Funnel Control for Relative Degree One Nonlinear Systems With Input Saturation

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Abstract— **The dilemma between transient behavior and accuracy in tracking control arises in both theoretical research and engineering practice and funnel control has shown great potential in solving that problem. Apart from the controlled system, the performance of funnel control strongly depends on the reference signal and the choice of prescribed funnel boundary. In this paper, we will present a new form of funnel controller for systems with control saturation. Compared to former research, the new controller is more reliable, and the closed-loop system can even achieve asymptotic tracking. Besides that, a new concept called constrained funnel boundary is introduced. Together with the new controller and the constrained funnel boundary, the application range of funnel control is extended significantly.**

I. INTRODUCTION

The study of adaptive control which do not require accurate identification goes back to the early 1980s [1], [2]. For the stabilization of the output y of a nonlinear system a classical adaptive control strategy takes the form

$$u(t) = -k(t)y(t),$$

with a time-varying gain given by $\dot{k}(t) = y(t)^2$, $k(0) = k_0 > 0$. In the context of tracking of a given reference signal r, the adaptive controller above can be modified into

$$u(t) = -k(t)e(t), \tag{1}$$

where $e(t) = y(t) - r(t)$ denotes the tracking error. Similarly, the adaptive gain in tracking control can be written as $\dot{k}(t) = e(t)^2$. The study of these control strategies (and their variants) can be found in [3]–[5], and a good survey of this field is shown in [6]. Nevertheless, the application of this control strategy is still limited: the adaptive gain $k(t)$ is non-decreasing; and the transient behavior of the closed-loop system is not addressed.

These above limitations are resolved with the so-called funnel controller which was proposed by Ilchmann et al. [7]. The principle of funnel control is: with a high-gain property of the nonlinear system, a large value of the time-varying $k(t)$ in the control law (1) is able to drive the tracking error $e(t)$ towards zero arbitrarily fast, and furthermore $k(t)$ is only large if needed, resulting in a non-monotonic gain. In fact, the time-varying gain in funnel control is

$$k(t) = \mathcal{K}\big(e(t), t\big) = \frac{1}{\psi(t) - |e(t)|}, \tag{2}$$

where $\psi(t)$ denotes the funnel boundary, which can be interpreted as a strict time-varying error bound. Under certain feasibility assumptions the funnel controller (1)+(2) ensures that the error evolves within the funnel, i.e.

$$|e(t)| < \psi(t) \quad \forall t \geq 0.$$

The study of funnel controller has been flourishing over the last two decades. Funnel controller was successfully extended to work for system with relative degree two [8] and for system with known strict relative degree [9]. A bang-bang funnel controller for nonlinear system with arbitrary relative degree was proposed in [10]. Related to adaptive high-gain observer, a funnel pre-compensator was designed in [11]. Some applications of funnel control can be found in [8], [12], [13].

An important limitation in many practical stabilization and tracking problems are input saturations as shown in Figure 1.

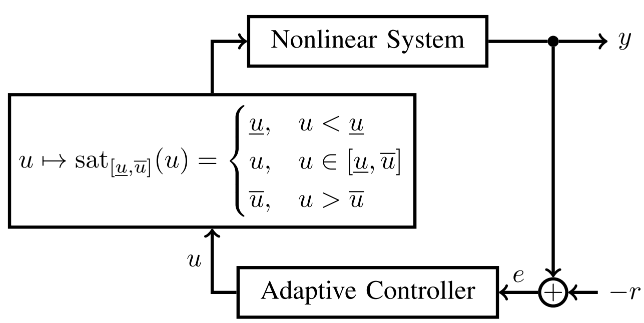

Fig. 1. Closed loop with input saturations $\underline{u} < 0 < \overline{u}$.

Funnel control with input saturation was first studied in [14] for chemcial reactor models and more general in [15] for single-input single-output (SISO) nonlinear system with relative degree one and in [16] for linear multiple-input multiple-output (MIMO) systems. For synchronous machines, PI-funnel controller with input saturation is considered in [17]. For unknown input saturation some results are obtained in [18].

To our best knowledge, all current funnel controllers for input saturation system contain control scheme (2) as the adaptive component. Or even one can say that most funnel controllers contain equation (2) in their controller structures [7]–[9], [19]. But when the control input is saturated, one need to make strict assumptions for the saturation value to guarantee $|e(t)| < \psi(t)$. Otherwise the entire closed-loop system collapses.

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Compared to the above pioneering contributions, and partly inspired by [20], we will introduce a new funnel control approach for relative degree one systems with control input saturation. The main contributions of this paper are: 1) a novel ratio based funnel controller design, and 2) the introduction of a new concept named constrained funnel boundary. With these two tools, the application range of funnel control can be extended significantly. Not only can the new funnel control approach increase the robustness property of closed-loop system, but we can also show that funnel control can achieve asymptotic tracking in the presence of input saturations.

After the initial submission of this manuscript we became aware of [21] which also considers input saturation and an "outer funnel", however, the control law is rather different and a detailed comparison is a topic of future research.

