

An averaged model for switched systems with state jumps applicable for PWM descriptor systems

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Abstract—Switched descriptor systems with pulse width modulation are characterized by modes whose dynamics are described by differential algebraic equations; this type of models can be viewed as switched impulsive systems, i.e. switched systems with ordinary differential equations as modes dynamics and state jumps at the switching time instants. The presence of possible jumps in the state makes the application of the classical averaging technique nontrivial. In this paper we propose an averaged model for switched impulsive systems. The state trajectory of the proposed averaged model is shown to approximate the one of the original system with an error of order of the switching period. The model reduces to the classical averaged model when there are no jumps in the state. The practical interest of the theoretical averaging result is demonstrated through numerical simulations of a switched capacitor electrical circuit.

I. INTRODUCTION

Switching represents the natural behavior of many systems of practical interest, e.g. mechanical systems [1], electronic circuits [2], [3], piecewise affine systems [4], [5], [6]. In switched descriptor systems, in particular, the dynamics are determined by switching among different modes, where each mode is characterized by a set of linear differential equations and algebraic constraints. A mathematical representation of this class of systems can be obtained in terms of switched linear differential algebraic equations (DAE). Switched DAE belongs to the class of so called switched impulsive systems where each mode is represented by ordinary differential equations (ODE) together with rules for state jumps at the switching time instants.

In this paper we propose a discrete-time averaged model for switched impulsive systems with pulse width modulation (PWM). Averaging theory for switched systems has received a great deal of interest in the control literature considering different approaches and points of view related to the switched system characteristics: non-periodic switching functions [7], pulse modulations [8], [9], [10], dithering [11], hybrid systems [12], [13]. Averaging is a widely used technique in the power electronics community since 1970s [14] and it has been also applied to other switched systems of practical interest, see [15], [16] and the references therein.

The presence of state discontinuities at the switching time instants introduces several difficulties for the averaging analysis. An averaged model for homogeneous switched DAE

systems is presented in [17], [18] while an averaging result for the non homogeneous case is proposed in [19]. In these contributions strict algebraic conditions (commutativity) on the matrices characterizing the state jumps, which are not assumed in our analysis, were required. The results in [17], [18], [19] were extended in [20] by considering an averaged model whose dynamic matrix depends on the switching period, however also that averaging result requires conditions on the kernel and the image of the matrices of the system which are not always satisfied by practical systems [2], [21]. In this paper we propose a discrete-time averaged model for switched impulsive systems under milder assumptions with respect to those required by previous averaged models. In particular, the operative conditions used for obtaining the theoretical findings in [22], where no averaging analysis was proposed, are here exploited as a preliminary step for proving a new averaging result.

We show that the difference between the solution of the proposed averaged model and the moving average of the solution of the original system is of the same order as the switching period. We also conjecture a possible structure for a corresponding continuous-time version of the averaged model. In the work we focus on homogeneous switched systems with two modes and constant duty cycles. Nevertheless non-homogeneous systems can be represented in our framework when inputs can be incorporated as new state variables of the switched impulsive system. A switched capacitor electrical circuit which can be modeled as a PWM switched descriptor system is considered as a practical example; numerical simulations validate the effectiveness of the proposed model.

The paper is organized in several sections. In Section II some preliminary definitions and properties of switched impulsive systems are recalled. Section III and Section IV present the proposed averaged model and the main approximation result, respectively. In Section V a numerical verification of the theoretical results is proposed. In Section VI conclusions and future works are summarized.

II. SWITCHED IMPULSIVE SYSTEMS

The switched impulsive system considered in our analysis is characterized by a PWM between two modes with a switching period $p \in \mathbb{R}_+$, where \mathbb{R}_+ is the set of positive real numbers. The system commutes from the mode 2 to the mode 1 at the time instants $t_k \in \mathbb{R}_+$, $k \in \mathbb{N}$ with \mathbb{N} being the set of natural numbers, being the multiple of the period p , and from the mode 1 to the mode 2 at the switching time instants s_k within the k -th period. Then we have $t_k = kp$

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and $s_k = t_k + dp$, where $d \in (0, 1)$ is the duty cycle. In the sequel we consider a finite time interval $t \in [0, T]$, we assume that T is a multiple of the switching period p and we indicate $\ell(p) = T/p$. The generalization of the proposed results when T is not a multiple of p is straightforward.

