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On geometric and differentiation index of nonlinear differential-algebraic equations

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Abstract: We discuss two notions of index, i.e., the geometric index and the differentiation index for nonlinear differential-algebraic equations (DAEs). First, we analyze solutions of nonlinear DAEs by revising a geometric reduction method (see e.g. Rabier and Rheinboldt (2002),Riaza (2008)). Then we show that although both of the geometric index and the differentiation index serve as a measure of difficulties for solving DAEs, they are actually related to the existence and uniqueness of solutions in a different manner. It is claimed in (Campbell and Gear, 1995) that the two indices coincide when sufficient smoothness and assumptions are satisfied, we elaborate this claim and show that the two indices indeed coincide if and only if a condition of uniqueness of solutions is satisfied (under certain constant rank assumptions). Finally, an example of a pendulum system is used to illustrate our results on the two indices.

Keywords: differential-algebraic equations, geometric method, differentiation index, geometric index, existence and uniqueness of solutions

1. INTRODUCTION

We consider a nonlinear differential-algebraic equation (DAE) of the following form

$$\Xi: \; E(x)\dot{x} = F(x), \tag{1}$$

where $x \in X$ is called the generalized state and X is an open subset of $\mathbb{R}^n$, and $E : TX \to \mathbb{R}^l$ and $F : X \to \mathbb{R}^l$ are $\mathcal{C}^\infty$-smooth maps, where TX is the tangent bundle of X. We denote a DAE of the form (1) by $\Xi_{l,n} = (E, F)$ or, simply, Ξ. The DAE (1) is sometimes called a quasi-linear DAE (see e.g., Rabier and Rheinboldt (2002); Riaza (2008)), which is a special case of DAEs in the general form

$$\Xi^{gen} : G(t, x, x') = 0, \tag{2}$$

where $G : I \times TX \to \mathbb{R}^l$ is $\mathcal{C}^\infty$-smooth, $I \subseteq \mathbb{R}$ is an open interval. Notice that Ξ^{gen} can be transformed into a DAE of the form (1) by extending the generalized state to $(t, x, z) = (t, x, x')$, i.e., $\dot{t} = 1$, $\dot{x} = z$, $0 = G(t, x, z)$. which is a DAE of form (1) with $E = \left[\begin{smallmatrix} I_{n+1} & 0 \\ 0 & 0 \end{smallmatrix}\right] \in \mathbb{R}^{(n+l+1)\times(2n+1)}$ and $F(t, x, z) = \left[\begin{smallmatrix} 1 \\ z \\ G(t,x,z) \end{smallmatrix}\right]$.

Two main streams of researches on solutions of nonlinear DAEs are geometric methods (Rheinboldt, 1984; Reich, 1990, 1991; Rabier and Rheinboldt, 1994) and numerical methods (Gear, 1988; Brenan et al., 1996; Kunkel and Mehrmann, 2006). Moreover, some crossover results of the two methods can be consulted in Griepentrog (1991); Campbell and Griepentrog (1995); Rabier and Rheinboldt (2002) and the references therein. To characterize different properties of DAEs, various notions of index are proposed, see the survey or survey-like papers on index of DAEs as Griepentrog et al. (1992); Campbell (1995); Campbell and Gear (1995); Mehrmann (2015). The most commonly used notions of index seem to be the geometric index (Reich, 1990; Rabier and Rheinboldt, 2002) and the differentiation index (see the two different definitions of differentiation index in Definition 4 (Campbell and Gear, 1995) and in Section 3 (Griepentrog, 1991)). Our formulations of the definitions of the two indices are given in Definition 6 and Definition 10 below, respectively.

The purpose of the present paper is to have a comprehensive understanding of the two notions of DAE index by analyzing their relations and differences. Moreover, Campbell and Gear (1995) have claimed that the geometric index and the differentiation index coincide when sufficient smoothness and assumptions are satisfied, we aim to elaborate this claim and present our results about the relations of the two indices. We give our analysis on the solvability of DAEs by revising the geometric reduction method and show how the method leads to the definition of geometric index in Section 2. We discuss differentiation index and show its relations and differences with geometric index in Section 3. The conclusions of this paper are given in Section 4. Throughout we use the following notations: We use x' or $\dot{x}$ to denote the derivate of x with respect to t. The symbol $\mathcal{C}^k$ denotes the class of functions which are k-times differentiable. For a map $A : X \to \mathbb{R}^{l\times n}$, $\ker A(x)$, $\operatorname{Im} A(x)$ and $\operatorname{rank} A(x)$ are the kernel, the image and the rank of A at x, respectively. For a smooth map $f : X \to \mathbb{R}$, we denote its differential by $\mathrm{d}f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\mathrm{d}x_i = [\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}]$ and for a vector-valued map $f : X \to \mathbb{R}^m$, where $f = [f_1, \dots, f_m]^T$, we denote its differential by $\mathrm{D}f$ and its Jacobian matrix by $\mathrm{D}_x f$. For two column vectors $v_1 \in \mathbb{R}^m$ and $v_2 \in \mathbb{R}^n$, we write $(v_1, v_2) = [v_1^T, v_2^T]^T \in \mathbb{R}^{m+n}$.

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2. GEOMETRIC REDUCTION METHOD AND GEOMETRIC INDEX OF DAES

A solution of a DAE Ξ, given by (1), is a $\mathcal{C}^1$-curve $x : I \to X$ with an open interval I such that for all $t \in I$, $E(x(t))\dot{x}(t) = F(x(t))$. A point $x_a \in X$ is called *admissible* (or *consistent*) if there exists a solution $x(\cdot)$ such that $x(t_a) = x_a$ for a certain $t_a \in I$. Denote by S_a the set of all admissible (consistent) points of Ξ. It is clear that S_a is the set on which the solutions of DAEs exist.

The main idea of geometric analysis of DAE solutions is to view a DAE as a vector field defined on a submanifold[1]. which we will call the maximal invariant submanifold, its formal definition is given as follows.

Definition 1. (maximal invariant submanifolds). Consider a DAE $\Xi = (E, F)$ defined on X, fix an admissible point $x_a \in X$. A smooth connected submanifold M of X is called *locally invariant* if there exists an open neighborhood $U \subseteq X$ of x_a such that for any point $x_0 \in M \cap U$, there exists a solution $x : I \to X$ of Ξ such that $x(t_0) = x_0$ for a certain $t_0 \in I$ and $x(t) \in M \cap U$ for all $t \in I$. A locally invariant submanifold M^* is called *maximal* if there exists a neighborhood V of x_a such that for any other locally invariant submanifold M, we have $M \cap V \subseteq M^* \cap V$.