II. Problem Setting

A. System class

Consider the following nonlinear system

$$\dot{y} = f\big(p_f, y, z\big) + g\big(p_g, y, z\big) \cdot \mathrm{sat}_{[\underline{u},\overline{u}]}(u), \quad y(0) = y^0, \tag{3a}$$

$$\dot{z} = h\big(p_h, y, z\big), \qquad z(0) = z^0, \tag{3b}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}$ denotes the system's output, $u : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is the control input and $z : \mathbb{R}_{\geq 0} \to \mathbb{R}^{n-1}$ is the internal state of order $n-1 \in \mathbb{N}$. The functions f, g: $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}$ and h: $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ are assumed to be locally Lipschitz continuous and $p_f, p_g, p_h : \mathbb{R}_{\geq 0} \to \mathbb{R}^d$ are locally integrable perturbations (and/or unknown d-dimensional, time-varying paramters). The following additional assumptions are made for (3):

A1 $g(p_g, y, z) > 0$ for all p_g, y and z.

A2 BIBO-stability of zero-dynamics: for all bounded and continuous p_h, y the solutions of (3b) satisfy

$$\|z(t)\| \leq b_z\big(\|p_{h[0,t)}\|_\infty, \|y_{[0,t)}\|_\infty, \|z(0)\|\big),$$

for some continuous function b_z. Furthermore, assume $z_0 \in Z_0$ for some bounded $Z_0 \subset \mathbb{R}^{n-1}$.

A3 The perturbations p_f, p_g and p_h are bounded by $p_f^{\max}$, $p_g^{\max}$, $p_h^{\max}$, respectively.

The main idea of funnel control is to establish an output feedback control rule such that the tracking error $e(t) = y(t) - r(t)$ evolves within prescribed funnel boundaries $\psi^-(t)$ and $\psi^+(t)$, see Figure 2. Correspondingly, the time-varying region for tracking error is given by

$$\mathcal{F}_{\psi^-,\psi^+} := \big\{\ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}\ \big|\ \psi^-(t) \leq e \leq \psi^+(t)\ \big\} \tag{4}$$

The following assumptions are made for the funnel boundaries and the reference signal.

A4 $\psi^+ : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $\psi^- : \mathbb{R}_{\geq 0} \to \mathbb{R}_{<0}$ are continuously differentiable, bounded and with bounded derivative.

A5 Reference signal $r : \mathbb{R}_{\geq 0} \to \mathbb{R}$ is continuously differentiable, bounded with bounded derivative.

A6 Initial error: $e(0) := y^0 - r(0) \in (\psi^-(0), \psi^+(0))$.

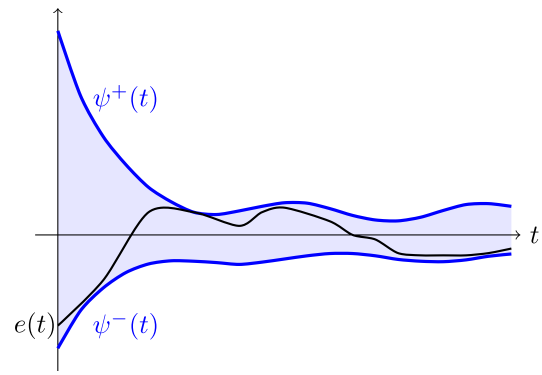

Fig. 2. Prescribed funnel boundary

B. Traditional funnel control

Classical funnel controllers [7]–[9], [11], [12], [15]–[19] have the form

$$u(t) = -\mathcal{K}\big(e(t), t\big) \cdot e(t),$$

in which $\mathcal{K}$ is a positive adaptive gain which ensures that the tracking error $e(t)$ evolves within the prescribed boundaries. A typical choice of adaptive rule $\mathcal{K}$ is shown in (2). Inevitably, this control strategy raises two kinds of issue

i) The closed-loop system can not achieve asymptotic tracking: $\psi(t) \to 0$ as $t \to \infty$ leads to $\mathcal{K} \to \infty$, which is undesirable.

ii) The structure of controller (2) is quite fragile: once the needed input value exceeds the saturation bounds and the error leaves the funnel then the gain (2) has the wrong sign and further destabilizes the closed loop.

III. Controller Design

Partly inspired by [20], we design a new funnel control approach for control input saturated system. This control approach includes two parts: a ratio based funnel controller and constrained funnel boundary. Funnel controllers for different kinds of system will be discussed in this section.

A. Controller design for SISO system

The new funnel controller for system (3) is defined in terms of the ratios $\eta^\pm(t) := e(t)/\psi^\pm(t)$ and is given by

$$u(t) = \begin{cases} \alpha\big(\eta^+(t)\big) \cdot \underline{u} & 0 \leq e(t) \leq \psi^+(t) \\ \alpha\big(\eta^-(t)\big) \cdot \overline{u} & \psi^-(t) \leq e(t) \leq 0, \end{cases} \tag{5}$$

where

A7 $\alpha : [0, 1] \to [0, 1]$ is continuous, $\alpha(0) = 0$, $\alpha(1) = 1$.

The above mentioned shortcomings in classical funnel control used in combination with saturated inputs can be overcome by (5); indeed, $\psi^\pm(t) \to 0$ as $t \to \infty$ doesn't automatically lead to infinite signals and if the saturation is too restrictive to keep the error in the funnel, then the definition of α according to **A**7 can easily be extended for arguments larger than one with value equal to one, i.e. the input just continues to use the maximal effort to get the error back into the funnel (without any crossing of poles), see Section IV for details.