A continuous-time model of the switched impulsive system of interest can be represented as follows

$$\dot{x}(t) = F_1 x(t), \quad t \in (t_k, s_k) \quad (1a)$$

$$x(t_k^+) = \Pi_1 x(t_k^-) \quad (1b)$$

$$\dot{x}(t) = F_2 x(t), \quad t \in (s_k, t_{k+1}) \quad (1c)$$

$$x(s_k^+) = \Pi_2 x(s_k^-) \quad (1d)$$

with $t \in [0, T]$, $x(0^-) = x_0 \in \mathbb{R}^n$ the initial condition of the state vector and $\dot{x}(t)$ represents the time derivative of the state. The flow matrices $F_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, characterize the dynamics of the two modes and the matrices $\Pi_i \in \mathbb{R}^{n \times n}$, $i = 1, 2$, called consistency projectors in the DAE terminology, determine the possible jumps of the state variables at the switching time instants. The switched impulsive system (1) includes several practical systems and, among them, switched descriptor systems which can be represented in the form of homogeneous switched DAE with regular matrix pairs [22].

The solution of the switched system (1) can be obtained by an iterative process. The solution at the switching time instants can be written as

$$x_k^+ = \Pi_1 x_k^- \quad (2a)$$

$$x(s_k^-) = G_1(dp)x_k^+ \quad (2b)$$

$$x(s_k^+) = \Pi_2 x(s_k^-) \quad (2c)$$

$$x_{k+1}^- = G_2((1-d)p)x(s_k^+) \quad (2d)$$

for $k \in \{0, \dots, \ell(p)\}$, where $G_i(s) = e^{F_i s}$, $s \in [0, p]$, $i \in \{1, 2\}$. By combining (2), one obtains that the left solution of (1) at the time instants multiple of the switching period must satisfy the following iterative equation

$$x_{k+1}^- = \Theta(p)x_k^- \quad (3)$$

for $k \in \{0, \dots, \ell(p)\}$, where

$$\Theta(p) = G_2((1-d)p)\Pi_2 G_1(dp)\Pi_1. \quad (4)$$

By iteratively applying of (3), the left solution of (1) at the time instants multiple of p can be written as

$$x_k^- = \Theta(p)^k x_0^- \quad (5)$$

for all $k \in \{0, \dots, \ell(p)\}$.

III. DISCRETE-TIME AVERAGED MODEL

In this section we present the proposed averaged model and we prove a corresponding approximation result with respect to the solution of (1) at the time instants multiple of the switching period.

The proposed discrete-time averaged model has the following structure

$$z_{k+1} = \Phi(p)z_k \quad (6a)$$

$$\mu_k = \Gamma z_k \quad (6b)$$

with $k \in \{0, \dots, \ell(p) - 1\}$, with

$$\Phi(p) = \Pi_\cap + \Lambda p \quad (7a)$$

$$\Gamma = \Pi_1 d + \Pi_\cap(1-d) \quad (7b)$$

where

$$\Pi_\cap = \Pi_2 \Pi_1, \quad (8a)$$

$$\Lambda = \Pi_2 F_1 \Pi_1 d + F_2 \Pi_2 \Pi_1 (1-d). \quad (8b)$$

It should be noticed that in the case of a switched ODE, the matrices Π_1 and Π_2 are equal to the identity matrix and the matrix Λ reduces to the dynamic matrices of the classical continuous-time averaged model of PWM systems with two modes, i.e. $F_1 d + F_2(1-d)$. Even if the matrices Π_1 and Π_2 are idempotent, i.e. $\Pi_i^2 = \Pi_i$, $i = 1, 2$, the matrix Π_\cap may not be, i.e. in general $\Pi_\cap^2 \neq \Pi_\cap$. Note that if Π_1 and Π_2 are idempotent and commutative matrices, then Π_\cap is idempotent. The solution of (6) can be written as

$$z_k = \Phi(p)^k z_0 \quad (9)$$

for all $k \in \{0, \dots, \ell(p) - 1\}$. In particular, one can write

$$\Phi(p)^k = (\Pi_\cap + \Lambda p)^k = \sum_{i=0}^k p^i \sum_{j=1}^{N_{i,k}} \Psi_{i,j,k} \quad (10)$$

where

$$N_{i,k} = \frac{k!}{i!(k-i)!} = \frac{1}{i!} \prod_{\xi=0}^{i-1} (k-\xi) \leq \frac{k^i}{i!} \quad (11)$$

and $\Psi_{i,j,k}$ are suitable matrices where in each of them the matrix Λ and the matrix Π_\cap will appear i and $k-i$ times, respectively, with different combinations of the product of their powers. Remind that $N_{i,k}$ is not monotone in i and has its maximum at $i = k/2$.

We are interested to compare (5) and (9). To this aim we need to recall the Big O notation and some of its properties.

Definition 1 ($O(p^r)$): For any finite integer $r \in \mathbb{N}$, a matrix function $G : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ is said to be an $O(p^r)$ function as $p \rightarrow 0$ ($G(p) = O(p^r)$ for short), if there exist positive constants α and \bar{p} such that

$$\|G(p)\| \leq \alpha p^r, \quad \forall p \in (0, \bar{p}).$$

Any linear combination of functions which are $O(p^r)$ is an $O(p^r)$ function itself.

