists $\mu_p > 0$, such that

$$M_{\sigma(t_i)}(t_i) \leq \mu_{\sigma(t_i^-)} M_{\sigma(t_i^-)}(t_i^-), \quad \forall t_i \in \mathscr{S}. \tag{30}$$

Then the switched nonlinear system (1) is contracting for any MDADT/MDALT switching signals satisfying (20).

Proof. By taking a Lyapunov function in the form of $V_p(x, \xi, t) = \xi_p^\top M_p(t)\xi_p$, it follows that (28) and (30) satisfy (22) and (19) in Theorem 4.4, respectively. By differentiating $V_p(x, \xi, t)$ along the trajectory of system (4), we have $\dot{V}_p(x, \xi, t) = \xi_p^\top \Big(F_p(x,t)^\top M_p(t) + \dot{M}_p(t) + M_p(t) F_p(x,t)\Big)\xi_p$. Using (29), it follows that $\dot{V}_p(x, \xi, t) \leq \eta_p V_p(x, \xi, t)$, e.g. (18) holds. By Theorem 4.4, it implies that (1) is contracting for any switching signals satisfying (20). □

We note that the most popular quadratic Lyapunov function in contraction analysis literature is $V_p(x, \xi, t) = \xi^\top M_p \xi$, where M_p is a positive definite constant matrix [15]. In this case, $\dot{M}_p(t)$ in (29) is vanished. However, in the contraction analysis problem, since $F(x)$ in (4) is time-varying and state-dependent, the existence of such a *constant* matrix M_p is not always possible. In addition, in this paper, we allow subsystems are all non-contracting, M_p should be time-dependent. Hence, in general, allowing for time-varying matrix $M_p(t)$ in Lemma 4.7 leads to a significantly less conservative stability condition. For a general time dependent matrix $M_p(t)$, the inequality (30) is not trivial to solve. Another well-known technique to solve such a problem is the discretized Lyapunov function technique which is widely used in the stabilization of linear switched systems [8], [9]. The basic idea of the discretized Lyapunov function technique is to linearize $M_p(t)$ into the form of $\frac{t-t_i}{\tau_{dp}} P_p + (1 - \frac{t-t_i}{\tau_{dp}}) Q_p$. However, it can be difficult to find such $M_p(t)$ for some simple systems, e.g. for the switched system $p = 1 : \begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2, \\ \dot{x}_2 = 0.5x_1 + 0.7x_2, \end{cases} \quad p = 2 : \begin{cases} \dot{x}_1 = 0.5x_1 - 0.9x_2, \\ \dot{x}_2 = 0.1x_1 - 1.4x_2. \end{cases}$. If we apply discretized Lyapunov function technique as presented in [8] to this switched system, the corresponding LMIs are not feasible or $\tau_{dp} > -\frac{\ln \mu_p}{\eta_p}$. We will present later in Corollary 4.12 a method to design stabilizing switching signals for this switched system.

In order to compensate the conservativity brought by the Matrix Young inequality, in the following, we propose a construction of $M_p(t)$ in a nonlinear fashion by the addition of $\phi_p(t)\Big(1 - \phi_p(t)\Big)G_p$ to $M_p(t)$, which is more general than the discretized Lyapunov function proposed in [8], [9]. by considering the class of switching signals with mode dependent strict dwell time $\tau_{dp} > 0$, i.e., each mode p is active at least for τ_{dp} time before switching to another mode, we can transform the inequality condition of (28)-(30) into LMI conditions in Theorem 4.10 presented below. This is achieved by introducing a time-varying Lyapunov function that interpolates two quadratic constant Lyapunov functions in a prescribed dwell time τ_{dp}. Before stating our main result, we first recall two technical lemmas on matrix algebra.

Lemma 4.8: (Matrix Young inequality): For any $X, Y \in \mathbb{R}^{n\times m}$ and any symmetric positive-definite matrix $S \in \mathbb{R}^{n\times n}$,

$$X^\top Y + Y^\top X \le X^\top S X + Y^\top S^{-1} Y \tag{31}$$

holds.