In order to check the forthcoming feasibility condition, knowledge of the following constants is required.

A8 There exists constants $Y_{\max}$, $Y_{\min}$, $Z_{\max}$, $G_{\min}$, $F_{\max}$, $F_{\min} \in \mathbb{R}$ such that

$$Y_{\max} \geq \sup_{t\geq 0}(r(t)+\psi^+(t)),\ Y_{\min} \leq \inf_{t\geq 0}(r(t)+\psi^-(t)),$$

$$Z_{\max} \geq \max_{\|p_h\|\leq p_h^{\max}, y\in[Y_{\min},Y_{\max}], z_0\in Z_0} b_z(\|p_h\|,|y|,\|z_0\|),$$

$$0 < G_{\min} \leq \min_{\|p_g\|\leq p_g^{\max}, y\in[Y_{\min},Y_{\max}], |z|\leq Z_{\max}} g\big(p_g,y,z\big),$$

$$F_{\max} \geq \max_{\|p_f\|\leq p_f^{\max}, y\in[Y_{\min},Y_{\max}], |z|\leq Z_{\max}} f\big(p_f,y,z\big).$$

$$F_{\min} \leq \min_{\|p_f\|\leq p_f^{\max}, y\in[Y_{\min},Y_{\max}], |z|\leq Z_{\max}} f\big(p_f,y,z\big).$$

Theorem 1: Consider the nonlinear SISO system (3) satisfying assumptions **A**1-**A**3 with prescribed funnel boundaries ψ^+, ψ^- and reference signal r satisfying **A**4-**A**6. The output feedback controller (5) that satisfies **A**7 ensures that

$$\psi^-(t) < e(t) < \psi^+(t) \quad \forall t\geq 0 \tag{6}$$

if the control input saturation values satisfy

$$\underline{u} < \frac{\min_{t\geq 0}\big(\dot{\psi}^+(t)+\dot{r}(t)\big)-F_{\max}}{G_{\min}} \quad \text{and} \qquad \overline{u} > \frac{\max_{t\geq 0}(\dot{\psi}^-(t)+\dot{r}(t))-F_{\min}}{G_{\min}}, \tag{7}$$

where $F_{\max}$, $F_{\min}$, $G_{\min}$ satisfy **A**8.

Proof: By continuing the definition $u(t)$ continuously also outside the funnel by $u(t) := \overline{u}$ and $u(t) = \underline{u}$, resp., standard arguments from ODE theory, ensure that the closed-loop system has a unique maximal solution $(y,z) : [0,\overline{\omega}) \to \mathbb{R}\times\mathbb{R}^{n-1}$ for some $\overline{\omega} > 0$. Seeking a contradiction, assume now that there is a minimal $\omega \in (0,\overline{\omega})$ such that $e(\omega) = \psi^+(\omega)$ or $e(\omega) = \psi^-(\omega)$.

The tracking error satisfies

$$\begin{aligned}\dot{e} =&\dot{y}-\dot{r}\\ =&f(p_f,y,z)+g(p_g,y,z)\,\mathrm{sat}_{[\underline{u},\overline{u}]}(u)-\dot{r}\end{aligned} \tag{8}$$

By assumption e is contained within $[\psi^-,\psi^+]$ and r is bounded on $[0,\omega]$, hence $y = e + r \in [Y_{\min}, Y_{\max}]$ and therefore by **A**2 together with **A**8 we can conclude that $|z(t)| \leq Z_{\max}$ for all $t \in [0,\omega]$. Furthermore, from $u(\omega) = \alpha(1)\underline{u} = \underline{u}$ or $u(\omega) = \alpha(1)\overline{u} = \overline{u}$ together with **A**8 it follows from (8) that either

$$\dot{e}(\omega) \leq F_{\max} - \dot{r}(\omega) + G_{\min}\underline{u} \quad \text{or} \qquad \dot{e}(\omega) \geq F_{\min} - \dot{r}(\omega) + G_{\min}\overline{u}. \tag{9}$$

Plugging (7) into (9) we obtain either

$$\dot{e}(\omega) < \dot{\psi}^+(\omega) \quad \text{or} \quad \dot{e}(\omega) > \dot{\psi}^-(\omega).$$

This is in contradiction that the error reaches the funnel boundary from the interior. In particular, this shows that the funnel $\mathcal{F}_{\psi^+,\psi^-}$ is positively invariant and finite escape time cannot occur, i.e. $\overline{\omega} = \infty$, which concludes the proof. ■

B. Controller design for MIMO system

Consider the following relative degree one MIMO nonlinear system

$$\dot{y}(t) = F\big(p_F(t),y(t),z(t),u(t)\big), \qquad y(0)=y^0, \tag{10a}$$

$$\dot{z}(t) = H\big(p_h(t),y(t),z(t)\big), \qquad z(0)=z^0, \tag{10b}$$

where $y : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are input and output signals resp, for some $m > 1$. The control input is assumed saturated in energy sense: $\|u\|_2 \leq \hat{u}$, for some $\hat{u} > 0$. $F : \mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{n-m}\times\mathbb{R}^m \to \mathbb{R}^m$ and $H : \mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{n-m} \to \mathbb{R}^{n-m}$ are locally Lipschitz continuous functions and p_F, $p_h : \mathbb{R} \to \mathbb{R}^d$ are locally integrable perturbations. We make the following assumptions for (10).

