Model reduction for switched linear systems

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Problem formulation

System class: Switched linear ODEs

$$\dot{x} = A_\sigma x + B_\sigma u$$
$$y = C_\sigma x$$

$\sigma : [t_0, t_f) \to \{0, 1, 2, \ldots, \mathtt{m}\}$
$A_0, A_1, \ldots, A_{\mathtt{m}} \in \mathbb{R}^{n \times n}$
$B_0, B_1, \ldots, B_{\mathtt{m}} \in \mathbb{R}^{n \times m}$
$C_0, C_1, \ldots, C_{\mathtt{m}} \in \mathbb{R}^{p \times n}$

Reduced model

$$\dot{z} = \widehat{A}_\sigma z + \widehat{B}_\sigma u$$
$$y = \widehat{C}_\sigma z$$

$\widehat{A}_0, \widehat{A}_1, \ldots, \widehat{A}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n} \times \widehat{n}}$
$\widehat{B}_0, \widehat{B}_1, \ldots, \widehat{B}_{\mathtt{m}} \in \mathbb{R}^{\widehat{n} \times m}$
$\widehat{C}_0, \widehat{C}_1, \ldots, \widehat{C}_{\mathtt{m}} \in \mathbb{R}^{p \times \widehat{n}}$

Related research

- Simultaneous balancing (Monshizadeh et al. 2012)
- Output-depending switching (Papadopoulus&Prandini 2016)
- Enveloping (non-switched) system (Schulze & Unger 2018)
- Gramian-based approaches (Petrezcky, Gosea, ...)

Novel viewpoint

Consider switched linear ODE as special case of time-varying linear system

$$\dot{x} = A(t)x + B(t)u$$
$$y = C(t)x$$

In particular, consider switching signal as given time-varying system parameter

Existing approaches unsuitable

Existing approaches (Imae, Shokoohi, Silverman, Verriest):

- Smoothness of coefficients assumed
- Reduced model is fully time-varying (not piecewise-constant)

Challenge: Mode-wise reduction

Naive mode-wise reduction is not working

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

Each mode is input-output equivalent to same scalar system

$$\dot{z} = u, \quad y = z$$

But outputs do not match anymore after switch!

Reducability of modes is effected by other modes

In example:
Second state is unobservable in first mode, but becomes observable in second mode

Challenge: Different reduced state-dimensions

Reduced switched system with non-equal state-dimensions

Example:

on $[t_0, t_1)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix} x$$

on $[t_1, t_f)$:

$$\dot{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

Reduced system (with identical input-output behavior):

on $[t_0, t_1)$:

$$\dot{z}^0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} z^0 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} z^0$$

on $[t_1, t_f)$:

$$\dot{z}^1 = 0 \cdot z^1 + u$$

$$y = z^1$$

with concatination condition: $z^1(t_1) = \begin{bmatrix} 1 & 0 \end{bmatrix} z^0(t_1)$

New system class: Switched ODEs with jumps

Model reduction leaves original system class in general.

Challenge: Duration depend reduction

Reducability may depend on mode durations

Example:

on $[t_0, t_1)$:
$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

on $[t_1, t_2)$:
$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$
$$y = 0$$

on $[t_2, t_f)$
$$\dot{x} = 0$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

For $t_2 - t_1 = 2k\pi$ reduction possible to

on $[t_0, t_1)$:
$$\dot{z}^0 = 0 \cdot z^0 + u$$
$$y = z^0$$

on $[t_1, t_2)$:
no state
$$y = 0$$

on $[t_2, t_f)$
$$\dot{z}^2 = 0$$
$$y = z^2$$

For almost all other switching durations: First two modes not reducible!

Duration-dependent reduction methods?