By considering the Taylor approximation we can write, for any matrix $F \in \mathbb{R}^{n \times n}$ and any $s \in [0, p]$

$$G(s) = e^{F s} = I + F s + O(p^2) = I + O(p) \quad (12)$$

where I is the identity matrix.

In (5) the power of the matrix $\Theta(p)$ appears. Since k represents the number of switching periods within a given time interval, when p goes to zero k tends to infinity. For the averaging result it is crucial to analyze what happens to the matrix $\Theta(p)^k$ when p goes to zero.

In order to compare (5) and (9) we are interested to relate the matrix $\Theta(p)^k$ to the matrix $\Phi(p)^k$. As a preliminary step let us consider the following lemma.

Lemma 2: Consider a finite $T \in \mathbb{R}_+$ multiple of $p \in \mathbb{R}_+$, $\ell(p) = T/p$, a duty cycle $d \in (0, 1)$ and generic Lipschitz continuous matrix functions $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$ and $M : \mathbb{R}_+ \rightarrow \mathbb{R}^{n \times n}$. Assume that there exists a $\gamma \geq 0$ such that

$$\|\Phi(p)\| \leq 1 + \gamma p \quad (13a)$$

$$M(p) = O(p^2), \quad (13b)$$

for all $k \in \{0, \dots, \ell(p)\}$. Then

$$\Phi(p)^k = O(1) \quad (14a)$$

$$(\Phi(p) + M(p))^k = \Phi(p)^k + O(p). \quad (14b)$$

for all $k \in \{0, \dots, \ell(p)\}$.

Proof: The proof of this lemma follows as a direct application of Lemma 3 in [22]. ■

We are now ready to prove that the difference between (5) and (9) is of order of the switching period.

Lemma 3: Consider the systems (1) and (6) with the corresponding solutions (5) and (9), respectively. Assume that there exists $\gamma \geq 0$ such that (13a) holds. Then, for any $z_0 = x_0^- + O(p)$ the following condition

$$x_k^- = z_k + O(p) \quad (15)$$

holds for all $k \in \{0, \dots, \ell(p) - 1\}$.

Proof: By using (12) in (4) we can write

$$\begin{aligned} \Theta(p) &= G_2((1-d)p)\Pi_2 G_1(dp)\Pi_1 \\ &= (I + F_2(1-d)p)\Pi_2(I + F_1 dp)\Pi_1 + O(p^2) \\ &= \Phi(p) + O(p^2) \end{aligned} \quad (16)$$

By applying Lemma 2 with (13a), one can write

$$\Theta(p)^k = \Phi(p)^k + O(p) \quad (17)$$

for all $k \in \{0, \dots, \ell(p)\}$. By subtracting (9) to (5) one obtains

$$\begin{aligned} x_k^- &= z_k + \Theta(p)^k x_0^- - \Phi(p)^k z_0 \\ &\stackrel{a}{=} z_k + \Phi(p)^k (x_0^- - z_0) + O(p) \\ &\stackrel{b}{=} z_k + O(p) \end{aligned} \quad (18)$$

where in $\stackrel{a}{=}$ we used (17) and in $\stackrel{b}{=}$ we used (14a) together with the hypothesis on the initial conditions. ■

The verification of (13a) is not easy to be checked in a formal way. It is useful to consider the following operative sufficient conditions which allow us to verify the assumption.

Lemma 4: Consider a finite $T \in \mathbb{R}_+$ multiple of $p \in \mathbb{R}_+$, $\ell(p) = T/p$, a duty cycle $d \in (0, 1)$, and the matrices $\Phi(p)$ and Π_\cap given by (7a) and (8). If there exists a symmetric matrix P such that

$$P \succ 0, \quad \Pi_\cap^\top P \Pi_\cap - P \preceq 0 \quad (19)$$

then (13a) holds for any $k \in \{0, \dots, \ell(p)\}$ and for all p , with $\|\cdot\|$ being the norm induced by P .

Proof: By using (7a) one can write

$$\Phi(p) = \Pi_\cap + O(p) \quad (20)$$

Let us consider the difference equation $\xi_{k+1} = \mathcal{P}_k \xi_k$ with $\xi_k \in \mathbb{R}^n$, $k \in \mathbb{N}_0$, $\mathcal{P}_k \in \mathcal{F}$ and $\mathcal{F} = \{\Pi_1, \Pi_2\}$. By using the piecewise quadratic stability based on Lyapunov theory [5] it follows that the existence of a matrix P which solves the linear matrix inequalities (19) implies that the system is absolutely stable for any sequence of matrices in \mathcal{F} , see Sec. 5 in [23]. Then, by using Theorem 3 in [24] it follows that

$$\|\Pi_\cap\| \leq 1 \quad (21)$$

with $\|\cdot\|$ being the norm induced by the matrix P . Therefore, by applying such norm to (20) and by using (21), the condition (13a) follows. ■

IV. APPROXIMATION OF THE MOVING AVERAGE

In this section we show that the solution of the proposed averaged model approximates the moving average (6) of the solution of (1) with a state error which is $O(p)$. The section is concluded with a candidate continuous-time averaged model of (1) derived from (6) by exploiting the approximation result.