B1 F is differentiable with respect to u. There exist a continuous function $\gamma : \mathbb{R}^d\times\mathbb{R}^m\times\mathbb{R}^{n-m} \to \mathbb{R}$ and $\underline{\alpha} \in (0,1)$ such that the following implication holds for all $\eta,\tilde{\eta} \in \mathbb{R}^m$, $\mu \in \mathbb{R}^d$, $\nu \in \mathbb{R}^m$, $\xi \in \mathbb{R}^{n-m}$, $\zeta \in \mathbb{R}^m$:

$$\|\eta\|_2 \leq 1 \ \wedge\ \|\tilde{\eta}\|_2 \leq 1 \ \wedge\ \eta^\top\tilde{\eta} \geq \underline{\alpha}\|\eta\|_2^2$$

$$\implies \eta^\top\Big[\frac{\partial}{\partial u}F(\mu,\nu,\xi,\zeta)\Big]\tilde{\eta} \geq \gamma(\mu,\nu,\xi)\|\eta\|_2^2 > 0.$$

B2 BIBO zero dynamics: all solutions of $\dot{z} = H(p_h,y,z)$ satisfy the following inequality

$$\|z(t)\| \leq B_z(\|p_{h[0,t)}\|_\infty, \|y_{[0,t)}\|_\infty, \|z(0)\|)$$

for some continuous and bounded function B_z. Furthermore, assume $z_0 \in Z_0$ for some bounded $Z_0 \in \mathbb{R}^{n-m}$.

B3 The perturbations p_F, p_h are bounded by $p_F^{\max}$, $p_h^{\max}$, respectively.

B4 Reference signal $r : \mathbb{R} \to \mathbb{R}^m$ is continuous differentable, bounded and with bounded derivative. There exists $C_{\dot{r}} > 0$, such that $\|\dot{r}\|_2 \leq C_{\dot{r}}$.

The control object is to keep $\|e(t)\|_2$ within the funnel for all $t \geq 0$, i.e.

$$\mathcal{F}_\psi := \{\ (t,e) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^m \ \mid \|e\|_2 \leq \psi(t)\ \}$$

Following assumptions are made for funnel boundary and initial value.

B5 ψ: $\mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is bounded and Lipschitz continuous, with Lipschitz constant Λ.

B6 Initial error $e(0) := y^0 - r(0)$, and $\|e(0)\|_2 \in [0,\psi(0)]$.

Let $\mathbb{B}^m_{\leq 1} := \{\ \eta \in \mathbb{R}^m \ \mid \|\eta\|_2 < 1\ \}$, then the to SISO-controller (5) is adjusted as follows for the MIMO case:

$$u(t) = -\alpha\Big(\frac{e(t)}{\psi(t)}\Big)\hat{u}, \qquad \|e(t)\|_2 < \psi(t). \tag{11}$$

B7 $\alpha : \mathbb{B}^m_{\leq 1} \to \mathbb{B}^m_{\leq 1}$ is continuous and satisfies

$$\eta^\top\alpha(\eta) \geq \underline{\alpha}\|\eta\|_2^2$$

with $\underline{\alpha} \in (0,1)$ given by **B**1.

Similar to the SISO case, knowledge of the following constants is required to check the upcoming feasibility condition.

B8 There exists constants $Y_{\max}$, $Z_{\max}$, C_F, $C_\gamma > 0$ such that

$$Y_{\max} \geq \sup_{t\geq 0}\|r(t)+\psi^+(t)\|,$$

$$Z_{\max} \geq \max_{\|p_h\|\leq p_h^{\max}, \|y\|\leq Y_{\max}, z_0\in Z_0} b_z(\|p_h\|,\|y\|,\|z_0\|),$$

$$C_F \geq \max_{\|p_F\|\leq p_F^{\max}, \|y\|\leq Y_{\max}, \|z\|\leq Z_{\max}} \|F(p_F,y,z,0)\|$$

$$0 < C_\gamma \leq \min_{\|p_F\|\leq p_F^{\max}, \|y\|\leq Y_{\max}, \|z\|\leq Z_{\max}} \gamma(p_F,y,z)$$

Theorem 2: Consider the nonlinear MIMO system (10) satisfying assumptions **B**1–**B**3 with reference signal r and ψ satisfying **B**4–**B**6. The closed-loop system with funnel controller (11) satisfying **B**7 ensures that the error evolves within the funnel for all times, i.e. $\|e(t)\|_2 < \psi(t)$ for all $t \geq 0$ if the saturation value satisfies

$$\hat{u} \geq \frac{C_F + C_{\dot{r}} + \Lambda}{C_\gamma}. \tag{12}$$

Proof: Analogously as in the proof of Theorem 1, there exists a maximal solution $(y, z) : [0, \overline{\omega}) \to \mathbb{R}^m \times \mathbb{R}^{n-m}$ of the closed loop. Seeking a contradiction, assume there is $\omega \in (0, \overline{\omega})$ such that $\|e(\omega)\|_2 = \psi(\omega)$. Then, with $\eta := e/\psi$

$$\tfrac{d}{dt}\left(\|e\|_2^2\right) = 2e^\top \dot{e} = 2\psi\eta^\top F(p_F, y, z, u) - 2\psi\eta^\top \dot{r}.$$