Duration dependent methods not practical (numerically expensive, non-robust)

More general system class

Switched linear ODEs with jumps

$$\Sigma_\sigma : \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0,1,2,\ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Key features:

- states $x^k : (t_k, t_{k+1}) \to \mathbb{R}^{n_k}$ may have mode-dependent dimension
- $J_k : \mathbb{R}^{n_{k-1}} \to \mathbb{R}^{n_k}$ defines jumps at switch
- Certain switched DAEs $E_\sigma \dot{x} = A_\sigma x + B_\sigma u$, $y = C_\sigma x$ fall into this class
- Reduced model in same system class

Content

Introduction

Reduced realization

Time-varying Gramian-based reduction

Reduced realization

Question

Given: Switched ODE with jumps and known mode sequence
Sought: Reduced switched ODE with jumps with same input-output behavior

Key reduction intuition

States which are unreachable and unobservable can be removed
BUT: Reachability and observability are not local properties anymore
AND: Exact reachability and observability spaces are time-varying during modes

Definition (Reachability + Unobservability spaces)

$$\mathcal{R}_{[t_0,t)} := \left\{ x(t^-) \;\middle|\; \exists \text{ sol. } (x,u) \text{ of } \Sigma_\sigma \text{ with } x(t_0^-) = 0 \right\}$$
$$\mathcal{U}_{[t,t_f)} := \left\{ x(t^+) \;\middle|\; \exists \text{ sol. } (x,u=0) \text{ of } \Sigma_\sigma \text{ with } y = 0 \text{ on } [t,t_f) \right\}$$

Recursive expressions for reach./unobs. spaces

Theorem

For $t \in (t_k, t_{k+1}]$*:*

$$\mathcal{R}_{[t_0,t)} = e^{A_k(t-t_k)} J_k \mathcal{M}_{k-1} + \mathcal{R}_k$$

where $\mathcal{M}_{k-1} := \mathcal{R}_{[t_0,t_k)}$ *and* $\mathcal{R}_k = \operatorname{im}[B_k, A_k B_k, \ldots, A_k^{n_k-1} B_k] = \langle A_k | \operatorname{im} B_k \rangle$.

For $t \in [t_k, t_{k+1})$*:*

$$\mathcal{U}_{[t,t_f)} = e^{-A_k(t_{k+1}-t)} J_{k+1}^{-1} \mathcal{N}_{k+1} \cap \mathcal{U}_k$$

where $\mathcal{N}_{k+1} := \mathcal{U}_{[t_{k+1},t_f)}$ *and* $\mathcal{U}_k = \ker[C_k / C_k A_k / \ldots / C_k A_k^{n_k-1}] = \langle \ker C_k | A_k \rangle$

Eliminate time-dependence

For any subspace $\mathcal{S} \subseteq \mathbb{R}^n$ and any matrix $A \in \mathbb{R}^{n \times n}$:

$$\forall t \in \mathbb{R} : \langle \mathcal{S} \mid A \rangle \subseteq e^{At} \mathcal{S} \subseteq \langle A \mid \mathcal{S} \rangle$$

Extended reach. / restricted unobs. space

$$\mathcal{M}_k = e^{A_k \tau_k} J_k \mathcal{M}_{k-1} + \mathcal{R}_k \qquad \mathcal{N}_k = e^{-A_k \tau_k} J_{k+1}^{-1} \mathcal{N}_{k+1} \cap \mathcal{U}_k$$

Extended reachable space

$\overline{\mathcal{R}}_0 := \mathcal{R}_0$
$\overline{\mathcal{R}}_k := \langle A_k \mid J_k \overline{\mathcal{R}}_{k-1} \rangle + \mathcal{R}_k$

Restricted unobservable space

$\overline{\mathcal{U}}_\mathtt{m} := \mathcal{U}_\mathtt{m}$
$\overline{\mathcal{U}}_k := \langle J_{k+1}^{-1} \overline{\mathcal{U}}_{k+1} \mid A_k \rangle + \mathcal{U}_k$

Theorem (Properties of extended reach. / restricted unobs. space)