A. Sampled data moving average

In order to compare the variable μ_k given by (6b) with the averaged of the state $x(t)$ computed over the switching period p , we consider as a preliminary step the error between μ_k and the discrete-time average of the state at the multiple of the switching period, say m_k . This variable can be written as

$$m_k = \frac{1}{p} \int_{kp}^{(k+1)p} x(t) dt \quad (22)$$

for all $k \in \{0, \dots, \ell(p) - 1\}$. The following lemma shows that the difference between (6b) and (22) is of order of the switching period.

Lemma 5: Consider the systems (1) and (6) with the corresponding solutions (5) and (9), respectively and the discrete-time average (22). Assume that there exists $\gamma \geq 0$ such that (13a) holds. Then, for any $z_0 = x_0^- + O(p)$ the following condition

$$m_k = \mu_k + O(p) \quad (23)$$

holds for all $k \in \{0, \dots, \ell(p) - 1\}$.

Proof: By using (2) in (22) one can write

$$\begin{aligned} pm_k &= \int_0^{dp} G_1(t)\Pi_1 x_k^- dt + \int_0^{(1-d)p} G_2(t)\Pi_2 x(s_k^-) dt \\ &= \left[\int_0^{dp} G_1(t) dt + \int_0^{(1-d)p} G_2(t)\Pi_2 G_1(dp) dt \right] \Pi_1 x_k^- \end{aligned} \quad (24)$$

for all $k \in \{0, \dots, \ell(p) - 1\}$. Then, from (24) by using (12) one has:

$$\begin{aligned} pm_k &= [Id + \Pi_2(1-d)] p \Pi_1 x_k^- + O(p^2) \\ &= \Gamma p x_k^- + O(p^2). \end{aligned} \quad (25)$$

Then by using Lemma 3 and (6b) it follows that (23) holds. ■

B. Continuous-time moving average

Under some extra conditions on the matrices Π_1 , Π_2 and Λ it can be proved that the continuous-time moving average of the state variable defined as

$$m(t) = \frac{1}{p} \int_t^{t+p} x(\tau) d\tau \quad (26)$$

for any $t \in (0, T-p]$, remains close to (22). This is proved by the following lemma.

Lemma 6: Consider the system (1) with the corresponding solution (5), the discrete-time average (22) and the continuous-time moving average (26). Assume that (13) hold. Consider (10) and (11) and assume that there exist positive constants α and β such that

$$\left\| \sum_{j=1}^{N_{i,k}} (\Pi_\cap - I) \Psi_{i,j,k} \right\| \leq \alpha \frac{k^{i-1}}{i!} \beta^i \quad (27)$$

for all k and $i = 0, \dots, k$, where the matrices $\Psi_{i,j,k}$ are defined through (10). Then the following condition

$$m(t) = m_k + O(p) \quad (28)$$

holds for any $t \in (0, T-p]$ and where $k = \lfloor \frac{t}{p} \rfloor$.

Proof: Since $m(t) = m_k$ for any $t = t_k = kp$, $k \in \{0, \dots, \ell(p) - 1\}$, in the time instants multiple of the switching period the condition (28) is trivially satisfied.

Let us consider the moving average over a time interval of length p which starts in the first mode. For any $t \in [t_k, s_k]$, $k \in \{0, \dots, \ell(p) - 1\}$, $\tau_1 = t - t_k$, i.e. $\tau_1 \in [0, dp]$, by using (2) in (26) one can write

$$\begin{aligned} pm(t) &= \int_\tau^{dp} G_1(\xi) \Pi_1 x_k^- d\xi \\ &+ \int_0^{(1-d)p} G_2(\xi) \Pi_2 G_1(dp) \Pi_1 x_k^- d\xi \\ &+ \int_0^{\tau_1} G_1(\xi) \Pi_1 x_{k+1}^- d\xi \end{aligned} \quad (29)$$

By taking the difference between (24) and (29) one obtains

$$\begin{aligned} p(m(t) - m_k) &= \int_0^{\tau_1} G_1(\xi) \Pi_1 (x_{k+1}^- - x_k^-) d\xi \\ &= \Pi_1 (\Pi_\cap - I) \tau_1 x_k^- + O(p^2) \end{aligned} \quad (30)$$

for any $t \in [t_k, s_k]$, $\tau_1 = t - t_k$, for all $k \in \{0, \dots, \ell(p) - 1\}$.