The following recursive method is called the geometric reduction method, which was frequently used (see e.g. Reich (1990); Rabier and Rheinboldt (2002); Riaza (2008); Berger (2016, 2017); Chen and Respondek (2021b)) to construct the locally maximal invariant submanifold M^* and to study the existence of solutions.

Definition 2. (geometric reduction method). For a DAE $\Xi_{l,n} = (E, F)$, fix a point $x_p \in X$ and let U_0 be an open connected subset of X containing x_p. Set $M_0 = X$, $M_0^c = U_0$. Suppose that there exist an open neighborhood U_{k-1} of x_p and a sequence of smooth connected submanifolds $M_{k-1}^c \subsetneq \cdots \subsetneq M_0^c$ of U_{k-1} for a certain $k \geq 1$, has been constructed. Define recursively

$$M_k := \left\{x \in M_{k-1}^c : F(x) \in E(x)T_x M_{k-1}^c\right\}. \qquad (3)$$

As long as $x_p \in M_k$, assume that there exists a neighborhood U_k of x_p such that $M_k^c = M_k \cap U_k$ is a smooth connected submanifold.

Remark 3. If we apply the above method to a linear DAE $E\dot{x} = Ax$, where $E, A \in \mathbb{R}^{l\times n}$, and denote $\mathscr{V} = M$, we get the following sequence of subspaces: $\mathscr{V}_0 := \mathbb{R}^n$ and for $k \geq 1$, set

$$\mathscr{V}_k = \{x \in \mathscr{V}_{k-1} \mid Ax \in E\mathscr{V}_{k-1}\} = A^{-1}E\mathscr{V}_{k-1}.$$

The above sequence $\mathscr{V}_k$ is one of the Wong sequences (Wong, 1974), which plays an important role in the geometric theory of linear DAEs (see e.g., (Berger and Trenn, 2012) and (Chen and Respondek, 2021a)). Thus the sequence of submanifolds M_k is, clearly, a nonlinear generalization of the Wong sequence $\mathscr{V}_k$. Note that the limits of $\mathscr{V}_k$, i.e., $\mathscr{V}^* = \mathscr{V}_n$ is the largest subspace such that $A\mathscr{V}^* \subseteq E\mathscr{V}^*$, which coincides with the consistency space of the linear DAE.

The geometric descriptions of Definition 2 can be implemented through Algorithm 1 below.

[1] Throughout when talking about submanifolds, we will always mean embedded submanifolds, see (Lee, 2001)

Algorithm 1 Geometric reduction algorithm

Initiatlization: Consider a DAE $\Xi_{l,n} = (E, F)$, fix $x_p \in X$ and let $U_0 \subseteq X$ be an open connected subset containing x_p. Set $z_0 = x$, $E_0(z_0) = E(x)$, $F_0(z_0) = F(x)$, $M_0^c = U_0$, $r_0 = l$, $n_0 = n$, and $\Xi_0 = (E_0, F_0)$.

Step k ($k > 1$): Suppose that we have defined at Step $k-1$: an open neighborhood $U_{k-1} \subseteq X$ of x_p, a smooth connected submanifold M_{k-1}^c of U_{k-1} and a DAE $\Xi_{k-1} = (E_{k-1}, F_{k-1})$ with

$$E_{k-1} : M_{k-1}^c \to \mathbb{R}^{r_{k-1}\times n_{k-1}}, \quad F_{k-1} : M_{k-1}^c \to \mathbb{R}^{r_{k-1}},$$

whose arguments are denoted $z_{k-1} \in M_{k-1}^c$. Rename $\tilde{E}_k = E_{k-1}$, $\tilde{F}_k = F_{k-1}$ and define $\tilde{\Xi}_k := (\tilde{E}_k, \tilde{F}_k)$.

Assumption (A1): There exists an open neighborhood $U_k \subseteq U_{k-1} \subseteq X$ of x_p such that $\dim E(x)T_x M_{k-1}^c = \operatorname{rank} \tilde{E}_k(z_{k-1}) = const. = r_k$, $\forall z_{k-1} \in W_k = U_k \cap M_{k-1}^c$.

1: Find a smooth map $Q_k : W_k \to GL(r_{k-1}, \mathbb{R})$, such that $\tilde{E}_k^1$ of $Q_k\tilde{E}_k = \begin{bmatrix} \tilde{E}_k^1 \\ 0 \end{bmatrix}$ is of full row rank and denote $Q_k\tilde{F}_k = \begin{bmatrix} \tilde{F}_k^1 \\ \tilde{F}_k^2 \end{bmatrix}$, where $\tilde{E}_k^1 : W_k \to \mathbb{R}^{r_k\times n_{k-1}}$, $\tilde{F}_k^2 : W_k \to \mathbb{R}^{r_{k-1}-r_k}$ (so all the matrices depend on z_{k-1}).

2: Following (3), define $M_k = \left\{z_{k-1} \in W_k \mid \tilde{F}_k^2(z_{k-1}) = 0\right\}$.

Assumption (A2): $x_p \in M_k$ and $\operatorname{rank} \mathrm{D}\tilde{F}_k^2(z_{k-1}) = const. = n_{k-1} - n_k$ for $z_{k-1} \in M_k \cap U_k$.

3: By Assumption 2, $M_k \cap U_k$ is a smooth submanifold and by taking a smaller U_k, we may assume that $M_k^c = M_k \cap U_k$ is connected and choose new coordinates $(z_k, \bar{z}_k) = \psi_k(z_{k-1})$ on W_k, where $\bar{z}_k = (\bar{\varphi}_k^1(z_{k-1}), ..., \bar{\varphi}_k^{n_{k-1}-n_k}(z_{k-1}))$, with $\mathrm{d}\bar{\varphi}_k^1(z_{k-1}), ..., \mathrm{d}\bar{\varphi}_k^{n_{k-1}-n_k}(z_{k-1})$ being all independent rows of $\mathrm{D}\tilde{F}_k^2(z_{k-1})$, and z_k are any complementary coordinates such that ψ_k is a local diffeomorphism.