- $\overline{\mathcal{R}}_k \supseteq \mathcal{M}_k = \mathcal{R}_{[t_0,t_{k+1})}$, *in fact*, $\forall t \in (t_k, t_{k+1}) : \overline{\mathcal{R}}_k \supseteq \mathcal{R}_{[t_0,t)}$
- $\overline{\mathcal{U}}_k \subseteq \mathcal{N}_k = \mathcal{U}_{[t_k,t_f)}$, *in fact*, $\forall t \in (t_k, t_{k+1}) : \overline{\mathcal{U}}_k \subseteq \mathcal{U}_{[t,t_f)}$
- $\overline{\mathcal{R}}_k \neq \mathbb{R}^n \rightsquigarrow$ *uniformly unreachable states in mode* k
- $\overline{\mathcal{U}}_k \neq \{0\} \rightsquigarrow$ *uniformly unobservable states in mode* k
- $\overline{\mathcal{R}}_k$ *and* $\overline{\mathcal{U}}_k$ *are both* A_k*-invariant*

Weak Kalman decomposition

Theorem (Weak Kalman decomposition)

For (A,B,C) *let* $\overline{\mathcal{R}} \supseteq \operatorname{im} B$ *and* $\overline{\mathcal{U}} \subseteq \ker C$ *be two* *A-invariant* *subspaces.*
Let $T = [T_1, T_2, T_3, T_4]$ *invertible with*

$$\operatorname{im} T_1 = \overline{\mathcal{R}} \cap \overline{\mathcal{U}}, \quad \operatorname{im}[T_1, T_2] = \overline{\mathcal{R}}, \quad \operatorname{im}[T_1, T_3] = \overline{\mathcal{U}}$$

then

$$(T^{-1}AT, T^{-1}B, CT) = \left(\begin{bmatrix} A^{11} & A^{12} & A^{13} & A^{14} \\ 0 & A^{22} & 0 & A^{24} \\ 0 & 0 & A^{33} & A^{34} \\ 0 & 0 & 0 & A^{44} \end{bmatrix}, \begin{bmatrix} B^1 \\ B^2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & C^2 & 0 & C^4 \end{bmatrix} \right).$$

In particular, $Ce^{At}B = C_2 e^{A_{22}t} B_2$

With above notation let $V := T_2$ be the weak-KD-right-projector and W the corresponding rows in T^{-1} be the weak-KD-left projector.

Proposed reduction method based on weak KD

$$\Sigma_\sigma: \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0, 1, 2, \ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Reduction algorithm

Step 1a: Calculate extended reachable spaces $\overline{\mathcal{R}}_0, \overline{\mathcal{R}}_1, \ldots, \overline{\mathcal{R}}_{\mathtt{m}}$
Step 1b: Calculate restricted unobservable spaces $\overline{\mathcal{U}}_{\mathtt{m}}, \overline{\mathcal{U}}_{\mathtt{m}-1}, \ldots, \overline{\mathcal{U}}_0$

Step 2: Calculate weak-KD-left/right-projectors W_k, V_k

Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (W_k A_k V_k, W_k B_k, C_k V_k)$

Step 4: Calculate reduced jump map $\widehat{J}_k := W_k J_k V_{k-1}$

Properties of this reduction method

Reduction method is implementable

Method only depends on mode sequence of switching signal, not mode duration.

Theorem

Original and reduced systems have identical input-output behavior.

Theorem

Applying procedure on reduced system doesn't lead to further reduction.

Open question

Does this procedure lead to minimal realization for almost all switching durations?

Content

Further reduction

Next step: Remove difficult to observe and difficult to reach states

Define suitable reachability and observability Gramians

$$\mathcal{P}_0^\sigma(t) := \int_{t_0}^{t} e^{A_0(\tau-t_0)} B_0 B_0^\top e^{A_0^\top(\tau-t_0)} d\tau, \qquad t \in [t_0, t_1],$$

$$\mathcal{P}_k^\sigma(t) := e^{A_k(t-t_k)} J_k \mathcal{P}_{k-1}^\sigma(t_k) J_k^\top e^{A_k^\top(t-t_k)} + \int_{s_k}^{t} e^{A_k(\tau-t_k)} B_k B_k^\top e^{A_k^\top(\tau-t_k)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

