Switched differential algebraic equations: Jumps and impulses

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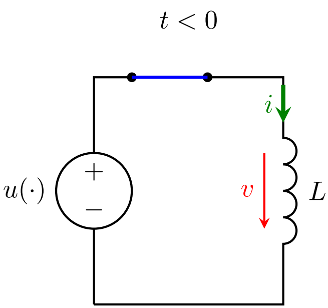
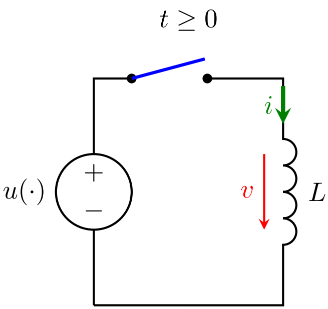
Motivating example



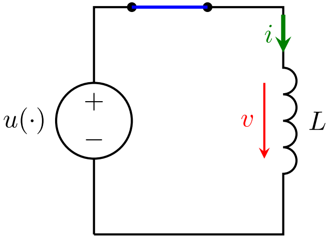


inductivity law: $L\frac{\mathrm{d}}{\mathrm{d}t}i = v$

switch dependent: $0 = v - u$ $\quad$ $0 = i$

Motivating example

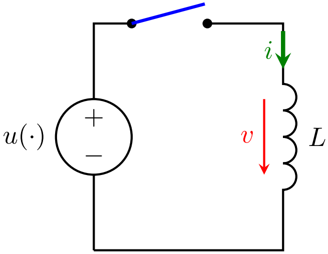
$t < 0$



$x = [i, v]^\top$

$$\begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} u$$

$t \geq 0$



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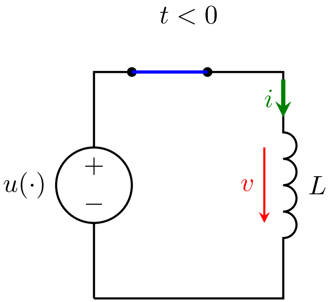
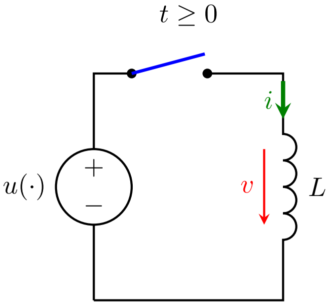
Motivating example





$E_1\dot{x} = A_1x + B_1u$
on $(-\infty, 0)$

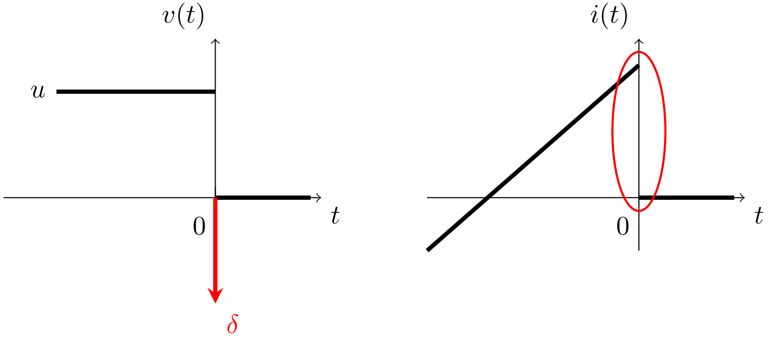
$E_2\dot{x} = A_2x + B_2u$
on $[0, \infty)$

$\rightarrow$ switched differential-algebraic equation

Solution of circuit example

$t < 0$	$t \geq 0$
$v = u$	$i = 0$
$L\frac{\mathsf{d}}{\mathsf{d}t} i = v$	$v = L\frac{\mathsf{d}}{\mathsf{d}t} i$

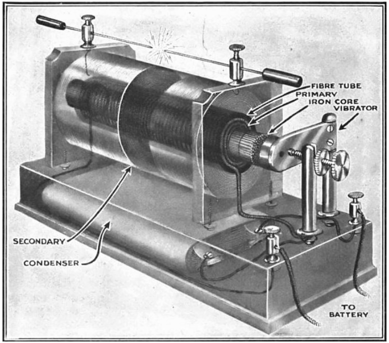
Solution (assume constant input u):