4: Set $\hat{E}_k = Q_k\tilde{E}_k\left(\frac{\partial\bar{\varphi}_k}{\partial z_{k-1}}\right)^{-1}$, $\hat{F}_k = Q_k\tilde{F}_k$. Define

$$\hat{\Xi}_k : \begin{bmatrix} \hat{E}_k^1(z_k, \bar{z}_k) & \bar{E}_k^1(z_k, \bar{z}_k) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{z}_k \\ \dot{\bar{z}}_k \end{bmatrix} = \begin{bmatrix} \hat{F}_k^1(z_k, \bar{z}_k) \\ \hat{F}_k^2(z_k, \bar{z}_k) \end{bmatrix} \qquad (4)$$

with $\hat{E}_k^1 : W_k \to \mathbb{R}^{r_k\times n_k}$, $\hat{F}_k^1 \circ \psi_k = \tilde{F}_k^1$, $\hat{F}_k^2 \circ \psi_k = \tilde{F}_k^2$ and $[\hat{E}_k^1 \circ \psi_k \ \ \bar{E}_k^1 \circ \psi_k] = \tilde{E}_k^1\left(\frac{\partial\psi_k}{\partial z_{k-1}}\right)^{-1}$.

5: Set $\bar{z}_k = 0$ to define the following restricted DAE on $M_k^c = \{z_{k-1} \in W_k \mid \bar{z}_k = 0\}$ by

$$\hat{\Xi}_k|_{M_k^c} = \Xi_k : E_k(z_k)\dot{z}_k = F_k(z_k),$$

where $E_k(z_k) = \hat{E}_k^1(z_k, 0)$, $F_k(z_k) = \hat{F}_k^1(z_k, 0)$ are matrix-valued maps and $E_k : M_k^c \to \mathbb{R}^{r_k\times n_k}$, $F_k : M_k^c \to \mathbb{R}^{r_k}$.

Repeat: Step k for $k = 1, 2, 3, \ldots$, **until** $n_{k+1} = n_k$, set $k^* = k$.

Result: Set $n^* = n_{k^*} = n_{k^*+1}$, $r^* = r_{k^*+1}$, $M^* = M_{k^*+1}^c$, $z^* = z_{k^*+1} = z_{k^*}$ and $\Xi^* = (E^*, F^*)$ with $E^* = E_{k^*+1}$, $F^* = F_{k^*+1}$.

The following theorem shows that under Assumptions (A1) and (A2) of Algorithm 1, the sequence of submanifolds M_k converges to the locally maximal invariant submanifold M^*, which coincides locally with the admissible set S_a.

Theorem 4. Consider a DAE $\Xi_{l,n} = (E, F)$ and fix a point $x_p \in X$. Suppose that Assumptions (A1) and (A2) in each Step $k \geq 1$ of Algorithm 1 are satisfied. Then there always exists $k^* \leq n$ such that k^* is the smallest integer such that $x_p \in M_{k^*+1}^c$ and $M_{k^*+1}^c \cap U_{k^*+1} = M_{k^*}^c \cap U_{k^*+1}$. We have that x_p is an admissible point, i.e., $x_p = x_a$ and $M^* = M_{k^*+1}^c$ is a locally maximal invariant submanifold around x_p. Moreover, there exists a neighborhood $U^* \subseteq U_{k^*+1}$ of x_p such that

(i) $S_a \cap U^* = M^* \cap U^*$;

(ii) there exist a local diffeomorphism defined on U^* mapping solutions of Ξ into solutions of

$$E^*(z^*)\dot{z}^* = F^*(z^*), \bar{z}_1 = 0, \ldots, \bar{z}_{k^*} = 0 \qquad (5)$$

where $z^* = z_{k^*+1} = z_{k^*}$ are local coordinates on M^*, $E^* = E_{k^*+1} : M^* \to \mathbb{R}^{r^* \times n^*}$, $F^* = F_{k^*+1} : M^* \to \mathbb{R}^{r^*}$, and $\operatorname{rank} E^*(z^*) = r^*$, $\forall z^* \in M^*$, i.e., $E^*(z^*)$ is of full row rank.

(iii) for any $x_0 \in M^* \cap U^*$, there passes only one solution if and only if $\dim M^* = \dim E(x)T_xM^*$ for all $x \in M^* \cap U^*$, i.e., $n^* = r^*$.

The proof of Theorem 4 can be consulted in the proofs of Propositions 2.7 and 3.3, and Theorem 3.11(ii) of (Chen and Respondek, 2021b).

Remark 5. Item (i) of Theorem 4 illustrates that for all x in a neighborhood U^* of an admissible point $x_a = x_p$, the solutions of nonlinear DAEs exist on the maximal invariant submanifold M^* only. Therefore, for any point $x_0 \in U^* \backslash M^*$, there exist no solutions passing through x_0. From Theorem 4(ii), it is seen that solutions of Ξ are isomorphic to those of $E^*(z^*)\dot{z}^* = F^*(z^*)$ with E^* being of full row rank, which is an "under-determined" DAE and can be expressed as an ordinary differential equation (ODE) with free variables. Item (iii) of Theorem 4 is a geometric characterization of the uniqueness of solutions for nonlinear DAEs.

The definition of geometric index is given as follows (see also Rabier and Rheinboldt (2002); Riaza (2008)).

Definition 6. (geometric index). Consider the sequence of submanifolds M_k^c constructed via Definition 2 around a point x_p, then the (local) geometric index ν_g of a DAE Ξ is defined by

$$\nu_g := \min \left\{ k \geq 0 \,|\, (M_{k+1}^c = M_k^c) \wedge (M_k^c \neq \emptyset) \right\}.$$

Now combining the results of Theorem 4, we give some comments on the geometric index defined above.

Remark 7. (i) The definition of the geometric index needs only the assumption that for each $k \geq 0$, M_k^c is locally a smooth connected submanifold. The constant rank assumptions (A1) and (A2) are not necessarily required for the existence of geometric index. Take for example the following DAE

$$\Xi : x\dot{x} = x^2, \tag{6}$$

where $x \in X = \mathbb{R}$. Clearly, $M^* = M_1^c = M_0^c = X$ is a maximal invariant submanifold and any point $x_a \in X$ is admissible. The geometric index of (6) is $\nu_g = 0$. However $\dim E(x)T_xM^* \neq const.$, for all $x \in M^*$ since $\dim E(x)T_xM^*$ equals 1 for $x \neq 0$ and is 0 for $x = 0$.

(ii) The geometric index of DAEs allows for a conclusion about the existence of solutions. Suppose that Ξ has a well-defined geometric index ν_g around x_p. For any $x_0 \in U_k \backslash M_k^c$, $0 \leq k \leq \nu_g$, around x_p, we can conclude that there does not exist any solution passing through x_0.

(iii) The geometric index does not concern uniqueness of solutions. As an example, consider two DAEs

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_2, \end{cases} \quad \text{and} \quad \tilde{\Xi}_{2,3} : \begin{cases} \dot{x}_1 = f(x_1, x_2) \\ 0 = x_3, \end{cases}$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth. Observe that both Ξ and $\tilde{\Xi}$ have the same geometric index $\nu_g = 1$. Nevertheless, the DAE Ξ has a unique solution for any admissible initial point $(x_{10}, x_{20}) = (x_{10}, 0)$, the DAE $\tilde{\Xi}$ has infinite numbers of solutions for any admissible initial point $(x_{10}, x_{20}, x_{30}) = (x_{10}, x_{20}, 0)$ since x_2 is a free variable of the ODE $\dot{x}_1 = f(x_1, x_2)$.