According to the Mean Value Theorem (applied to the scalar map $\lambda \mapsto F(p_F, y, z, \lambda u)$), for each, p_F, y, z, u, there exists $\lambda \in (0, 1)$ such that

$$F(p_F, y, z, u) = F(p_F, y, z, 0) + \frac{\partial}{\partial u} F(p_F, y, z, \lambda u) \cdot u.$$

From **B**7, **B**1 and $\|\eta(\omega)\|_2 = 1$ it follows that

$$\begin{aligned}
&\eta(\omega)^\top \frac{\partial}{\partial u} F\big(p_F(\omega), y(\omega), z(\omega), \lambda u(\omega)\big) \cdot u(\omega) \\
&= -\eta(\omega)^\top \frac{\partial}{\partial u} F\big(p_F(\omega), y(\omega), z(\omega), \lambda u(\omega)\big) \cdot \alpha(\eta(\omega))\hat{u} \\
&\leq -\gamma\big(p_F(\omega), y(\omega), z(\omega)\big) \cdot \hat{u}.
\end{aligned}$$

Hence using the Cauchy-Schwartz-Inequality together with **B**8, we obtain

$$\tfrac{d}{dt}(\|e(\omega)\|_2^2) \leq 2\psi(\omega)(C_F + C_{\dot{r}} - C_\gamma \cdot \hat{u}) \tag{13}$$

From (12) it now follows that

$$\tfrac{d}{dt}\big(\|e(\omega)\|_2^2\big) \leq 2\psi(\omega)\dot{\psi}(t_\varepsilon) = \tfrac{d}{dt}(\psi(\omega)^2),$$

which implies that at (and around) $t = \omega$ the function $\|e(\cdot)\|_2$ decreases faster than the funnel boundary $\psi(\cdot)$; this contradicts the assumption that up to $t = \omega$ the error approached the funnel boundary from the interior. In particular, this shows that the funnel $\mathcal{F}_\psi$ is positively invariant and hence $\overline{\omega} = \infty$ which concludes the proof. ■

IV. Constrained funnel boundary design

Ideally, we prefer that the tracking error $e(t)$ evolves strictly within the prescribed funnel. However, if the saturation bounds of the control input are not sufficiently large to follow a fastly changing reference trajectory or quickly react to a rapidly shrinking funnel then we want at least that the error returns to the interior of the desired funnel region as quickly as possible. Furthermore, we want to be able to provide some outer safety region for which we can guarantee that the error evolves within even if the desired funnel region is left.

In the following we will focus on the SISO-case; similar ideas will also be applicable to the MIMO case. Our approach has two main ingredients: 1) Extending the domain of the controller definition (5) also for $e \notin [\psi^-, \psi^+]$ by simply setting $\alpha(\eta) = 1$ for all $\eta > 1$; 2) by defining a *constraint funnel*

$$\mathcal{F}_{\text{out}} := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R} \;\middle|\; \psi^-_{\text{out}}(t) \leq e \leq \psi^+_{\text{out}}(t) \right\}$$

which contains the original funnel and is the "smallest" region which is feasible with respect to the input constraints, see Figure 3.

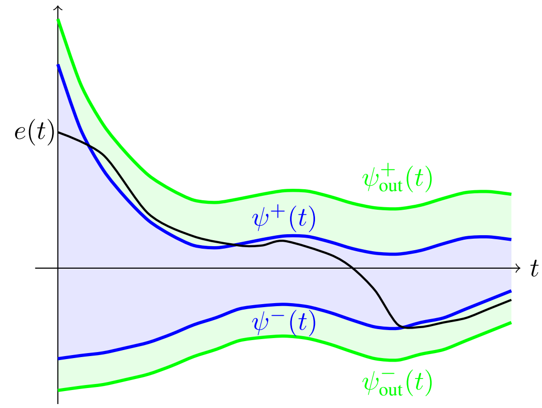

Fig. 3. Constrained funnel boundary.

In the following we will make assumptions as before, however, we consider a slight variation of **A**8 as follows

$\overline{\mathbf{A}8}$ There exists constants $\Psi_{\max}$, $\Psi_{\min}$, $\overline{Y}_{\max}$, $\overline{Y}_{\min}$, $\overline{Z}_{\max}$, $\overline{G}_{\min}$, $\overline{F}_{\max}$, $\overline{F}_{\min} \in \mathbb{R}$ such that

$$\Psi_{\max} \geq \sup_{t\geq 0} \psi^+(t), \qquad \Psi_{\min} \leq \inf_{t\geq 0} \psi^-(t),$$

$$\overline{Y}_{\max} \geq \Psi_{\max} + \sup_{t\geq 0} r(t), \quad \overline{Y}_{\min} \leq \Psi_{\min} + \inf_{t\geq 0} r(t),$$

and $\overline{Z}_{\max}$, $\overline{G}_{\min}$, $\overline{F}_{\max}$, $\overline{F}_{\min}$ satisfy the corresponding inequalities from **A**8 where the interval $[Y_{\min}, Y_{\max}]$ is replaced by the larger interval $[\overline{Y}_{\min}, \overline{Y}_{\max}]$.