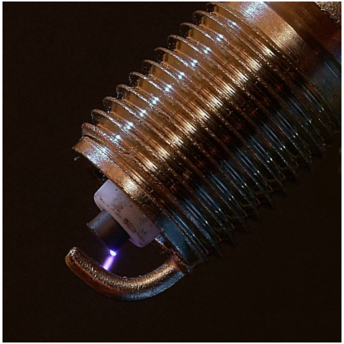
Dirac impulse is “real”

Dirac impulse

Not just a mathematical artifact!



Drawing: Harry Winfield Secor, public domain



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Contents

Definition

Switch $\to$ Different DAE models (=modes)
depending on time-varying position of switch

Definition (Switched DAE)

Switching signal $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ picks mode at each time $t \in \mathbb{R}$:

$$\begin{aligned} E_{\sigma(t)}\dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) \\ y(t) &= C_{\sigma(t)}x(t) + D_{\sigma(t)}u(t) \end{aligned} \tag{swDAE}$$

Attention

Each mode might have different consistency spaces
$\Rightarrow$ inconsistent initial values at each switch
$\Rightarrow$ Dirac impulses, in particular distributional solutions

Definition

Switch $\rightarrow$ Different DAE models (=modes)
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Switching signal $\sigma : \mathbb{R} \to \{1, \ldots, N\}$ picks mode at each time $t \in \mathbb{R}$:

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Distribution theory - basic ideas

Distributions - overview

- Generalized functions
- Arbitrarily often differentiable
- Dirac-Impulse δ is “derivative” of Heaviside step function $\mathbb{1}_{[0,\infty)}$

Two different formal approaches

1) Functional analytical: Dual space of the space of test functions (L. Schwartz 1950)
2) Axiomatic: Space of all “derivatives” of continuous functions (J. Sebastião e Silva 1954)

Distributions - formal

Definition (Test functions)

$\mathcal{C}_0^\infty := \{\varphi : \mathbb{R} \to \mathbb{R} \mid \varphi \text{ is smooth with compact support}\}$

Definition (Distributions)

$\mathbb{D} := \{D : \mathcal{C}_0^\infty \to \mathbb{R} \mid D \text{ is linear and continuous}\}$

Definition (Regular distributions)

$f \in \mathcal{L}_{1,\text{loc}}(\mathbb{R} \to \mathbb{R})$: $\quad f_{\mathbb{D}} : \mathcal{C}_0^\infty \to \mathbb{R},\ \varphi \mapsto \int_{\mathbb{R}} f(t)\varphi(t)\mathrm{d}t \in \mathbb{D}$

Definition (Derivative)

$D'(\varphi) := -D(\varphi')$

Dirac Impulse at $t_0 \in \mathbb{R}$

$\delta_{t_0} : \mathcal{C}_0^\infty \to \mathbb{R}, \quad \varphi \mapsto \varphi(t_0)$

$$(\mathbb{1}_{[0,\infty)_{\mathbb{D}}})'(\varphi) = -\int_{\mathbb{R}} \mathbb{1}_{[0,\infty)}\varphi' = -\int_0^\infty \varphi' = -(\varphi(\infty) - \varphi(0)) = \varphi(0)$$

Multiplication with functions

Definition (Multiplication with smooth functions)

$\alpha \in \mathcal{C}^\infty : \quad (\alpha D)(\varphi) := D(\alpha\varphi)$

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \tag{swDAE}$$

Coefficients not smooth

Problem: $E_\sigma, A_\sigma, C_\sigma \notin \mathcal{C}^\infty$

Observation, for $\sigma_{[t_i,t_{i+1})} \equiv p_i$, $i \in \mathbb{Z}$:

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \quad \Leftrightarrow \quad \forall i \in \mathbb{Z} : \begin{aligned} (E_{p_i}\dot{x})_{[t_i,t_{i+1})} &= (A_{p_i}x + B_{p_i}u)_{[t_i,t_{i+1})} \\ y_{[t_i,t_{i+1})} &= (C_{p_i}x + D_{p_i}u)_{[t_i,t_{i+1})} \end{aligned}$$

New question: Restriction of distributions

Desired properties of distributional restriction

Distributional restriction:

$$\{ M \subseteq \mathbb{R} \mid M \text{ interval } \} \times \mathbb{D} \to \mathbb{D}, \quad (M, D) \mapsto D_M$$

and for each interval $M \subseteq \mathbb{R}$

- $D \mapsto D_M$ is a projection (linear and idempotent)
- $\forall f \in \mathcal{L}_{1,\text{loc}}: \quad (f_{\mathbb{D}})_M = (f_M)_{\mathbb{D}}$
- $\forall \varphi \in \mathcal{C}_0^\infty: \quad \begin{bmatrix} \operatorname{supp}\varphi \subseteq M & \Rightarrow & D_M(\varphi) = D(\varphi) \\ \operatorname{supp}\varphi \cap M = \emptyset & \Rightarrow & D_M(\varphi) = 0 \end{bmatrix}$
- $(M_i)_{i\in\mathbb{N}}$ pairwise disjoint, $M = \bigcup_{i\in\mathbb{N}} M_i$:

$$D_M = \sum_{i\in\mathbb{N}} D_{M_i}, \quad D_{M_1 \dot\cup M_2} = D_{M_1} + D_{M_2}, \quad (D_{M_1})_{M_2} = 0$$

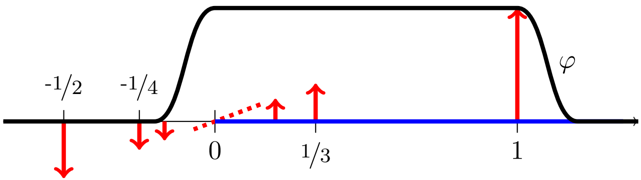
Theorem ([T. 2009, T. 2021])

Such a distributional restriction does not exist.

Proof of non-existence of restriction

Consider the following (well defined!) distribution:

$$D := \sum_{i\in\mathbb{N}} d_i\, \delta_{d_i}, \quad d_i := \frac{(-1)^i}{i+1}$$



Restriction should give

$$D_{[0,\infty)} = \sum_{k\in\mathbb{N}} d_{2k}\, \delta_{d_{2k}}$$

Choose $\varphi \in \mathcal{C}_0^\infty$ such that $\varphi_{[0,1]} \equiv 1$:

$$D_{[0,\infty)}(\varphi) = \sum_{k\in\mathbb{N}} d_{2k} = \sum_{k\in\mathbb{N}} \frac{1}{2k+1} = \infty$$

Dilemma

Switched DAEs

- Examples: distributional solutions
- Multiplication with non-smooth coefficients
- Or: Restriction on intervals

Distributions

- Distributional restriction not possible
- Multiplication with non-smooth coefficients not possible
- *Initial value problems cannot be formulated*

Underlying problem

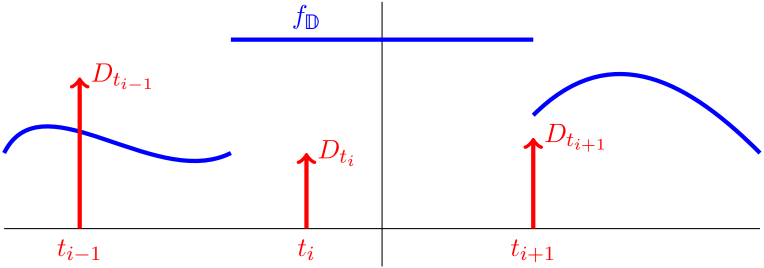
Space of distributions too big.