Example 8. Consider the following DAE $\Xi_{5,5} = (E, F)$, borrowed from Rabier and Rheinboldt (1994),

$$\Xi : \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5x_1 \\ x_4 \\ -x_5x_3 - g \\ x_1^2 + x_3^2 - l^2 \end{bmatrix}. \tag{7}$$

This DAE describes the mathematical model of a pendulum with a mass attached to its end and where the length of the pendulum is given by the constant l. We consider a point $x_p = (x_{1p}, x_{2p}, x_{3p}, x_{4p}, x_{5p})$, where $x_{1p} = 0$, $x_{2p} = 0$, $x_{3p} = -l$, $x_{4p} = 0$, $x_{5p} = g/l$ and apply Algorithm 1 to $\Xi_1 = (E_1, F_1) = \Xi$.

Step 1: since E_1 is already in the desired form, we set $Q_1 = I_5$. It follows that

$$M_1 = \left\{ x \in X \,|\, x_1^2 + x_3^2 - l^2 = 0 \right\}.$$

We have $x_p \in M_1$ and $M_1^c = M_1 \cap U_1$ is a smooth connected submanifold, where $U_1 = \{x \in X \,|\, x_3 < 0\}$. Then choose a new coordinate $\bar{z}_1 = \bar{x}_3 = x_1^2 + x_3^2 - l^2$ and keep the remaining coordinates $z_1 = (x_1, x_2, x_4, x_5)$ unchanged, the DAE Ξ_1 represented in the new coordinates (defined on U_1) is

$$\hat{\Xi}_1 : \begin{bmatrix} 1&0&0&0&0 \\ 0&1&0&0&0 \\ -2x_1&0&0&0&1 \\ 0&0&0&1&0 \\ 0&0&0&0&0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{\bar{x}}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5x_1 \\ 2x_3x_4 \\ -x_5x_3 - g \\ \bar{x}_3 \end{bmatrix},$$

where $x_3 = -(l^2 - \bar{x}_3^2 - x_1^2)^{1/2}$. Set $\bar{x}_3 = 0$ to get

$$\hat{\Xi}_1|_{M_1^c} : \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ -2x_1&0&0&0 \\ 0&0&0&1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5x_1 \\ -2(l^2 - x_1^2)^{1/2}x_4 \\ x_5(l^2 - x_1^2)^{1/2} - g \end{bmatrix}.$$

Step 2: consider $\Xi_2 = (E_2, F_2) = \hat{\Xi}_1|_{M_1^c}$. Find Q_2 such that

$$Q_2E_2(z_1) = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \; Q_2F_2(z_1) = \begin{bmatrix} x_2 \\ -x_5x_1 \\ x_5(l^2 - x_1^2)^{1/2} - g \\ -(l^2 - x_1^2)^{1/2}x_4 + x_1x_2 \end{bmatrix}.$$

It follows that

$$M_2 = \left\{ x \in M_1^c \,|\, -(l^2 - x_1^2)^{1/2}x_4 + x_1x_2 = 0 \right\}.$$

We have $x_p \in M_2$ and $M_2^c = M_2 \cap U_2$ is a smooth connected submanifold, where $U_2 = \{x \in U_1 \,|\, x_1 < l\}$. Then we define new local coordinates on M_1^c via the local diffeomorphism $\psi_2(z_1) = (z_2, \bar{z}_2) = (x_1, x_2, x_5, \bar{x}_4)$, where $\bar{z}_2 = \bar{x}_4 = -(l^2 - x_1^2)^{1/2}x_4 + x_1x_2$. The DAE Ξ_2 in the new coordinates is

$$\hat{\Xi}_2 : \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ -a(z_1)&-x_1&0&1 \\ 0&0&0&0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \\ \dot{\bar{x}}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5x_1 \\ g(l^2 - x_1^2)^{1/2} - x_5(l^2 - x_1^2) \\ \bar{x}_4 \end{bmatrix},$$

where $a(z_1) = x_1^2x_2(l^2 - x_1^2)^{-1} + x_2$. Set $\bar{x}_4 = 0$ to have

$$\hat{\Xi}_2|_{M_2^c} : \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ -a(x)&-x_1&0 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5x_1 \\ -x_5(l^2 - x_1^2) + g(l^2 - x_1^2)^{1/2} \end{bmatrix}.$$

Step 3: Consider $\Xi_3 = (E_3, F_3) = \hat{\Xi}_2|_{M_2^c}$. Then via a similar procedure as in Step 1 and 2, we get

$$M_3 = \left\{ x \in M_2^c \,|\, x_1^2x_2^2(l^2 - x_1^2)^{-1} + x_2^2 - x_5l^2 - gx_3 = 0 \right\}$$

$= \{x \in M_2^c \,|\, x_4^2 + x_2^2 - x_5 l^2 - g x_3 = 0\}$.

It follows that $x_p \in M_3$ and $M_3^c = M_3 \cap U_3$, where $U_3 = U_2$, is a smooth connected submanifold. Set $\bar{x}_5 = x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x^2)^{1/2} - l^2 x_5$ and define the new local coordinates $(z_3, \bar{z}_3) = (x_1, x_2, \bar{x}_5)$ on M_2^c, then we denote Ξ_3 in the new coordinates by $\hat{\Xi}_3$. It follows that

$$\hat{\Xi}_3|_{M_3^c} : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_5 x_1 \end{bmatrix}, \tag{8}$$

where $x_5 = \frac{1}{l^2}(x_1^2 x_2^2 (l - x_1^2)^{-1} + x_2^2 + g(l^2 - x_1^2)^{1/2})$.

Step 4: Since $\Xi_4 = \tilde{\Xi}_3|_{M_3^c}$ is an ODE, it is seen that $k^* = 3$ and $M^* = M_4^c = M_3^c$.