In order that tracking control makes sense at all under input constraints, it is intuitively clear that the following implications need to hold:

$$u(t) = \overline{u} \implies \dot{e}(t) > 0,$$
$$u(t) = \underline{u} \implies \dot{e}(t) < 0,$$

i.e. if we apply the maximal available input the output is moving at least a little bit into the direction of the reference signal. If only the above system constants are known it is easily seen that this intuitive condition is satisfied if

$$\begin{aligned}
\overline{d} &:= \overline{F}_{\min} + \overline{G}_{\min}\overline{u} - \sup_{t\geq 0} \dot{r} > 0, \\
\underline{d} &:= \overline{F}_{\max} + \overline{G}_{\min}\underline{u} - \inf_{t\geq 0} \dot{r} < 0.
\end{aligned} \tag{14}$$

In fact, it then follows that the interval $[\Psi_{\min}, \Psi_{\max}]$ is controlled invariant for the error signal.

Given the actual desired funnels $\psi_\pm$ and the feasibility constants $\underline{d} < 0 < \overline{d}$ we can now construct the constrained funnel as follows:

$$\begin{aligned}
\psi^+_{\text{out}}(t) &:= \min\left\{ s \geq \psi^+(\tau) + \underline{d}(t-\tau) \;\middle|\; \tau \in [0,t] \right\} \\
\psi^-_{\text{out}}(t) &:= \max\left\{ s \leq \psi^-(\tau) + \overline{d}(t-\tau) \;\middle|\; \tau \in [0,t] \right\}
\end{aligned} \tag{15}$$

The intuition behind this definition is, that the constrained funnels tries to follow the desired funnel as long as possible, but whenever the desired funnels slope is too steep, the constrained funnel only shrinks with a rate $\underline{d}$ or $\overline{d}$, see Figure 4.

With very similar arguments as in the proof of Theorem 1 it can now be shown that the constrained funnel is positively

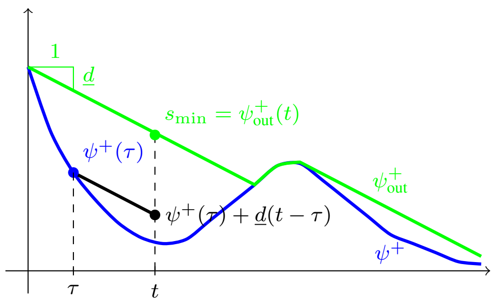

Fig. 4. Illustration of the design of ψ^+_{out}.

invariant under the extended funnel rule (5). These observation are summarized in the following theorem.

Theorem 3: Consider nonlinear SISO system (3) with controller (5) (defined in terms of the desired funnel boundaries $\psi^{\pm}$) satisfying **A**1-**A**7 and $\overline{\mathbf{A}8}$ with $\underline{d} < 0 < \overline{d}$ given by (14). Then

$$e(t) \in [\psi^-_{\text{out}}, \psi^+_{\text{out}}],$$

where $\psi^{\pm}_{\text{out}}$ are given by (15).

Remark 1: Since $\overline{d} < 0$, $\underline{d} > 0$, **A**4 indicates that $e(t)$ always returns the to desired funnel in finite time. In particular, for $\psi(t) \to 0$ as $t \to \infty$, asymptotic tracking remains feasible even if the error leaves the funnel.

V. SIMULATIONS

Taking the controlled system from previous research [15] as

$$\begin{aligned}\dot{y}(t) =& p_f(t) + |y(t)|y(t) + z(t) + \text{sat}_{[-\hat{u},\hat{u}]}\big(u(t)\big),\\ \dot{z}(t) =& -z(t) - z^3(t) + [1 + z^2(t)]y(t),\\ y^0 =& 3.4,\ \ z^0 = 1,\end{aligned}$$

with reference signal $r(t) = \xi_1(t)$ and perturbation $p_f(t) = -\xi_2(t)$ from the Lorenz system

$$\begin{aligned}\dot{\xi}_1(t) =& \xi_2(t) - \xi_1(t),\\ \dot{\xi}_2(t) =& \Big(\frac{28\xi_1(t)}{10}\Big) - \Big(\frac{\xi_2(t)}{10}\Big) - \xi_1(t)\xi_3(t),\\ \dot{\xi}_3(t) =& \xi_1(t)\xi_2(t) - \Big(\frac{8\xi_3(t)}{30}\Big),\\ \xi_1(0) =& 1,\ \ \xi_2(0) = 0,\ \ \xi_3(0) = 3.\end{aligned}$$

In particular, the reference signal and perturbation satisfy $\|r\|_\infty \le 2$, $\|\dot{r}\|_\infty \le 1$ and $\|p_f\|_\infty \le 2.4$.