Lemma 4.9: (Lemma 2 in [9]) Consider the matrix polynomial $f : [0,1]^n \to \mathbb{R}$ defined by

$$f(\tau_1,\tau_2,\cdots,\tau_n) = \Sigma_0 + \tau_1\Sigma_1 + \tau_1\tau_2\Sigma_2 + \cdots + (\prod_{k=1}^{n}\tau_k)\Sigma_n, \quad \forall \tau_k \in [0,1]. \tag{32}$$

If the matrices Σ_k, $k \in \mathbb{N}$, are symmetric and satisfy $\sum_{k=0}^{d}\Sigma_k < 0$ (or $\sum_{k=0}^{d}\Sigma_k > 0$) for all $d = 0, 1, \cdots, n$, then $f(\tau_1,\tau_2,\cdots,\tau_n) < 0$ (or $f(\tau_1,\tau_2,\cdots,\tau_n) > 0$).

Theorem 4.10: Consider switched nonlinear system (1) with globally Lipschitz f_p, $p \in \mathcal{M}$ and with given switching times $\mathscr{S} := \{t_0, t_1, \ldots, t_i, \ldots t_n, \ldots\}$ generated by $\sigma : [0,\infty) \to \mathcal{M}$. Assume that the modes can be classified as stable or unstable, i.e. $\mathcal{M} = \mathcal{S} \dot\cup \mathcal{U}$ and assume that for every mode p there exists $\tau_{ap} > 0$ such that (16) for $p \in \mathcal{S}$ or (17) for $p \in \mathcal{U}$ holds. Suppose that for each mode $p \in \mathcal{M}$ there exist a minimum mode dependent dwell time $\tau_{dp} > 0$, a constant matrix A_p, a semipositive definite matrix Γ_p, symmetric constant matrices P_p, Q_p, G_p, and positive constants $\overline{m}_p > 0$, $\epsilon_p \ge 0$ such that f_p is decomposed[2] into the following form

$$f_p(x,t) = A_p x + g_p(x,t), \tag{33}$$

with $\nabla_x g_p(x,t)^\top \nabla_x g_p(x,t) \le \Gamma_p$, for all $x \in \mathbb{R}^n$, $t \ge 0$, and

$$0 < Q_p < \overline{m}_p I, \quad 0 < P_p < \overline{m}_p I, \quad 0 < P_p + G_p < \overline{m}_p I, \tag{34}$$

$$A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) + \epsilon_p^{-1}\Gamma_p + \epsilon_p \overline{m}_p Q_p \le \eta_p Q_p, \tag{35}$$

$$A_p^\top (P_p + G_p) + (P_p + G_p)A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) + \epsilon_p^{-1}\Gamma_p + \epsilon_p\overline{m}_p(P_p + G_p) \le \eta_p(P_p + G_p), \tag{36}$$

$$A_p^\top P_p + P_p A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) + \epsilon_p^{-1}\Gamma_p + \epsilon_p\overline{m}_p P_p \le \eta_p P_p, \tag{37}$$

$$A_p^\top P_p + P_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p\overline{m}_p P_p \le \eta_p P_p, \tag{38}$$

hold with $\eta_p \ge 0$ if $p \in \mathcal{U}$ or $\eta_p < 0$ otherwise. Assume that for every $p \in \mathcal{M}$, there exists $\mu_p > 0$ such that

$$Q_{\sigma(t_i)} \le \mu_{\sigma(t_i^-)} P_{\sigma(t_i^-)}, \quad \forall t_i \in \mathscr{S}. \tag{39}$$

Then the switched nonlinear system (1) is contracting for any MDADT/MDALT switching signals satisfying (20), and which have mode dependent dwell time $\tau_{dp} > 0$.