Let us consider the moving average over a time interval of length p which starts in the second mode. For any $t \in [s_k, t_{k+1}]$, $k \in \{0, \dots, \ell(p) - 1\}$, $\tau_2 = t - s_k$, i.e. $\tau_2 \in [0, (1-d)p]$, one can write

$$\begin{aligned} pm(t) &= \int_{\tau_2}^{(1-d)p} G_2(\xi) \Pi_2 G_1(dp) \Pi_1 x_k^- d\xi \\ &+ \int_0^{\tau_2} G_2(\xi) \Pi_2 G_1(dp) \Pi_1 x_{k+1}^- d\xi \\ &+ \int_0^{dp} G_1(\xi) \Pi_1 x_{k+1}^- d\xi \\ &= [\Pi_\cap ((1-d)p - \tau_2) + \Pi_\cap \tau_2 \\ &+ \Pi_1 \Pi_\cap dp] x_k^- + O(p^2) \end{aligned} \quad (31)$$

By taking the difference between (24) and (31) one obtains

$$\begin{aligned} p(m(t) - m_k) &= \int_0^{dp} G_1(\xi) \Pi_1 (x_{k+1}^- - x_k^-) d\xi \\ &+ \int_0^{\tau_2} G_2(\xi) \Pi_2 G_1(dp) \Pi_1 (x_{k+1}^- - x_k^-) d\xi \\ &= (\Pi_2 \tau_2 + Idp) \Pi_1 (\Pi_\cap - I) x_k^- + O(p^2) \end{aligned} \quad (32)$$

for any $t \in [s_k, t_{k+1}]$, $\tau_2 = t - s_k$, i.e. $\tau_2 \in [0, (1-d)p]$, for all $k \in \{0, \dots, \ell(p) - 1\}$.

By using (5) and (17), the expression (30) can be rewritten as

$$p(m(t) - m_k) = \Pi_1 \tau_1 (\Pi_\cap - I) \Phi(p)^k x_0^- + O(p^2) \quad (33)$$

for any $t = \tau_1 + t_k$, $k \in \{0, \dots, \ell(p) - 1\}$, $\tau_1 \in [0, dp]$, and (31) can be rewritten as

$$\begin{aligned} p(m(t) - m_k) &= (\Pi_2 \tau_2 + Id_1 p) \Pi_1 (\Pi_\cap - I) \Phi(p)^k x_0^- \\ &+ O(p^2) \end{aligned} \quad (34)$$

for any $t = \tau_2 + s_k$, $k \in \{0, \dots, \ell(p) - 1\}$, $\tau_2 \in [0, (1-d)p]$.

By considering (33) and (34) it will appear the term

$$(\Pi_\cap - I) \Phi(p)^k = \sum_{i=0}^k p^i (\Pi_\cap - I) \sum_{j=1}^{N_{i,k}} \Psi_{i,j,k} \quad (35)$$

By taking the norms of the second term on the right hand side of (35) and by using (27) it follows

$$\begin{aligned} \left\| \sum_{i=0}^k p^i (\Pi_\cap - I) \sum_{j=1}^{N_{i,k}} \Psi_{i,j,k} \right\| &\leq \sum_{i=0}^k p^i \left\| (\Pi_\cap - I) \sum_{j=1}^{N_{i,k}} \Psi_{i,j,k} \right\| \\ &\leq \sum_{i=0}^k p^i \alpha \beta^i \frac{k^{i-1}}{i!} \end{aligned} \quad (36)$$

$$= \frac{\alpha}{k} \sum_{i=0}^k \tilde{p}^i \frac{k^i}{i!} \leq \frac{\alpha}{k} e^{\tilde{t}} \quad (37)$$

with $\tilde{p} = p\beta$, $\tilde{t} = \tilde{p}k$ and we used the assumption (27).

By substituting (37) in (35) one has that:

$$(\Pi_\cap - I) \Phi(p)^k = O(p) \quad (38)$$

with $k = \lfloor \frac{T}{p} \rfloor$. Then by using (38) in (33) and (34), by dividing (33) and (34) by p and by considering that $\tau_1 = O(p)$ and $\tau_2 = O(p)$ it follows the proof. ■

In case Π_\cap is idempotent and commutes with Λ then it is easily seen that $(\Pi_\cap - I) \Psi_{i,j,k} = 0$ so that (27) is trivially satisfied. Hence the intuition behind (27) is that (sufficiently high powers of) Π_\cap should behave similar to an idempotent matrix and ‘‘approximately should commute’’ with Λ , thus allowing the relaxation of conditions required in [17], [18], [19], [20] in order to obtain the averaging result.

We can now prove that the error between the continuous moving average $m(t)$ of the solution of (1) and the output μ_k of (6) is of order of the switching period.