The saturation values for the control input are $-\underline{u} = \overline{u} = 34$. The prescribed funnel boundaries are chosen as $\psi^+(t) = 2.5e^{-20t}$, and $\psi^-(t) = -\psi^+(t)$. The closed-loop system adopts controller (5) with $\alpha(\eta) = \eta$. To obtain a bound for the zero-dynamics, define $V(t) := z^2(t)/2$ for all $t \in [0, \omega)$, then

$$\begin{aligned}\dot{V}(t) =& -2V(t) - z^2(t) + z(t)y(t) + z^3(t)y(t)\\ \le& -2V(t) + \frac{y^2(t)}{2} + \frac{y^4(t)}{4},\ \ \forall t \in [0, \omega).\end{aligned}$$

Utilizing Gronwall's lemma [22] it follows that

$$V(t) \le e^{-2t}V(0) + \frac{1}{4}\int_0^4 \big(2y^2(s) + y^4(s)\big)e^{-2(t-s)}ds$$

and hence for all $t \ge 0$

$$|z(t)| \le e^{-t}|z^0| + \frac{1}{2\sqrt{2}}\text{ess-sup}_{s\in[0,t]}\big(|y(s)|\sqrt{2 + |y(s)|^2}\big).$$

Consequently $F_{\max} = 31.2$ satisfies

$$\begin{aligned}F_{\max} \ge& \|p_f\|_\infty + (\|\psi^+\|_\infty + \|r\|_\infty)^2 + \|z^0\|\\ &+ \frac{1}{2\sqrt{2}}(\|\psi^+\|_\infty + \|r\|_\infty)\sqrt{2 + (\|\psi^+\|_\infty + \|r\|_\infty)^2}\end{aligned}$$

$e(0) = y^0 - r(0) = 2.4$. Meanwhile, $F^+_{\max} + \|\dot{r}\|_\infty \le 32.2 < 34 = -\underline{u}$. Based on (14), $\underline{d} = 32.2 - 34 = -1.8$. Simulations are shown in Figure 5, Figure 6 and Figure 7.

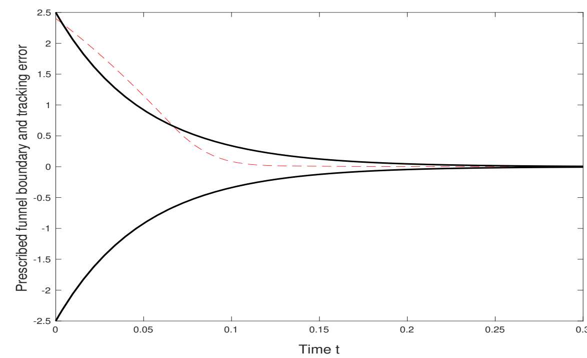

Fig. 5. Prescribed funnel boundary and tracking error.

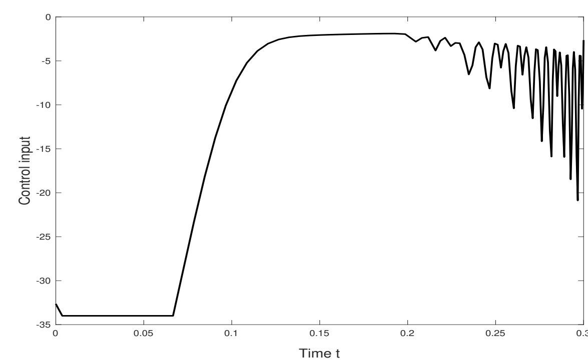

Fig. 6. Control input under the saturation.

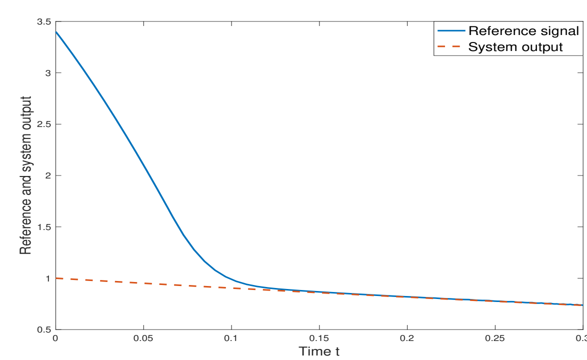

Fig. 7. Reference input and system output.

As shown in Figure 5, tracking error is allowed to leave the prescribed funnel boundary and the closed-loop system achieve asymptotic tracking under the controller (5). The tracking error returns to ψ^+ at $0.06s$. The chattering visible in the control input in Figure 6 is a numerical artifact due to the small funnel values and could be resolved by smaller time-steps (i.e. higher sampling in applications) or by adjusting the control law (which is ongoing research).

Then the constrained funnel boundary is given by $\psi^+_{\text{out}}(t) = \max\{(e(0) + \underline{d} \cdot t), \psi^+(t)\}$, see Figure 8.

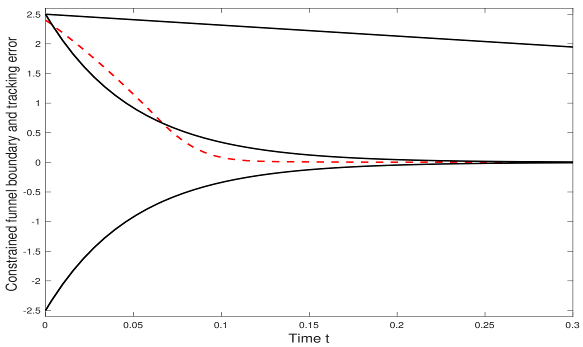

Fig. 8. Constrained funnel boundary and tracking error.