PROOF. Let us define $M_p(t)$ in the following form

$$M_p(t) = \begin{cases} \phi_p(t)\Big(1-\phi_p(t)\Big)G_p + \phi_p(t)P_p + \Big(1-\phi_p(t)\Big)Q_p, \\ \qquad t \in [t_i, t_i + \tau_{dp}), \\ P_p, \qquad t \in [t_i + \tau_{dp}, t_{i+1}), \end{cases} \tag{40}$$

where $\phi_p(t) = \frac{t - t_i}{\tau_{dp}}$, so that $M_p(t_i) = Q_p$ and $M_p(t_i + \tau_{dp}) = P_p$. Note that $M_p(t)$ is positive definite according to (34) and Lemma 4.9. Now, let us consider $M_p(t)$ in the time interval $[t_i, t_i + \tau_{dp})$. The time derivative of $M_p(t)$ is given by

$$\dot M_p(t) = \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \phi_p(t)\frac{2}{\tau_{dp}}G_p. \tag{41}$$

For $t \in [t_i, t_i + \tau_{dp})$, we obtain from (29), (40) and (41) that

$$F_p(x,t)^\top M_p(t) + \dot M_p(t) + M_p(t)F_p(x,t) - \eta_p M_p(t) = \Sigma_1 + \phi_p(t)\Sigma_2 + \phi_p^2(t)\Sigma_3, \tag{42}$$

where

$$\begin{aligned} \Sigma_1 &= F_p^\top Q_p + Q_p F_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p, \\ \Sigma_2 &= F_p^\top (G_p + P_p - Q_p) + (G_p + P_p - Q_p)F_p \\ &\quad - \frac{2}{\tau_{dp}}G_p - \eta_p(G_p + P_p - Q_p), \\ \Sigma_3 &= -F_p^\top G_p - G_p F_p + \eta_p G_p. \end{aligned} \tag{43}$$

According to (33), and Lemma 4.8 (for $S = \epsilon_p I$), we have

$$\begin{aligned} \Sigma_1 &= \Big(A_p + \nabla_x g_p\Big)^\top Q_p + Q_p\Big(A_p + \nabla_x g_p\Big) + \\ &\quad \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p \\ &\le A_p^\top Q_p + Q_p A_p + \epsilon_p^{-1}\nabla_x g_p^\top \nabla_x g_p + \epsilon_p Q_p Q_p + \\ &\quad \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p \\ &\le A_p^\top Q_p + Q_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p\overline{m}_p Q_p + \\ &\quad \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p, \end{aligned} \tag{44}$$

Similarly we have

$$\begin{aligned} \Sigma_1 + \Sigma_2 &= F_p^\top (G_p + P_p) + (G_p + P_p)F_p + \\ &\quad \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p(G_p + P_p) \\ &\le A_p^\top (G_p + P_p) + (G_p + P_p)A_p + \epsilon_p^{-1}\Gamma_p + \\ &\quad \epsilon_p\overline{m}_p(G_p + P_p) + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \\ &\quad \eta_p(G_p + P_p), \end{aligned} \tag{45}$$

and

$$\begin{aligned} \Sigma_1 + \Sigma_2 + \Sigma_3 &= F_p^\top P_p + P_p F_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p P_p \\ &\le A_p^\top P_p + P_p A_p + \epsilon_p^{-1}\Gamma_p + \epsilon_p\overline{m}_p P_p + \\ &\quad \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) - \eta_p P_p, \end{aligned} \tag{46}$$

Using the hypotheses (35), (36), and (37) of the theorem, it follows that $\Sigma_1 < 0$, $\Sigma_1 + \Sigma_2 < 0$, $\Sigma_1 + \Sigma_2 + \Sigma_3 < 0$. Since $\phi_p(t) \in [0,1]$, it follows from Lemma 4.9 that (42) is negative definite. Similarly, for $t \in [t_i + \tau_{dp}, t_{i+1})$, (42) is negative definite according to (38). Consequently, in combination with (34), (39), and (20), all hypotheses in Lemma 4.7 are satisfied and the claim of the theorem follows immediately. □

We remark that there are a number of families of systems that can be written in the form of (33). This includes Lipschitz systems [27], Lorentz systems, Lur'e systems, and Persidskii systems.Note that, the assumption of g_p after (33) is uniformly in x. This is because for a time-varying system, contracting property does not guarantee the boundedness of x (we refer to Example 5.1 later where one of the states can diverge to infinity). However, this condition is less conservative then the global Lipschitz condition

[2]This decomposition is well-posed since the vector field f_p is assumed to be globally Lipschitz. The matrix A_p in this decomposition can be non-Hurwitz, which is relevant for the unstable modes.