Theorem 7: Consider the switched system (1) and the discrete-time averaged model (6). If there exist constants

$\gamma \geq 0$ and $\alpha > 0$ such that (13a) and (27) hold, then for all $z_0 = x_0^- + O(p)$ the following condition

$$m(t) = \mu_k + O(p) \quad (39)$$

holds for any $t \in (0, T - p]$ and where $k = \lfloor \frac{t}{p} \rfloor$.

Proof: The proof directly follows by applying in sequence Lemma 6 and Lemma 5. ■

C. Continuous-time model

Theorem 7 shows that the discrete-time model (6) provides an approximation of the moving average of the solution of the switched impulsive system (1). One may be interested to a continuous-time averaged model. In order to conjecture a structure for this model we consider the following system

$$\dot{\tilde{z}}(t) = A(p)\tilde{z}(t), \quad t \in \mathbb{R}_+ \quad (40a)$$

$$\mu(t) = \Gamma\tilde{z}(t) \quad (40b)$$

with $\tilde{z}(0) = z_0 \in \mathbb{R}^n$ initial condition, the dynamic matrix

$$A(p) = \frac{1}{p}(\Phi(p) - I) \quad (41)$$

where $\Phi(p)$ and Γ are given by (7a)–(8).

A motivation for the choice of the matrices in (40) can be obtained by discretizing the model (40) with the forward Euler method. The expression (41) clearly shows that this discretization procedure leads to the discrete-time averaged model (6). A further motivation is the fact that the models (6) and (40) have the same steady state subspaces.

The model (40) has an interesting interpretation. Indeed, in the particular case $\Pi_i = I$ for $i = 1, 2$, it is $\Phi(p) = I + [F_1 d + F_2(1-d)]p$ which implies $A = F_1 d + F_2(1-d)$ and $\Gamma = I$, i.e. the model (40) reduces to the classical continuous-time averaged model for switched ODE. Unfortunately, in the case of switched impulsive ODE to found a bound on the error $\mu(t) - \mu_k$ is a nontrivial issue because the dynamic matrix of the continuous-time system (40) depends on p which is the sampling period used for the discretization. The solution of this problem is left for future investigations.

V. EXAMPLE

In this section we verify the averaging result by considering the switched capacitor electrical circuit whose scheme is shown in Fig. 1. The circuit represents the typical elementary cell of a ladder step-up switched capacitor and it consists of two capacitors and four electronic switches that are controlled in a complementary way. The two modes of the system correspond to the pair $\{\mathcal{S}_1, \mathcal{S}_2\}$ turned on together with the pair $\{\mathcal{S}_3, \mathcal{S}_4\}$ turned off and viceversa.

The constant input voltage u is modeled by using the state variable x_1 with zero time-derivative. The voltages on the capacitors C_1 and C_2 , say x_2 and x_3 respectively, are chosen as further state variables. By applying the Kirchoff rules to the circuit the matrices of the model (1) can be written as

$$F_1 = \frac{1}{R\rho^2} \begin{bmatrix} 0 & 0 & 0 \\ -\rho & C_2 & -C_1 \\ -\rho & C_2 & -C_1 \end{bmatrix}, \quad \Pi_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_2\rho & C_1\rho \\ 0 & C_2\rho & C_1\rho \end{bmatrix}$$

$$F_2 = \frac{1}{C_2\rho} \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

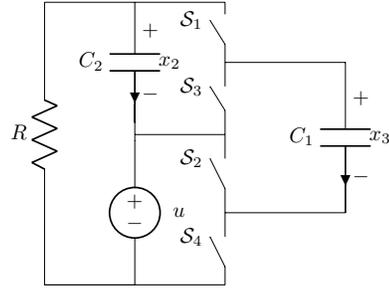


Fig. 1. Elementary cell of a ladder step-up switched capacitor converter.

where $\rho = \frac{1}{C_1 + C_2}$. It is easy to verify that Π_1 and Π_2 are idempotent, i.e. $\Pi_i^2 = \Pi_i$, $i = 1, 2$, but $\Pi_1 \Pi_2 \neq \Pi_2 \Pi_1$ it is not because Π_1 and Π_2 are not commutative matrices. We numerically verified the existence of a common matrix P that satisfies (19) and the validity of condition (36) for $p \in [0.01s, 1s]$ with $\alpha = 1$, $\beta = \|\Lambda\|$, $k = \lfloor \frac{T}{p} \rfloor$, $T = 1s$.

The following circuit parameters are considered: $C_1 = C_2 = 120\mu F$, $R = 10k\Omega$. In Fig. 2 and Fig. 3 are shown the simulation results for the state variables x_2 and x_3 with $p = 0.05s$ and $p = 0.1s$ respectively, over a time interval of 1s. The discrete-time averaged model well approximates the switching behavior (blue lines); indeed the error between the moving average and the discrete switching evolution decreases by reducing the switching period. Note that the amplitude of the jumps in the first part of the simulations are not related to the decreasing of the switching period. Nevertheless the discrete-time and the continuous-time averaged models are still able to capture the averaged behavior of the switched system.