Piecewise smooth distributions

Define a suitable smaller space:

Definition (Piecewise smooth distributions $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$, [T. 2009])

$$\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty} := \left\{ f_{\mathbb{D}} + \sum_{t\in T} D_t \;\middle|\; \begin{array}{l} f \in \mathcal{C}^\infty_{\mathrm{pw}}, \\ T \subseteq \mathbb{R} \text{ locally finite}, \\ \forall t \in T : D_t = \sum_{i=0}^{n_t} a_i^t \delta_t^{(i)} \end{array} \right\}$$



Properties of $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

- $\mathcal{C}^\infty_{\text{pw}}$ "$\subseteq$" $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$ and $D \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty} \Rightarrow D' \in \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$
- Well definded restriction $\mathbb{D}_{\text{pw}\mathcal{C}^\infty} \to \mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

$$D = f_{\mathbb{D}} + \sum_{t\in T} D_t \quad \mapsto \quad D_M := (f_M)_{\mathbb{D}} + \sum_{t\in T\cap M} D_t$$

- Multiplication with $\alpha = \sum_{i\in\mathbb{Z}} \alpha_{i[t_i,t_{i+1})} \in \mathcal{C}^\infty_{\text{pw}}$ well defined:

$$\alpha D := \sum_{i\in\mathbb{Z}} \alpha_i D_{[t_i,t_{i+1})}$$

- Evaluation at $t \in \mathbb{R}$: $D(t^-) := f(t^-)$, $D(t^+) := f(t^+)$
- Impulses at $t \in \mathbb{R}$: $D[t] := \begin{cases} D_t, & t \in T \\ 0, & t \notin T \end{cases}$

Application to (swDAE)

(x,u) solves (swDAE) $:\Leftrightarrow$ (swDAE) holds in $\mathbb{D}_{\text{pw}\mathcal{C}^\infty}$

Relevant questions

$$\begin{aligned} E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\ y &= C_\sigma x + D_\sigma u \end{aligned} \qquad \text{(swDAE)}$$

Piecewise-smooth distributional solution framework

$x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\mathrm{pw}\mathcal{C}^\infty}$

- Existence and uniqueness of solutions?
- Jumps and impulses in solutions?
- Conditions for impulse free solutions?
- Control theoretical questions
  - Stability and stabilization
  - Observability and observer design
  - Controllability and controller design

Existence and uniqueness of solutions for (swDAE)

$$E_\sigma \dot{x} = A_\sigma x + B_\sigma u \tag{swDAE}$$

Basic assumptions

- $\sigma \in \Sigma_0 := \left\{ \sigma : \mathbb{R} \to \{1, \ldots, N\} \;\middle|\; \begin{array}{l} \sigma \text{ is piecewise constant and} \\ \sigma|_{(-\infty,0)} \text{ is constant} \end{array} \right\}$.
- (E_p, A_p) is regular $\forall p \in \{1, \ldots, N\}$, i.e. $\det(sE_p - A_p) \not\equiv 0$

Theorem (T. 2009)

Consider (swDAE) *satisfying the basic assumptions. Then*

$$\forall\, u \in \mathbb{D}^m_{\mathrm{pw}\mathcal{C}^\infty}\ \forall\, \sigma \in \Sigma_0\ \exists \text{ solution } x \in \mathbb{D}^n_{\mathrm{pw}\mathcal{C}^\infty}$$

and $x(0^-)$ *uniquely determines* x.

Inconsistent initial values

$$E\dot{x} = Ax + Bu, \quad x(0) = x^0 \in \mathbb{R}^n$$

Inconsistent initial value = special switched DAE

$$\begin{aligned} \dot{x}_{(-\infty,0)} &= 0, & x(0^-) &= x^0 \\ (E\dot{x})_{[0,\infty)} &= (Ax+Bu)_{[0,\infty)} \end{aligned}$$

Corollary (Consistency projector)

Exist unique consistency projector $\Pi_{(E,A)}$ such that (for $u = 0$)

$$x(0^+) = \Pi_{(E,A)} x^0$$

$\Pi_{(E,A)}$ can easily be calculated via the Wong sequences [T. 2009].