presented in [27], and the references therein. For the global Lipschitz condition, one has $\nabla_x g_p(x,t)^\top \nabla_x g_p(x,t) \le \gamma^2 I$, where γ is the Lipschitz constant, while in our condition, Γ_p can be much smaller than $\gamma^2 I$. To illustrate this, let us consider $g_p(x,t) = \begin{bmatrix} \sin(x_1) \\ 0 \end{bmatrix}$, where the Lipschitz constant is given by $\gamma = 1$. For this example, we have $\nabla_x g_p(x,t)^\top \nabla_x g_p(x,t) = \begin{bmatrix} \cos^2(x_1) & 0 \\ 0 & 0 \end{bmatrix} \le \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Hence $\Gamma_p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, which is less than $\gamma^2 I$. In addition, if $\nabla_x g_p(x,t)$ is a symmetric matrix, such inequality reduces to the incremental monotonic condition present in [20], or the uniformly Lipschitz smooth condition introduced in [21].

Remark 4.11: Suppose that the hypotheses in Theorem 4.10 hold with $\mathcal{M} = \mathcal{U}$, i.e. all modes are non-contracting. Then the switched nonlinear system (1) is contracting for any MDALT switching signals satisfying (27).

As an interesting particular case of our main results above, let us consider the stabilization of linear switched systems where all modes are unstable. Using results in Theorem 4.10, we can stabilize such switched unstable systems. Consider a linear switched system given by

$$\dot{x}(t) = A_{\sigma(t)} x(t), \tag{47}$$

where $x(t)$ and $\sigma(t)$ are as in (1), and A_p, $p \in \mathcal{M}$, are unstable matrices for each mode p.

Corollary 4.12: Consider a linear switched system (47) with a given switching sequence $\mathscr{S} := \{t_0, t_1, \cdots, t_i, \cdots t_n\}$ generated by $\sigma(t)$. Assume that there exists $\tau_{ap} > 0$ such that (17) holds. Suppose that for each mode $p \in \mathcal{M}$ there exist a minimum mode dependent dwell time $\tau_{dp} > 0$, symmetric constant matrices P_p, Q_p, G_p, and scalars $\overline{m}_p > 0$ and $0 < \mu_p < 1$, such that (34), (39), and the following inequalities

$$A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) \le \eta_p Q_p, \forall p \in \mathcal{M}, \tag{48}$$

$$A_p^\top (P_p + G_p) + (P_p + G_p) A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) \le \eta_p (P_p + G_p), \forall p \in \mathcal{M}, \tag{49}$$

$$A_p^\top P_p + P_p A_p + \frac{1}{\tau_{dp}}(P_p - Q_p - G_p) \le \eta_p P_p, \forall p \in \mathcal{M}, \tag{50}$$

$$A_p^\top P_p + P_p A_p \le \eta_p P_p, \forall p \in \mathcal{M}, \tag{51}$$

hold. Then the switched system (47) is exponentially stable for any MDALT switching signals satisfying (27) and with mode-dependent dwell times $\tau_{dp} > 0$.

PROOF. The proof follows vis-á-vis with the proof of Theorem 4.10 adapted to the switched linear system (47). In this case, we have

$$\begin{aligned} A_p^\top M_p(t) + \dot{M}_p(t) + M_p(t) A_p - \eta_p M_p(t) = \\ \Sigma_1 + \phi_p(t)\Sigma_2 + \phi_p^2(t)\Sigma_3, \end{aligned} \tag{52}$$

where

$$\begin{aligned} \Sigma_1 &= A_p^\top Q_p + Q_p A_p + \frac{1}{\tau_{dp}}(G_p + P_p - Q_p) - \eta_p Q_p, \\ \Sigma_2 &= A_p^\top (G_p + P_p - Q_p) + (G_p + P_p - Q_p) A_p \\ &\quad - \frac{2}{\tau_{dp}} G_p - \eta_p (G_p + P_p - Q_p), \\ \Sigma_3 &= -A_p^\top G_p - G_p A_p + \eta_p G_p. \end{aligned} \tag{53}$$