Notice that $x_p \in M^*$ and assumptions (A1) and (A2) are satisfied around x_p, we can conclude by Theorem 4 that the solution of Ξ passing through any point $x_0 \in M^* \cap U^*$, where $U^* = U_3$, exists and is unique (since $\dim M^* = \dim E(x) T_x M^* = 2$ for all $x \in M^* \cap U^*$), and this unique solution is mapped via a diffeomorphism defined on U^* into the solution of $\Xi^* = \tilde{\Xi}_3|_{M_3^c}$, given by (8), with the constraints $\bar{x}_3 = \bar{x}_4 = \bar{x}_5 = 0$. Moreover, the geometric index $\nu_g = k^* = 3$. Note that our sequence of submanifolds M_k coincides with that of the projections W_k of the tangent bundles TW_k in (Rabier and Rheinboldt, 1994). But the two methods of constructing sequences of submanifolds are different, although it is interesting to compare them, it is not our aim to discuss their differences in details in the present paper.

3. GEOMETRIC INTERPRETATION OF THE DIFFERENTIATION INDEX

A classical definition of differentiation index (Gear, 1988) is given for DAEs of the general form (2): define the differential array of (2) by

$$H_k(t,x,x',w) = \begin{bmatrix} H \\ \mathrm{D}_t H + \mathrm{D}_x H x' + \mathrm{D}_{x'} H x'' \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k} H \end{bmatrix} (t,x,x',w) = 0, \tag{9}$$

where $w = [x^{(2)}, \ldots, x^{(k+1)}]$, the differentiation index ν_d is the least integer k such that equation (9) uniquely determines x' as a function of (x,t), i.e., $x' = v(x,t)$. Note that some other definition of differentiation index in, e.g., (Campbell and Gear, 1995), requires, a priori, the solvability of DAEs. We do not state such a requirement since an index is a measure of difficulties for solving a DAE, whose definition should be independent of the solvability of the DAE (Rabier and Rheinboldt, 2002). We now illustrate two deficiencies or ambiguities of the above classical definition of differentiation index.

Remark 9. (i) In contrast to the geometric index of Definition 6, the above classical definition of differentiation index does not allow for a conclusion about the existence of solutions. Take the following DAEs for example,

$$\Xi_{2,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \end{cases} \quad \text{and} \quad \tilde{\Xi}_{3,2} : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = f(x_1, x_2) \\ 0 = x_1 \end{cases},$$

where $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and we assume $f(0,0) \neq 0$. The two DAEs Ξ and $\tilde{\Xi}$ have the same differentiation index $\nu_d = 0$ by the definition above. However, Ξ has a unique solution and $\tilde{\Xi}$ has no solutions for the admissible initial point $(x_{10}, x_{20}) = (0,0)$. The reason is that the above definition of differentiation index only implies the existence and uniqueness of some vector field determined by the differential array but does not indicate where the solutions of the DAE should exist (solutions of DAE exists on the maximal invariant submanifold M^* which we discussed in Section 2). In particular, if the defined vector field $v(x,t)$ is not tangent to M^* at $x_0 \in M^*$ (as in our example, the vector field $(x_2, f(x_1,x_2))$ determined by $\tilde{\Xi}$ is not tangent to $M^* = \{x \in \mathbb{R}^2 \,|\, x_1 = x_2 = 0\}$ at $(x_{10}, x_{20}) = (0,0)$), then the trajectory of $\dot{x} = v(x,t)$ will leave M^*, indicating that the DAE does not have a solution at x_0.

(ii) The classical definition of differentiation index does not indicate the difference between ODEs and "over-determined" DAEs. Consider the DAEs $\Xi_{2,2}$ and $\tilde{\Xi}_{3,2}$ of item (i) above, we suppose now $f(0,0) = 0$. Then both Ξ and $\tilde{\Xi}$ have the same differentiation index $\nu_d = 0$. The DAE Ξ is clearly an ODE which does not need any differentiation to be solved, while $\tilde{\Xi}$ needs two times of differentiations (of the algebraic constraint $x_1 = 0$) in order to deduce a solution. Note that the "over-determined" (and also "under-determined") properties of nonlinear DAEs can be characterized by the notion of strangeness index, see e.g., (Kunkel and Mehrmann, 2006).

To give a geometric background to the notion of differentiation index, a new definition was proposed in (Griepentrog, 1991). In the present paper, in order to clear out the deficiencies mentioned in Remark 9, we reform Griepentrog's definition of differentiation index as follows. Consider a nonlinear DAE $\Xi_{l,n} = (E,F)$ and fix a point $x_p \in X$, let $U_0 \subseteq X$ be a neighborhood of x_p, set $H(x,\zeta_1) = E(x)\zeta_1 - F(x)$, denote $(\frac{\mathrm{d}^k}{\mathrm{d}t^k} H) = H^{(k)}$ and for $k \geq 0$, define

$$H_k(x, \bar{\zeta}_{k+1}) = \begin{bmatrix} H^{(0)}(x,\zeta_1) \\ H^{(1)}(x,\zeta_1,\zeta_2) \\ \vdots \\ H^{(k)}(x,\bar{\zeta}_{k+1}) \end{bmatrix} = 0, \tag{10}$$

where $\bar{\zeta}_{k+1} = (\zeta_1, \ldots, \zeta_{k+1})$. Set $\mathcal{M}_0 = X$, $M_0^c = U_0$, $\mathcal{Z}_1^0 = \mathbb{R}^n$ and for $k > 0$, define

$$\mathcal{M}_k := \{x \in X \,|\, H_{k-1}(x, \bar{\zeta}_k) = 0\}, \tag{11}$$

As long as $x_p \in \mathcal{M}_k$, assume that there exists a neighborhood $U_k \subseteq U_{k-1}$ such that $\mathcal{M}_k^c = \mathcal{M}_k \cap U_k$ is a smooth connected submanifold. Set

$$\mathcal{Z}_1^k := \{\zeta_1 \in \mathbb{R}^n \,|\, H_{k-1}(x, \bar{\zeta}_k) = 0, x \in \mathcal{M}_k^c\}.$$

Definition 10. (Differentiation index). The (local) differentiation index ν_d of a DAE Ξ around a point x_p is defined by $\nu_d :=$

$$\min \left\{ k \geq 0 \;\middle|\; \begin{matrix} (\mathcal{M}_k^c \neq \emptyset) \wedge (\mathcal{Z}_1^k = \mathcal{Z}_1^k(x) \text{ is a singleton}) \wedge \\ (\mathcal{Z}_1^k(x) \in T_x \mathcal{M}_k^c, \forall x \in \mathcal{M}_k^c) \end{matrix} \right\}.$$

Now we state our main theorem of this subsection.