VI. CONCLUSION

A new averaged model for switched impulsive systems that present state jumps at the switching time instants has been proposed. Pulse width modulated switched systems with two modes and a constant duty cycle have been considered. It is proved that the solution of the averaged model approximates the moving average of the solution of the original system with an error of order of the switching period. The averaging result requires weaker conditions with respect to the averaged models presented in the previous literature. Numerical results obtained by simulating a switched capacitor electrical circuit has validated the theoretical results.

Future work will be the formal analysis of the continuous-time averaged model whose idea has been presented and numerically validated in this paper and the extension of the averaging results to the case of more than two modes and time-varying duty cycles.

REFERENCES

- [1] S. Sajja, M. Corless, E. Zeheb, and R. Shorten, "Some stability tests for switched descriptor systems," *Automatica*, vol. 106, pp. 257–265, 2019.
- [2] E. Mostacciolo, F. Vasca, and S. Baccari, "Differential algebraic equations and averaged models for switched capacitor converters with state jumps," *IEEE Trans. Power Electron.*, vol. 33, no. 4, pp. 3472–3483, 2017.

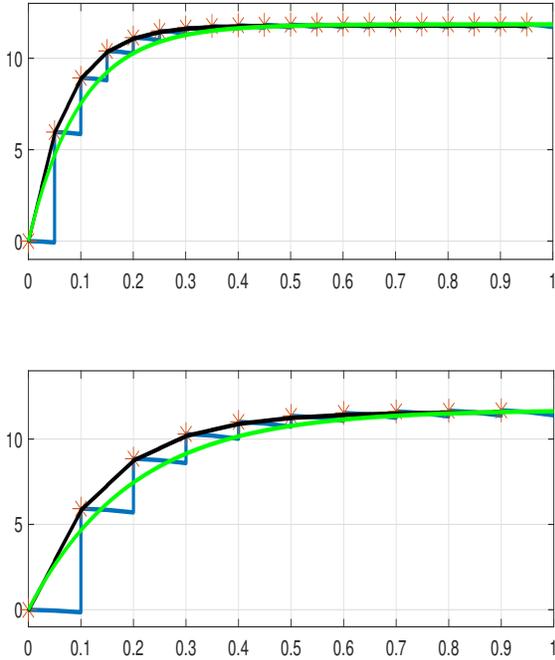


Fig. 2. Time evolution ([s]) of the second state variable ([V]) for different circuit models with $p = 0.05$ s (top) and $p = 0.1$ s (bottom): solution of the original switched system (1) (blue lines), solution of the discrete-time averaged model (6) (red stars), solution of the continuous-time averaged model (40) (green lines), moving average (26) (dark lines).

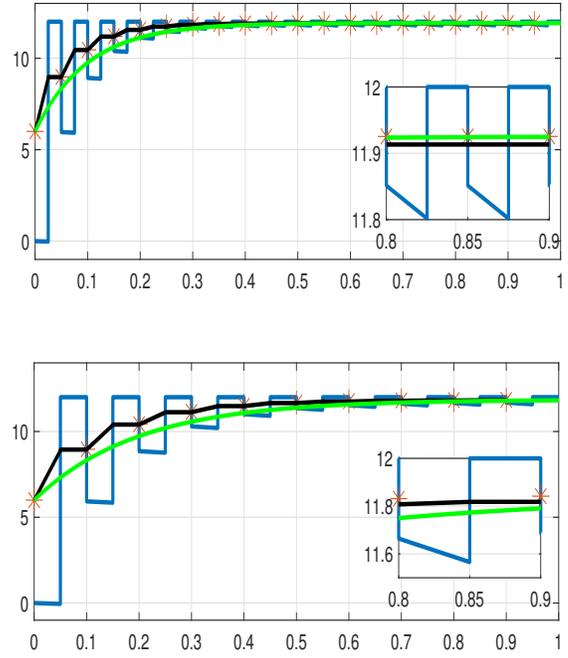


Fig. 3. Time evolution ([s]) of the third state variable ([V]) for different circuit models with $p = 0.05$ s (top) and $p = 0.1$ s (bottom): solution of the original switched system (1) (blue lines), solution of the discrete-time averaged model (6) (red stars), solution of the continuous-time averaged model (40) (green lines), moving average (dark lines).