Sufficient conditions for impulse-freeness

Question

When are all solutions of homogenous (swDAE) $E_\sigma \dot{x} = A_\sigma x$ impulse free?

Note: Jumps are OK.

Lemma (Sufficient conditions)

- *(E_p, A_p) all have index one (i.e. $(sE_p - A_p)^{-1}$ is proper)*
  $\Rightarrow$ *(swDAE) impulse free*
- *all consistency spaces of (E_p, A_p) coincide*
  $\Rightarrow$ *(swDAE) impulse free*

Characterization of impulse-freeness

Theorem (Impulse-freeness, [T. 2009])

The switched DAE $E_\sigma \dot{x} = A_\sigma x$ *is impulse free* $\forall \sigma \in \Sigma_0$

$$\Leftrightarrow \quad E_q(I - \Pi_q)\Pi_p = 0 \quad \forall p, q \in \{1, \ldots, N\}$$

where $\Pi_p := \Pi_{(E_p, A_p)}$, $p \in \{1, \ldots, N\}$ *is the* p*-th consistency projector.*

Remark

- Index-1-case $\Rightarrow$ $E_q(I - \Pi_q) = 0\ \forall q$
- Consistency spaces equal $\Rightarrow$ $(I - \Pi_q)\Pi_p = 0\ \forall p, q$

Contents

Global Observability of Switched DAEs

$$u \longrightarrow \boxed{\begin{array}{c}\textbf{System}\ \text{(swDAE)}\\ \text{known } \sigma \\ \text{unknown } x\end{array}} \longrightarrow y$$

Definition (Global observability)

(swDAE) with given σ is (globally) observable $:\Leftrightarrow$
$\forall$ solutions $(u_1, x_1, y_1), (u_2, x_2, y_2)$: $\quad (u_1, y_1) \equiv (u_2, y_2) \;\Rightarrow\; x_1 \equiv x_2$

Lemma (0-distinguishability)

(swDAE) *is observable if, and only if,* $\boxed{y \equiv 0 \text{ and } u \equiv 0 \quad \Rightarrow \quad x \equiv 0}$

Hence consider in the following (swDAE) without inputs:

$$\boxed{\begin{aligned} E_\sigma \dot{x} &= A_\sigma x \\ y &= C_\sigma x \end{aligned}} \quad \text{and observability question:} \quad \boxed{y \equiv 0 \overset{?}{\Rightarrow} x \equiv 0}$$

Motivating example

System 1:

$$\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\dot{x}=\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}x$$
$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$$y=x_3,\ \dot{y}=\dot{x}_3=0,\ x_2=0,\ \dot{x}_1=0$$
$\Rightarrow x_1$ unobservable

System 2:

$$\begin{bmatrix}0&0&0\\0&1&0\\1&0&0\end{bmatrix}\dot{x}=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}x$$
$$y=\begin{bmatrix}0&0&1\end{bmatrix}x$$

$$y=x_3=\dot{x}_1,\ x_1=0,\ \dot{x}_2=0$$
$\Rightarrow x_2$ unobservable

$\sigma(\cdot):1\to 2$

Jump in x_1 produces impulse in y
$\Rightarrow$ Observability

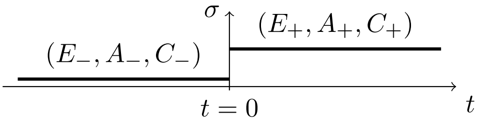
$\sigma(\cdot):2\to 1$

Jump in x_2 no influence in y
$\Rightarrow x_2$ remains unobservable

Question

$$\begin{aligned}E_p\dot{x}&=A_px+B_pu\\ y&=C_px+D_pu\end{aligned}\quad \text{not observable}\quad \overset{?}{\Rightarrow}\quad \begin{aligned}E_\sigma\dot{x}&=A_\sigma x+B_\sigma u\\ y&=C_\sigma x+D_\sigma u\end{aligned}\quad \text{observable}$$

The single switch result



Theorem (Unobservable subspace, Tanwani & T. 2010)

For (swDAE) *with **a single switch** the following equivalence holds*

$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathcal{M}$$

where

$$\boxed{\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}}$$

In particular: (swDAE) *observable* $\Leftrightarrow$ $\mathcal{M} = \{0\}$.