It follows from (48), (49), (50), (53) and Lemma 4.9 that (52) is negative definite. Then, following Theorem 4.10, the linear switched systems (47) is contracting. Since $x(t) = 0$ is one of admissible trajectories of (47) and it is contracting, it follows that all the trajectories will converge to $x(t) = 0$ exponentially. □

Discretized Lyapunov function technique for stabilizing switched systems with all unstable subsystems can be found in [8, Theorem 2]. The main differences with the results in Corollary 4.12 are as follows. Firstly, the construction of our Lyapunov functions is based on nonlinear interpolation that connects Q_p and P_p via G_p, as opposed to a linear interpolation used in [8]. Consequently, the derivative of $M_p(t)$ in (41) may be negative so that the corresponding Lyapunov function may decrease in $[t_i, t_i + \tau_{dp})$, in contrast to the non-decreasing Lyapunov function in [8]. We note that the discretized Lyapunov function technique in [8, Theorem 2] can be obtained by taking $G_p = 0$. Secondly, our approach consider MDALT condition which generalizes the DT condition assumed in [8]. For the previous linear case after Lemma 4.7, by using Corollary 4.12 we can fix $\eta_1 = \eta_2 = 1.7$, $\mu_1 = \mu_2 = 0.7$, then the switching law is given by $\tau_{d1} = \tau_{d2} = 0.2$.

V. Simulation Setup and Results

In this section, two numerical examples will be presented. In the first case, we analyze the contraction of a switched system with mixed contracting-noncontracting modes by using Theorem 4.4. In the second case, we apply Theorem 4.10 to design the switching law for the system whose subsystems are all noncontracting.

Example 5.1: Consider a switched system (1) consisting of two time-varying subsystems, whose dynamics take the form

$$\begin{aligned} p = 1 &: \begin{cases} \dot{x}_1 = -x_1 - x_1^3 + 3x_2 \sin t, \\ \dot{x}_2 = -2x_1 \sin t - x_2 + 2\cos t, \end{cases} \\ p = 2 &: \begin{cases} \dot{x}_1 = x_1 + x_2 + t, \\ \dot{x}_2 = -x_1 - 2x_2 + \cos x_2. \end{cases} \end{aligned} \tag{54}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. Subsystem $p = 2$ is non-contracting. The Lyapunov function can be selected as $V_1(\xi) = 2\xi_1^2 + 3\xi_2^2$, $V_2(\xi) = \xi_1^2 + \xi_2^2$. According to Theorem 4.4, we can fix $\eta_1 = -2$, $\eta_2 = 2$, $\mu_1 = 3$, $\mu_2 = 0.5$, the switched law (20) is given by $\tau_{a1} \ge 0.55$, $\tau_{a2} \le 0.35$. For the simulation shown Figure 1 we use a periodic switching signal with $\tau_1 = 0.65$ and $\tau_2 = 0.35$.

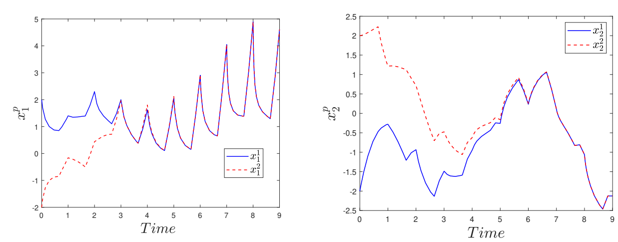

Fig. 1: The plot of trajectories of switched system in Example 5.1 initialized at $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 2 \end{bmatrix}$ for mode 1 and 2, respectively, and using a periodic switching signal with $\tau_1 = 0.65$ and $\tau_2 = 0.35$.

Example 5.2: Consider a switched system (1) consisting of two noncontracting subsystems, whose dynamics take the form

$$\begin{aligned} p = 1 &: \begin{cases} \dot{x}_1 = 0.1x_1 - 0.9x_2 - 0.2\cos(0.1x_1), \\ \dot{x}_2 = 0.1x_1 - 1.4x_2 - 0.7\cos(0.1x_2), \end{cases} \\ p = 2 &: \begin{cases} \dot{x}_1 = -1.9x_1 + 0.6x_2 + 0.7\cos(0.1x_2), \\ \dot{x}_2 = 0.6x_1 - 0.1x_2 + 0.2\cos(0.1x_2). \end{cases} \end{aligned} \tag{55}$$

where $x(t) \in \mathbb{R}^n$ is the state vector. It can be checked that for each mode, there exist a positive eigenvalue of $\nabla_x f_i(x,t)$ which satisfies $\lambda_1 \ge 0.0130$ (for the first mode) or $\lambda_2 \ge 0.0948$ (for the second mode). In other words, each individual system is non-contracting. As a result, the methods used in [23], [24] are no longer applicable in this particular case.