Theorem 11. For a DAE $\Xi_{l,n} = (E,F)$, fix a point x_p, assume that for each $k \geq 0$,

(A1)' $\dim E(x) T_x \mathcal{M}_k^c = const.$ for all $x \in \mathcal{M}_k^c$ around x_p.
(A2)' $\mathcal{M}_k^c$ is a smooth connected submanifold and $x_p \in \mathcal{M}_k^c$;

Then we have that for each $k \geq 0$, the set M_k^c of the geometric reduction method of Definition 2 is a smooth connected submanifold and

$$\mathcal{M}_k^c = M_k^c,$$

and there exists a smallest integer k^* such that $\mathcal{M}_{k^*+1}^c \cap U_{k^*+1} = \mathcal{M}_{k^*}^c \cap U_{k^*+1}$. It follows that $x_p \in \mathcal{M}^*$ is admissible and $\mathcal{M}^* = \mathcal{M}_{k^*+1}$ is a locally maximal invariant submanifold, and

(i) the geometric index $\nu_g = k^*$;
(ii) the differentiation index ν_d exists and $\nu_d = \nu_g$ if and only if $\dim \mathcal{M}^* = \dim E(x) T_x \mathcal{M}^*$ for all $x \in \mathcal{M}^*$ around x_p.

Proof. First, we construct $\mathcal{M}_k$ explicitly via the following procedure: Fix $x_p \in X$ and let U_0 be an open connected subset of X containing x_p. Set $\mathcal{M}_0^c = U_0$, $E_0 = E$, $F_0 = F$. Assume in Step $k-1$, we have constructed a smooth submanifold $\mathcal{M}_{k-1}^c$, and maps $E_{k-1} : \mathcal{M}_{k-1}^c \to \mathbb{R}^{l\times n}$ and $F_{k-1} : \mathcal{M}_{k-1}^c \to \mathbb{R}^{l\times n}$. Step k ($k > 0$): Rename $\hat{E}_k = E_{k-1}$, $\hat{F}_k = F_{k-1}$. Assume that $\operatorname{rank} \hat{E}_k(x) = const. = r_k$ for all $x \in \mathcal{M}_{k-1}^c$ around x_p. After a suitable permutation, it is possible to assume that the first r_k rows of $\hat{E}_k$ are linearly independent. Then we can rewrite $\hat{E}_k v + \hat{F}_k = 0$ as

$$\begin{bmatrix} \hat{E}_k^1 \\ \hat{E}_k^2 \end{bmatrix} v + \begin{bmatrix} \hat{F}_k^1 \\ \hat{F}_k^2 \end{bmatrix} = 0,$$

where $\hat{E}_k^1 : \mathcal{M}_{k-1}^c \to \mathbb{R}^{r_k\times n}$ is of full row rank and $\hat{F}_k^1 : \mathcal{M}_{k-1}^c \to \mathbb{R}^{r_k}$. Find $a_k : \mathcal{M}_{k-1} \to \mathbb{R}^{(l-r_k)\times r_k}$ such that $a_k \hat{E}_k^1 = \hat{E}_k^2$. Denote

$$v_k = \hat{E}_k^1 v + \hat{F}_k^1,$$

we have $\hat{E}_k^2 v + \hat{F}_k^2 = a_k(\hat{E}_k^1 v + \hat{F}_k^1) - a_k \hat{F}_k^1 + \hat{F}_k^2 = a_k v_k + \tilde{F}_k^2$, where $\tilde{F}_k^2 = \hat{F}_k^2 - a_k \hat{F}_k^1$. Define

$$\begin{aligned} \mathcal{M}_k &= \left\{ x \in \mathcal{M}_{k-1}^c \mid \hat{E}_k(x) v + \hat{F}_k(x) = 0 \right\} \\ &= \left\{ x \in \mathcal{M}_{k-1}^c \mid \tilde{F}_k^2(x) = 0 \right\}. \end{aligned} \tag{12}$$

Assume that there exists a neighborhood $U_k \subseteq U_{k-1}$ of x_p such that $\mathcal{M}_k^c = \mathcal{M}_k \cap U_k$ is a smooth connected submanifold. Set

$$E_k = \begin{bmatrix} \hat{E}_k^1 \\ D_x \tilde{F}_k^2 \end{bmatrix} \Big|_{\mathcal{M}_k^c} \quad \text{and} \quad F_k = \begin{bmatrix} F_{k-1}^1 \\ 0 \end{bmatrix} \Big|_{\mathcal{M}_k^c},$$

where $E_k : \mathcal{M}_k^c \to \mathbb{R}^{l\times n}$ and $F_k : \mathcal{M}_k^c \to \mathbb{R}^l$.

We now show that under assumptions (A1)' and (A2)', for each $k > 0$, the set $\mathcal{M}_k$, constructed by (12), is indeed the one defined by (11). Observe that for $k \geq 0$,

$$\ker E_k(x) = \ker E(x) \cap T_x \mathcal{M}_k^c, \ \forall x \in \mathcal{M}_k^c.$$

Thus assumption (A1)' implies that $\operatorname{rank} E_k(x) = const.$ for all $x \in \mathcal{M}_k^c$. Then consider the differentiation array H_k, given by (10), denote $\zeta_1 = v$ and use the same notations v_k as in the above procedure to have

$$\begin{aligned} H^{(0)}(x,v) = E(x)v + F(x) &= \begin{bmatrix} \hat{E}_1^1(x) \\ \hat{E}_1^2(x) \end{bmatrix} v + \begin{bmatrix} \hat{F}_1^1(x) \\ \hat{F}_1^2(x) \end{bmatrix} \\ &= \begin{bmatrix} v_1 \\ \tilde{a}_1(x, v_1) + \tilde{F}_1^2(x) \end{bmatrix}, \end{aligned}$$

where $\tilde{a}_1(x, v_1) = a_1(x) v_1$. Now by assumption (A2)' that $\mathcal{M}_k^c = \mathcal{M}_k \cap U_k$ is a smooth connected submanifold, we have (note that for any smooth map L defined on X, its restriction $L|_M$ to a smooth submanifold $M \subseteq X$ and the differentiation of L with respect to t are two commutative operations, i.e., $\frac{\mathrm{d}}{\mathrm{d}t}(L|_M) = (\frac{\mathrm{d}}{\mathrm{d}t} L)|_M$)

$$H^{(1)}|_{\mathcal{M}_1^c} = (H|_{\mathcal{M}_1^c})^{(1)} = \begin{bmatrix} v_1^{(1)} \\ \tilde{a}_1^{(1)} + \begin{bmatrix} \bar{v}_1 \\ \tilde{a}_2 + \tilde{F}_2^2 \end{bmatrix} \end{bmatrix},$$

where $\bar{a}_2(x, v_2) = a_2(x) v_2$ and $v_2 = (v_1, \bar{v}_1)$; in general,

$$H^k|_{\mathcal{M}_k^c} = \begin{bmatrix} v_1^{(k)} \\ \tilde{a}_1^{(k)} + (\tilde{F}_1^2)^{(k)} \end{bmatrix},$$