What are these four subspace?

The four subspaces

Unobservable subspace: $\mathcal{M} := \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}}$, i.e.

$$x(0^-) \in \mathcal{M} \quad \Leftrightarrow \quad y_{(-\infty,0)} \equiv 0 \ \wedge\ y[0] = 0 \ \wedge\ y_{(0,\infty)} \equiv 0$$

The four spaces

- Consistency: $x(0^-) \in \mathfrak{C}_-$
- Left unobservability: $y_{(-\infty,0)} \equiv 0 \ \Leftrightarrow\ x(0^-) \in \ker O_-$
- Right unobservability: $y_{(0,\infty)} \equiv 0 \ \Leftrightarrow\ x(0^-) \in \ker O_+^-$
- Impulse unobservability: $y[0] = 0 \ \Leftrightarrow\ x(0^-) \in \ker O_+^{\text{imp}}$

Question

How to calculate these four spaces?

Wong sequences

Definition

Let $E, A \in \mathbb{R}^{m\times n}$. The corresponding Wong sequences of the pair (E, A) are:

$$\begin{aligned}
\mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i), & i &= 0,1,2,3,\ldots\\
\mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{j+1} &:= E^{-1}A(\mathcal{W}_j), & j &= 0,1,2,3,\ldots
\end{aligned}$$

Note: $M^{-1}\mathcal{S} := \{x \mid Mx \in \mathcal{S}\}$ and $M\mathcal{S} := \{Mx \mid x \in \mathcal{S}\}$

Clearly, $\exists i^*, j^* \in \mathbb{N}$

$$\begin{aligned}
\mathcal{V}_0 \supset \mathcal{V}_1 \supset \ldots \supset \mathcal{V}_{i^*} = \mathcal{V}_{i^*+1} = \mathcal{V}_{i^*+2} = \ldots\\
\mathcal{W}_0 \subset \mathcal{W}_1 \subset \ldots \subset \mathcal{W}_{j^*} = \mathcal{W}_{j^*+1} = \mathcal{W}_{j^*+2} = \ldots
\end{aligned}$$

Wong limits:

$$\boxed{\mathcal{V}^* := \bigcap_{i\in\mathbb{N}} \mathcal{V}_i = \mathcal{V}_{i^*}} \qquad \boxed{\mathcal{W}^* = \bigcup_{j\in\mathbb{N}} \mathcal{W}_j = \mathcal{W}_{j^*}}$$

Wong sequences and the QWF

Theorem (QWF [Berger, Ilchmann & T. 2012])

The following statements are equivalent for square $E, A \in \mathbb{R}^{n\times n}$*:*

(i) (E, A) *is regular*

(ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{R}^n$

(iii) $E\mathcal{V}^* \oplus A\mathcal{W}^* = \mathbb{R}^n$

In particular, with $\operatorname{im} V = \mathcal{V}^*$, $\operatorname{im} W = \mathcal{W}^*$

$$(E, A) \text{ regular} \quad \Rightarrow \quad T := [V, W] \text{ and } S := [EV, AW]^{-1} \text{ invertible}$$

and S, T *yield* ***quasi-Weierstrass form (QWF)****:*

$$(SET, SAT) = \left(\begin{bmatrix} I & \\ & N \end{bmatrix}, \begin{bmatrix} J & \\ & I \end{bmatrix} \right), \quad N \text{ nilpotent}$$

Calculation of Wong sequences

Remark

Wong sequences can easily be calculated with Matlab even when the matrices still contain symbolic entries (like “R”, “L”, “C”).

```
function V=getPreImage(A,S)
% returns a basis of the preimage of A of the linear space spanned by
% the columns of S, i.e. im V = { x | Ax \in im S }

[m1,n1]=size(A); [m2,n2]=size(S);
if m1==m2
    H=null([A,S]);
    V=colspace(H(1:n1,:));
else
    error('Both matrices must have same number of rows');
end;
```

Consistency space

$$x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\mathsf{imp}-} \quad \Leftrightarrow \quad y \equiv 0$$

Corollary from QWF

$$\mathfrak{C}_- = \mathcal{V}_-^*$$

where $\mathcal{V}_-^*$ is the first Wong limit of (E_-, A_-).