$$\mathcal{Q}_\mathtt{m}^\sigma(t) := \int_{t}^{t_f} e^{A_\mathtt{m}^\top(t_f-\tau)} C_\mathtt{m}^\top C_\mathtt{m} e^{A_\mathtt{m}(t_f-\tau)} d\tau, \qquad t \in [t_\mathtt{m}, t_f],$$

$$\mathcal{Q}_k^\sigma(t) := e^{A_k^\top(s_{k+1}-t)} J_{k+1}^\top \mathcal{Q}_{k+1}^\sigma(t_{k+1}) J_{k+1} e^{A_k(t_{k+1}-t)} + \int_{t}^{t_{k+1}} e^{A_k^\top(t_{k+1}-\tau)} C_k^\top C_k e^{A_k(t_{k+1}-\tau)} d\tau, \qquad t \in [t_k, t_{k+1}].$$

Reachability/unobservability spaces and Gramians

Theorem

$\forall t \in [t_k, t_{k+1}):\quad \operatorname{im} \mathcal{P}_k^\sigma(t) = \mathcal{R}_{[t_0,t)}$ *and* $\ker \mathcal{Q}_k^\sigma(t) = \mathcal{U}_{[t,t_f)}$

Directly using these for balanced truncation?

- Fully time-varying: Balance-based projectors time-varying
  ⇝ leaves system class of switched ODEs (with jumps)
- Quantitative reachability / observability properties depend on mode durations
  ⇝ Duration dependence cannot be eliminated
- $\mathcal{P}_k^\sigma(t_k + \varepsilon)$ is dominated by reachability properties $\mathcal{P}_k^\sigma(t_k)$ of past
  ⇝ reachability properties of current mode not significantly visible in $\mathcal{P}_k^\sigma(t_k + \varepsilon)$
- Analogously for $\mathcal{Q}_k^\sigma(t_{k+1} - \varepsilon)$ which is dominated by the future ⇝ observability properties of current mode not significantly visible in $\mathcal{Q}_k^\sigma(t_{k+1} - \varepsilon)$

Midpoint based balanced truncation

$m_k := (t_k + t_{k+1})/2$

Use midpoint-Gramians

Use $\mathcal{P}_k^\sigma(m_k)$ and $\mathcal{Q}_k^\sigma(m_k)$ for balance truncation of mode k!

Reachability and observability assumption required

Invertibility (positive definiteness) of Gramians

$\iff \mathcal{R}_{[t_0,t)} = \mathbb{R}^{n_k}$ and $\mathcal{U}_{[t,t_f)} = \{0\}$ for all $t \in (t_0, t_f)$

Minimal realization

Similar as in time-invariant setup: Remove unobservable and unreachable states first

$\rightsquigarrow$ reduced realization discussed earlier

Overall reduction algorithm

$$\Sigma_\sigma: \quad \begin{aligned} \dot{x}^k &= A_k x^k + B_k u, && \text{on } (t_k, t_{k+1}) \\ x^k(t_k^+) &= J_k x^{k-1}(t_k^-), && k = 0, 1, 2, \ldots, \qquad x^{-1}(t_0^-) := 0 \\ y &= C_k x^k \end{aligned}$$

Algorithm

Step 0: If necessary reduce system via weak Kalman decomposition

Step 1: Calculate midpoint Gramians $\mathcal{P}_k^\sigma(m_k)$ and $\mathcal{Q}_k^\sigma(m_k)$

Step 2a: Based on singular values of $\mathcal{P}_k^\sigma(m_k)\mathcal{Q}_k^\sigma(m_k)$ decide on reduction order $\widehat{n}_k$
Step 2b: Calculate left/right projectors W_k, V_k via standard balanced truncation

Step 3: Calculate reduced modes $(\widehat{A}_k, \widehat{B}_k, \widehat{C}_k) := (W_k A_k V_k, W_k B_k, C_k V_k)$

Step 4: Calculate reduced jump map $\widehat{J}_k := W_k J_k V_{k-1}$