where $(\tilde{F}_1^2)^{(k)}$ is given by the following iterative formula:

$$(\tilde{F}_i^2)^{(k)} = \begin{bmatrix} \bar{v}_i^{(k-1)} \\ \tilde{a}_{i+1}^{(k-1)} + (\tilde{F}_{i+1}^2)^{(k-1)} \end{bmatrix},$$

where $\bar{a}_i = \bar{a}_i(x, v_i) = a_i(x) v_i$ and $v_i = (v_{i-1}, \bar{v}_{i-1})$ for $i > 1$. Notice that for each k, since $\hat{E}_k^1$ is of full row rank, the differentiation $v_1^{(k)}$ of v_1 and $\bar{v}_i^{(k-1)}$ of $\bar{v}_i$ for $i \geq 1$ do not create any constraint for x. So it is clear to see that for $k > 0$,

$$\begin{aligned} \mathcal{M}_k^c &= \left\{ x \in U_k \mid H_{k-1}(x, \bar{\zeta}_k) = 0 \right\} \\ &= \left\{ x \in \mathcal{M}_{k-1}^c \cap U_k \mid H^{(k-1)}(x, \bar{\zeta}_k) = 0, \right\} \\ &= \left\{ x \in \mathcal{M}_{k-1}^c \cap U_k \mid \tilde{F}_k^2(x) = 0 \right\}, \end{aligned}$$

$$\begin{aligned} \mathcal{Z}_1^k &= \left\{ \zeta_1 \mid H_{k-1}(x, \bar{\zeta}_k) = 0 \right\} \\ &= \left\{ \zeta_1 \mid E_{k-1}(x)\zeta_1 + F_{k-1}(x) = 0, x \in \mathcal{M}_k^c \right\} \\ &= \left\{ \zeta_1 \mid v_k(\zeta_1, x) = 0, x \in \mathcal{M}_k^c \right\}. \end{aligned} \tag{13}$$

Next we show that for each $k > 0$, $\mathcal{M}_k^c$ coincides with the submanifold M_k^c of the geometric reduction method of Definition 2. It is clear that $M_1^c = \left\{ x \in U_0 | \hat{F}_1(x) \in \operatorname{Im} \hat{E}_1(x) \right\}$ $= \left\{ x \in X \mid \tilde{F}_1^2(x) = 0 \right\} = \mathcal{M}_1^c$. For $k > 1$, suppose that $M_{k-1}^c = \mathcal{M}_{k-1}^c$, then we have

$$\begin{aligned} \mathcal{M}_k &= \left\{ x \in \mathcal{M}_{k-1}^c \mid \hat{E}_k(x) v + \hat{F}_k(x) = 0 \right\} \\ &= \left\{ x \in M_{k-1}^c \mid \begin{bmatrix} \hat{E}_k^1(x) \\ D_x \tilde{F}_k^2(x) \end{bmatrix} v + \begin{bmatrix} \hat{F}_k^1(x) \\ 0 \end{bmatrix} = 0 \right\} \\ &= \left\{ x \in M_{k-1}^c \mid E_{k-1}^1(x) v + F_{k-1}^1(x) = 0, v \in T_x M_{k-1}^c \right\} \\ &= \left\{ x \in M_{k-1}^c \mid F_{k-1}^1(x) \in E_{k-1}^1(x) T_x M_{k-1}^c \right\} \\ &= \left\{ x \in M_{k-1}^c \mid F(x) \in E(x) T_x M_{k-1}^c \right\} = M_k. \end{aligned}$$

So $\mathcal{M}_k^c = \mathcal{M}_k \cap U_k$ coincides with $M_k^c = M_k \cap U_k$. Then k^* is the smallest integer such that $\mathcal{M}_{k^*+1}^c \cap U_{k^*+1} = \mathcal{M}_{k^*}^c \cap U_{k^*+1}$ and the geometric index $\nu_g = k^*$ by Definition 6. It can be deduced from $\mathcal{M}^* = \mathcal{M}_{k^*+1}^c = \mathcal{M}_{k^*}^c$ on U_{k^*+1} that $\mathcal{Z}_1^{k^*}(x) = \mathcal{Z}_1^{k^*+1}(x) \in T_x \mathcal{M}^*$. Thus by (13),

$$\begin{aligned} \mathcal{Z}_1^{k^*}(x) &= \mathcal{Z}_1^{k^*+1}(x) \\ &= \left\{ \zeta_1 \in \mathbb{R}^n \mid E_{k^*}(x)\zeta_1 + F_{k^*}(x) = 0, x \in \mathcal{M}^* \right\} \end{aligned}$$

It follows that $\mathcal{Z}_1^{k^*} = \mathcal{Z}_1^{k^*}(x)$ is a singleton if and only if E_{k^*} is invertible, i.e., $\operatorname{rank} E_{k^*} = n$, or, equivalently, $\dim \ker E_{k^*}(x) = 0$. We can now conclude by Definition 10 that the differentiation index $\nu_d = k^* = \nu_g$ if and only if $\dim\ker E_{k^*}(x) = \dim(\ker E(x) \cap T_x\mathcal{M}^*) = 0$, i.e., $\dim \mathcal{M}^* = \dim E(x)T_x\mathcal{M}^*$ for all $x \in \mathcal{M}^*$ around x_p. □

Remark 12. (i) Assumptions (A1)' and (A2)' of Theorem 11 correspond to assumptions (A1) and (A2) of Theorem 4, respectively. If, a priory, we assume that for each $k \geq 0$, $M_k^c = \mathcal{M}_k^c$, then it is clear that (A1) coincides with (A1)', and (A2) is equivalent to (A2)' since $\mathcal{M}_k^c = M_k^c$ is a smooth (embedded) submanifold if and only if $\operatorname{rank} \mathrm{D}\tilde{F}_k^2(z_{k-1}) = const.$ for $z_{k-1} \in M_k \cap U_k$ (see e.g., (Lee, 2001)).

(ii) The differentiation index ν_d associated with a geometric background, given by Definition 10, allows to have a conclusion on the existence of solutions as the sequence of submanifolds $\mathcal{M}_k$ coincides with M_k and $\mathcal{M}^* = M^*$ is a locally maximal invariant submanifold, which coincides locally with the admissible set S_a.

(iii) The differentiation index and the geometric index differ from each other by their relations with uniqueness of solutions. Namely, the geometric index may exist even a DAE does not have a unique solution, while the differentiation index exists only if the DAE is uniquely solvable. As seen from Theorem 11, the geometric index ν_g always exists if the submanifolds $\mathcal{M}_k$ are not empty and the sequence of submanifolds converges. While in order to have a well-defined differentiation index, the condition $\dim \mathcal{M}^* = \dim E(x)T_x\mathcal{M}^*$ for uniqueness of solutions is necessary.