The differential projector

For regular (E, A) let $(SET, SAT) = \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \right)$.

Definition (Differential "projector")

$$\Pi^{\text{diff}}_{(E,A)} := T \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} S \quad \text{and} \quad \boxed{A^{\text{diff}} := \Pi^{\text{diff}}_{(E,A)} A}$$

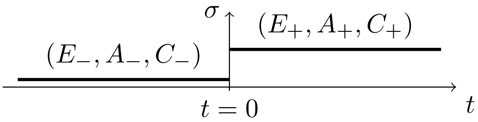
Following Implication holds:

$$x \text{ solves } E\dot{x} = Ax \quad \Rightarrow \quad \dot{x} = A^{\text{diff}} x$$

Hence, with $y = Cx$,

$$y \equiv 0 \quad \Rightarrow \quad x(0) \in \ker[C / CA^{\text{diff}} / C(A^{\text{diff}})^2 / \cdots / C(A^{\text{diff}})^{n-1}]$$

The spaces O_- and O_+



Hence

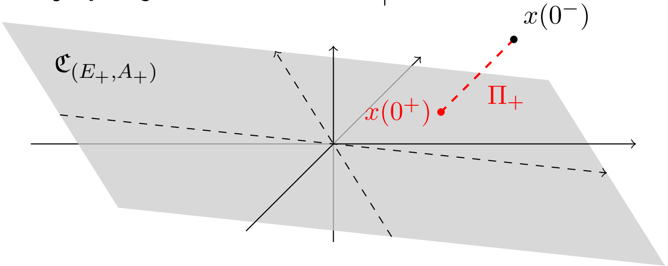
$$y_{(-\infty,0)} \equiv 0 \quad \Rightarrow \quad x(0^-) \in \ker \underbrace{[C_-/C_-A_-^{\mathsf{diff}}/C_-(A_-^{\mathsf{diff}})^2/\cdots/C_-(A_-^{\mathsf{diff}})^{n-1}]}_{:= O_-}$$

and

$$y_{(0,\infty)} \equiv 0 \quad \Rightarrow \quad x(0^+) \in \ker \underbrace{[C_+/C_+A_+^{\mathsf{diff}}/C_+(A_+^{\mathsf{diff}})^2/\cdots/C_+(A_+^{\mathsf{diff}})^{n-1}]}_{:= O_+}$$

Question: $x(0^+) \in \ker O_+ \quad \Rightarrow \quad x(0^-) \in \ ?$

Consistency projector and O_+^-



Assume $(S_+E_+T_+, S_+A_+T_+) = \left(\begin{bmatrix} I & 0 \\ 0 & N_+ \end{bmatrix}, \begin{bmatrix} J_+ & 0 \\ 0 & I \end{bmatrix}\right)$:

Consistency projector

$x(0^+) = \Pi_+ x(0^-)$ where

$$\Pi_+ := T_+ \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_+^{-1}$$

$$x(0^+) \in \ker O_+$$

$$\Rightarrow \quad x(0^-) \in \Pi_+^{-1} \ker O_+ = \ker \underbrace{O_+\Pi_+}_{=:\, O_+^-}$$

The impulsive effect

Assume $(S_+E_+T_+, S_+A_+T_+) = \left(\left[\begin{smallmatrix} I & 0 \\ 0 & N_+ \end{smallmatrix}\right], \left[\begin{smallmatrix} J_+ & 0 \\ 0 & I \end{smallmatrix}\right]\right)$:

Definition (Impulse "projector")

$$\Pi_+^{\text{imp}} := T_+ \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} S_+ \quad \text{and} \quad \boxed{E_+^{\text{imp}} := \Pi_+^{\text{imp}} E_+}$$