- [3] S. Markkassery, A. Saradagi, A. D. Mahindrakar, N. Lakshminarasamma, and R. Pasumathy, "Modeling, design and control of non-isolated single-input multi-output zeta-buck-boost converter," *IEEE Trans. Ind. Appl.*, vol. 56, no. 4, pp. 3904–3918, 2020.
- [4] R. Iervolino and F. Vasca, "Cone-copositivity for absolute stability of Lur'e systems," in *Proc. of 53rd IEEE Conf. on Decision and Control*, Los Angeles, CA, USA, 2014, pp. 6305–6310.
- [5] R. Iervolino, S. Trenn, and F. Vasca, "Stability of piecewise affine systems through discontinuous piecewise quadratic Lyapunov functions," in *Proc. of 56th IEEE Conf. on Decision and Control*, Melbourne, Australia, 2017, pp. 5894–5899.
- [6] S. Almér and U. Jönsson, "Harmonic analysis of pulse-width modulated systems," *Automatica*, vol. 45, no. 4, pp. 851–862, 2009.
- [7] S. Baldi, A. Papachristodoulou, and E. B. Kosmatopoulos, "Adaptive pulse width modulation design for power converters based on affine switched systems," *Nonlinear Analysis: Hybrid Systems*, vol. 30, pp. 306–322, 2018.
- [8] A. Teel, L. Moreau, and D. Nešić, "Input-to-state set stability of pulse width modulated systems with disturbances," *Syst. Control Lett.*, vol. 51, no. 1, pp. 23–32, 2004.
- [9] C. Pedicini, L. Iannelli, F. Vasca, and U. T. Jönsson, "An overview on averaging for pulse-modulated switched systems," in *Proc. of 50th IEEE Conf. on Decision and Control and European Control Conference*, Orlando, USA, 2011, pp. 1860–1865.
- [10] S. Rojas and A. Gensior, "Instantaneous average prediction of the currents in a PWM-controlled two-level converter," *IEEE Trans. Ind. Electron.*, vol. 66, no. 1, pp. 397–406, 2018.
- [11] L. Iannelli, K. Johansson, U. Jönsson, and F. Vasca, "Subtleties in the averaging of a class of hybrid systems with applications to power converters," *Control Eng. Practice*, vol. 18, no. 8, pp. 961–975, 2008.
- [12] W. Wang and D. Nešić, "Input-to-state set stability and averaging of linear fast switching systems," *IEEE Trans. Autom. Control*, vol. 55, no. 5, pp. 1274–1279, 2010.
- [13] W. Wang, D. Nešić, and A. Teel, "Input-to state stability for a class of hybrid dynamical systems via averaging," *Math. Control Signals Syst.*, vol. 23, pp. 223–256, 2012.
- [14] S. R. Sanders, J. M. Noworolski, X. Z. Liu, and G. C. Verghese, "Generalized averaging method for power conversion circuits," *IEEE Trans. Power Electron.*, vol. 6, no. 2, pp. 251–259, 1991.
- [15] C. Pedicini, L. Iannelli, and F. Vasca, "The averaging method for control design and stability analysis of practical switched systems," in *IEEE Conf. on Control Appl., part of IEEE Multi-Conference on Systems and Control*, Dubrovnik, Croatia, 2012, pp. 1285–1290.
- [16] M. A. Qureshi, F. Torelli, A. Mazza, and G. Chicco, "Application of artificial dynamics to represent non-isolated single-input multiple-output dc-dc converters with averaged models," in *56th Int. Universities Power Engineering Conf.* IEEE, 2021, pp. 1–6.
- [17] L. Iannelli, C. Pedicini, S. Trenn, and F. Vasca, "An averaging result for switched DAEs with multiple modes," in *Proc. of 52nd IEEE Conf. on Decision and Control*, Florence, Italy, 2013, pp. 1378–1383.
- [18] E. Mostacciolo, S. Trenn, and F. Vasca, "Partial averaging for switched DAEs with two modes," in *Proc. of 14th European Control Conf.*, Linz, Austria, 2015, pp. 2901–2906.
- [19] —, "Averaging for non-homogeneous switched DAEs," in *Proc. of 54th IEEE Conf. on Decision and Control*, Osaka, Japan, 2015, pp. 2951–2956.
- [20] —, "Averaging for switched DAEs: Convergence, partial averaging and stability," *Automatica*, vol. 82, pp. 145–157, 2017.
- [21] E. Mostacciolo and F. Vasca, "Averaged model for power converters with state jumps," in *Proc. of 15th European Control Conf.*, Aalborg, Denmark, 2016, pp. 301–306.
- [22] E. Mostacciolo, S. Trenn, and F. Vasca, "A smooth model for periodically switched descriptor systems," *Automatica*, vol. 136, p. 110082, 2022.
- [23] S. Boyd, L. E. Ghaoui, E. Feron, and V. Balakrishnan, *Linear matrix inequalities in system and control theory*. Philadelphia: SIAM, 1994.
- [24] V. S. Kozyakin, "Algebraic unsolvability of problem of absolute stability of desynchronized systems," *Autom. Remote Control*, vol. 51, no. 6, pp. 754–759, 1990.