Figure 8 depicts that, even though the control input can keep the tracking error within the prescribed funnel boundary due to input saturation, the closed-loop system quickly returns to the interior of the funnel and asymptotic tracking is achieved while all closed-loop signals remain bounded.

VI. CONCLUSIONS

Compared to traditional funnel control, a new control approach for nonlinear system with input saturation is established in this paper. The new funnel controller adopts a ratio form, and it appears more reliable in engineering practice by avoiding poles when $e(t)$ crosses $\psi(t)$.

Furthermore, a new concept called constrained funnel boundary is introduced and we provide a constructive method to describe this constrained funnel in terms of the original funnel and some system bounds. Based on this construction, it follows that even when the error leaves the desired funnel, the error remains in the constrained funnel and eventually returns to the desired funnel.

REFERENCES

[1] C. I. Byrnes and J. C. Willems, "Adaptive stabilization of multivariable linear systems," in *23rd IEEE Conference on Decision and Control (CDC)*, pp. 1574–1577, IEEE, 1984.

[2] B. Mårtensson, "The order of any stabilizing regulator is sufficient a priori information for adaptive stabilization," *Systems & Control Letters*, vol. 6, no. 2, pp. 87–91, 1985.

[3] E. P. Ryan, "A nonlinear universal servomechanism," *IEEE Transactions on Automatic Control*, vol. 39, no. 4, pp. 753–761, 1994.

[4] A. Ilchmann, *Non-identifier-based high-gain adaptive control*. Springer, 1993.

[5] A. Ilchmann and E. P. Ryan, "Universal λ-tracking for nonlinearly-perturbed systems in the presence of noise," *Automatica*, vol. 30, no. 2, pp. 337–346, 1994.

[6] A. Ilchmann and E. P. Ryan, "High-gain control without identification: a survey," *GAMM-Mitteilungen*, vol. 31, no. 1, pp. 115–125, 2008.

[7] A. Ilchmann, E. P. Ryan, and C. J. Sangwin, "Tracking with prescribed transient behaviour," *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 7, pp. 471–493, 2002.

[8] C. M. Hackl, N. Hopfe, A. Ilchmann, M. Mueller, and S. Trenn, "Funnel control for systems with relative degree two," *SIAM Journal on Control and Optimization*, vol. 51, no. 2, pp. 965–995, 2013.

[9] T. Berger, H. H. Lê, and T. Reis, "Funnel control for nonlinear systems with known strict relative degree," *Automatica*, vol. 87, pp. 345–357, 2018.

[10] D. Liberzon and S. Trenn, "The bang-bang funnel controller for uncertain nonlinear systems with arbitrary relative degree," *IEEE Transactions on Automatic Control*, vol. 58, no. 12, pp. 3126–3141, 2013.

[11] T. Berger and T. Reis, "The funnel pre-compensator," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 16, pp. 4747–4771, 2018.

[12] A. Ilchmann and H. Schuster, "PI-funnel control for two mass systems," *IEEE Transactions on Automatic Control*, vol. 54, no. 4, pp. 918–923, 2009.

[13] T. Berger, M. Puche, and F. L. Schwenninger, "Funnel control for a moving water tank," *Automatica*, vol. 135, p. 109999, 2022.

[14] A. Ilchmann and S. Trenn, "Input constrained funnel control with applications to chemical reactor models," *Syst. Control Lett.*, vol. 53, no. 5, pp. 361–375, 2004.

[15] N. Hopfe, A. Ilchmann, and E. P. Ryan, "Funnel control with saturation: Nonlinear SISO systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2177–2182, 2010.

[16] N. Hopfe, A. Ilchmann, and E. P. Ryan, "Funnel control with saturation: linear mimo systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 2, pp. 532–538, 2010.

[17] C. M. Hackl, "Current PI-funnel control with anti-windup for synchronous machines," in *54th IEEE Conference on Decision and Control (CDC)*, pp. 1997–2004, IEEE, 2015.

[18] Q. Chen, X. Tang, Y. Nan, and X. Ren, "Finite-time neural funnel control for motor servo systems with unknown input constraint," *Journal of Systems Science and Complexity*, vol. 30, no. 3, pp. 579–594, 2017.

[19] T. Berger, M. Puche, and F. Schwenninger, "Funnel control for a moving water tank," *arXiv preprint arXiv:1902.00586*, 2019.

[20] J. G. Lee and S. Trenn, "Asymptotic tracking via funnel control," in *58th IEEE Conference on Decision and Control (CDC)*, pp. 4228–4233, IEEE, 2019.

[21] C. Hackl, Y. Ji, and D. Schröder, "Nonidentifier–based adaptive control with saturated control input compensation," in *Proc. 15th Mediterian Conf. Control & Automation*, (Athens, Greece), 2007.

[22] W. Walter, *Ordinary differential equations*, vol. 182 of *Graduate Texts in Mathematics*. Springer, New York, 1998.