Using Theorem 4.10, where we fix $\overline{m}_1 = \overline{m}_2 = 0.1$, $\eta_1 = \eta_2 = 0.3$, $\mu_1 = 0.65$, $\mu_2 = 0.6$, $\epsilon_1 = \epsilon_2 = 1$, $\Gamma_1 = [\begin{smallmatrix} 0.0004 & 0 \\ 0 & 0.005 \end{smallmatrix}]$, $\Gamma_2 = [\begin{smallmatrix} 0.005 & 0 \\ 0 & 0.0004 \end{smallmatrix}]$, it can be checked that using the following symmetric constant matrices

$$\begin{aligned} P_i: &\begin{bmatrix} 0.0398 & -0.0071 \\ -0.0071 & 0.0933 \end{bmatrix}, \begin{bmatrix} 0.0881 & -0.0208 \\ -0.0208 & 0.0547 \end{bmatrix}, \\ Q_i: &\begin{bmatrix} 0.0493 & -0.0129 \\ -0.0129 & 0.0326 \end{bmatrix}, \begin{bmatrix} 0.0235 & -0.0013 \\ -0.0013 & 0.0554 \end{bmatrix}, \\ G_i: &\begin{bmatrix} -0.0038 & 0.0013 \\ 0.0013 & -0.0272 \end{bmatrix}, \begin{bmatrix} -0.0340 & 0.0107 \\ 0.0107 & -0.0064 \end{bmatrix}, \end{aligned} \tag{56}$$

the LMI problem given by (34)-(39) is feasible. Correspondingly, we have MDALTs as $\overline{\tau}_{a1} = 1.435$, $\overline{\tau}_{a2} = 1.702$, and the minimum dwell time for each mode as $\tau_{d1} = \tau_{d2} = 0.5$. To illustrate the contraction property, we consider switching signals with periodic switching time (each p mode has the same dwell time). Trajectories of the switching law: $\tau_{a1} = 0.5$, $\tau_{a2} = 1.7$ with two different initial conditions $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ are shown in Figure 2. The switching signal satisfies hypotheses of Theorem 4.10, we can conclude that the switched system is contracting. Figure 2 shows that despite each mode is noncontracting and the distance between the trajectories may increase in each mode (before the first switching, the distance are increasing), the increments are compensated by the switching behaviors, so that the trajectories converge to each other asymptotically.

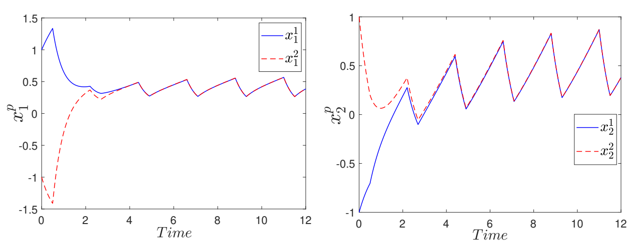

Fig. 2: The plot of trajectories of switched system in Example 5.2 initialized at $\left[\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ for mode 1 and 2, respectively, and using a periodic switching signal with $\tau_{a1} = 0.5$ and $\tau_{a2} = 1.7$.

VI. Conclusion

In this paper, the contraction property of switched systems with mixed contracting-noncontracting modes have been studied. It is established based on a necessary and sufficient condition that connects the contraction property of the original switched systems and the UGES of its variational systems. A time-dependent Lyapunov function and a mixed MDADT/MDALT method are introduced to study the UGES of the switched variational systems. Furthermore LMI conditions are presented that allow for numerical validation on the contraction property of switched systems with computable mode-dependent average dwell-time. Our results can be applied to stabilize linear switched systems with all unstable modes.

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