Impulsive part of solution:

$$x[0] = -\sum_{i=0}^{n-1} (E_+^{\text{imp}})^{i+1} x(0^-)\, \delta_0^{(i)}$$

Dirac impulses

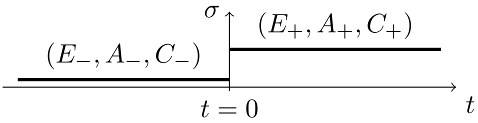
Conclusion:

$$y[0] = 0 \quad \Rightarrow \quad C_+ x[0] = 0 \quad \Rightarrow \quad \boxed{x(0^-) \in \ker O_+^{\text{imp}}}$$

where

$$O_+^{\text{imp}} := [C_+ E_+^{\text{imp}} \,/\, C_+ (E_+^{\text{imp}})^2 \,/\, \cdots \,/\, C_+ (E_+^{\text{imp}})^{n-1}]$$

Observability summary



$$y \equiv 0 \quad \Leftrightarrow \quad x(0^-) \in \mathfrak{C}_- \cap \ker O_- \cap \ker O_+^- \cap \ker O_+^{\text{imp}-}$$

with

- $\mathfrak{C}_- = \mathcal{V}_-^*$ (first Wong limit)
- $O_- = [C_-/C_-A_-^{\text{diff}}/C_-(A_-^{\text{diff}})^2/\cdots/C_-(A_-^{\text{diff}})^{n-1}]$
- $O_+^- = [C_+/C_+A_+^{\text{diff}}/C_+(A_+^{\text{diff}})^2/\cdots/C_+(A_+^{\text{diff}})^{n-1}]\Pi_+$
- $O_+^{\text{imp}} = [C_+E_+^{\text{imp}}\,/\,C_+(E_+^{\text{imp}})^2\,/\,\cdots\,/\,C_+(E_+^{\text{imp}})^{n-1}]$

Example revisited

System 1:

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\dot{x} = \begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}x$$

$$y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

$\sigma(\cdot): 1 \to 2$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_1, e_3\},\\ \ker O_- &= \operatorname{span}\{e_1, e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1, e_2, e_3\},\\ \ker O_+^{\mathsf{imp}} &= \operatorname{span}\{e_2, e_3\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M} = \{0\}$$

System 2:

$$\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 1 & 0 & 0\end{bmatrix}\dot{x} = \begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}x$$

$$y = \begin{bmatrix}0 & 0 & 1\end{bmatrix}x$$

$\sigma(\cdot): 2 \to 1$ gives

$$\begin{aligned}\mathfrak{C}_- &= \operatorname{span}\{e_2\},\\ \ker O_- &= \operatorname{span}\{e_1, e_2\}\\ \ker O_+^- &= \operatorname{span}\{e_1, e_2\},\\ \ker O_+^{\mathsf{imp}} &= \operatorname{span}\{e_1, e_2, e_3\}\end{aligned}$$

$$\Rightarrow \quad \mathcal{M} = \operatorname{span}\{e_2\}$$

Overall summary

$$
\begin{aligned}
E_\sigma \dot{x} &= A_\sigma x + B_\sigma u \\
y &= C_\sigma x + D_\sigma u
\end{aligned}
\qquad \text{(swDAE)}
$$

Piecewise-smooth distributional solution framework

$x \in \mathbb{D}^n_{\text{pw}\mathcal{C}^\infty}$, $u \in \mathbb{D}^m_{\text{pw}\mathcal{C}^\infty}$, $y \in \mathbb{D}^p_{\text{pw}\mathcal{C}^\infty}$

- Existence and uniqueness of solutions? ✓
- Jumps and impulses in solutions? ✓
- Conditions for impulse free solutions? ✓
- Control theoretical questions
  - Stability ✓ and stabilization ?
  - Observability ✓ and observer design ✓
  - Controllability ✓ and controller design ?
  - Extension to nonlinear case ?