Example 13. (continuation of Example 8) Consider the DAE $\Xi = (E, F)$, given by (7), since the derivatives $(\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4)$ are already determined by the first four equations of Ξ, we only differentiate the last equation $x_1^2 + x_3^2 - l^2 = 0$ to determine $\dot{x}_5$. It is not hard to verify that assumptions (A1)' and (A2)' are satisfied around x_0, and $\mathcal{M}_1^c = \left\{x \in U_0 \,|\, x_1^2 + x_3^2 - l^2 = 0\right\} = M_1^c$,

$$\begin{aligned}\mathcal{M}_2^c &= \{x \in \mathcal{M}_1^c \cap U_2 \,|\, 2x_1\dot{x}_1 + 2x_3\dot{x}_3 = 0\} \\ &= \{x \in \mathcal{M}_1^c \cap U_2 \,|\, x_1x_2 + x_3x_4 = 0\} = M_2^c,\end{aligned}$$

$$\begin{aligned}\mathcal{M}_3^c &= \{x \in \mathcal{M}_2^c \,|\, x_1\dot{x}_2 + x_2\dot{x}_3 + x_3\dot{x}_4 + x_4\dot{x}_3 = 0\} \\ &= \left\{x \in \mathcal{M}_2^c \,|\, -x_5x_1^2 + x_2^2 - x_5x_3^2 - x_3g + x_4^2 = 0\right\} \\ &= \left\{x \in \mathcal{M}_2^c \,|\, x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0\right\} = M_3^c.\end{aligned}$$

Moreover, by differentiating $x_4^2 + x_2^2 - x_5l^2 - gx_3 = 0$, we get that

$$\dot{x}_5 = \frac{1}{l^2}(-2x_4(x_5x_3 + g)2x_2(x_5x_1) - gx_4).$$

Combining the above equation with the first 4-equations of (7), we have

$$\mathcal{Z}_1^3 = \{\zeta_1 \,|\, \zeta_1 = (x_2, -x_5x_1, x_4, -x_5x_3 - g, \dot{x}_5),\ x \in \mathcal{M}_3^c\}.$$

We can conclude that $\mathcal{Z}_1^3 = \mathcal{Z}_1^3(x)$ is a singleton and $\mathcal{Z}_1^3(x) \in T_x\mathcal{M}_3^c, \forall x \in M_3^c$. Therefore the differentiation index $\nu_d = 3$, which coincides with the geometric index ν_g.

4. CONCLUSIONS

In this paper, we revised the geometric reduction method and give our conditions for the existence and uniqueness of the solutions of nonlinear DAEs. Then we discuss the notion of geometric index via its relations with solutions of DAEs. We modify the classical definition of the differentiation index of DAEs and show that under some constant rankness and smoothness assumptions, the sequence of submanifolds defined by the differentiation array coincides with those defined by the geometric reduction method and the differentiation index coincides with the geometric index if and only if a condition for uniqueness of solutions is satisfied.

REFERENCES

Berger, T. (2016). Controlled invariance for nonlinear differential-algebraic systems. *Automatica*, 64, 226–233.

Berger, T. (2017). The zero dynamics form for nonlinear differential-algebraic systems. *IEEE Transactions on Automatic Control*, 62(8), 4131–4137.

Berger, T. and Trenn, S. (2012). The quasi-Kronecker form for matrix pencils. *SIAM Journal on Matrix Analysis and Applications*, 33(2), 336–368.

Brenan, K.E., Campbell, S.L., and Petzold, L.R. (1996). *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*, volume 14. SIAM.

Campbell, S.L. (1995). High-index differential algebraic equations. *Journal of Structural Mechanics*, 23(2), 199–222.

Campbell, S.L. and Gear, C.W. (1995). The index of general nonlinear DAEs. *Numerische Mathematik*, 72(2), 173–196.

Campbell, S.L. and Griepentrog, E. (1995). Solvability of general differential algebraic equations. *SIAM Journal on Scientific Computing*, 16(2), 257–270.

Chen, Y. and Respondek, W. (2021a). Geometric analysis of differential-algebraic equations via linear control theory. *SIAM Journal on Control and Optimization*, 59(1), 103–130.

Chen, Y. and Respondek, W. (2021b). Geometric analysis of nonlinear differential-algebraic equations via nonlinear control theory. Submitted to be published, preprint available from https://arxiv.org/abs/2103.16711.

Gear, C.W. (1988). Differential-algebraic equation index transformations. *SIAM Journal on Scientific and Statistical Computing*, 9(1), 39–47.

Griepentrog, E. (1991). *Index Reduction Methods for Differential-Algebraic Equations.* Humboldt-Univ., Fachbereich Mathematik, Informationsstelle.

Griepentrog, E., Hanke, M., and März, R. (1992). *Toward a better understanding of differential algebraic equations (Introductory survey).* Humboldt-Universität zu Berlin, Mathematisch-Naturwissenschaftliche Fakultät ....

Kunkel, P. and Mehrmann, V. (2006). *Differential-Algebraic Equations: Analysis and Numerical Solution*, volume 2. European Mathematical Society.

Lee, J.M. (2001). *Introduction to Smooth Manifolds.* Springer.

Mehrmann, V. (2015). Index concepts for differential-algebraic equations. *Encyclopedia of Applied and Computational Mathematics*, 676–681.

Rabier, P.J. and Rheinboldt, W.C. (1994). A geometric treatment of implicit differential-algebraic equations. *Journal of Differential Equations*, 109(1), 110–146.

Rabier, P.J. and Rheinboldt, W.C. (2002). *Theoretical and Numerical Analysis of Differential-Algebraic Equations.* Elsevier.

Reich, S. (1990). On a geometrical interpretation of differential-algebraic equations. *Circuits, Systems, and Signal Processing*, 9(4), 367–382.

Reich, S. (1991). On an existence and uniqueness theory for nonlinear differential-algebraic equations. *Circuits, Systems and Signal Processing*, 10(3), 343–359.

Rheinboldt, W.C. (1984). Differential-algebraic systems as differential equations on manifolds. *Mathematics of Computation*, 43(168), 473–482.

Riaza, R. (2008). *Differential-Algebraic Systems: Analytical Aspects and Circuit Applications.* World Scientific.

Wong, K.T. (1974). The eigenvalue problem $\lambda Tx + Sx$. *Journal of Differential Equations*, 16(2), 270–280.